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GENERATING UNIFORMLY DISTRIBUTED POINTS IN A SPHERE OF NORM $\| \|_p$ IN $\mathbb{R}^n$

by

Stefan Corneliu STEFANESCU and Virgil CRAIU

Summary: In this paper an efficient algorithm is given for generating uniformly distributed points in a sphere in the space $\mathbb{R}^n$ with norm $\| \|_p$. A way is suggested for generating uniformly distributed points in a general sphere of norm $\| \|_p^{(\bar{C})}$, or in the intersection of such domains in $\mathbb{R}_+^n$, and uniformly distributed points on the surface of the Euclidean norm sphere or in a bounded domain of $\mathbb{R}^n$. The classical results are obtained as particular cases.

§1. **Introduction**. Let $\mathbb{R}^n$ the n-dimensional real space. For $p > 0$, we consider the norm

$$\|(x_1,x_2,\ldots,x_n\|_p = (|x_1|^p+|x_2|^p+\ldots+|x_n|^p)^{1/p} \tag{1}$$

and the sphere $S(n,p)$

$$S(n,p) = \{(x_1,x_2,\ldots,x_n) \mid x_i \in R,\ 1 \leq i \leq n,\ \|(x_1,x_2,\ldots,x_n)\|_p < 1\}. \tag{2}$$

DEFINITION 1. The random vector $(X_1,X_2,\ldots,X_n)$ is *uniformly distributed* over the domain $D \subset R^n$, D bounded, of nonzero volume (vol.(D) $\neq$ 0) if it has the density function

$$g(x_1,x_2,\dots,x_n) = \begin{cases} 1/\mathrm{vol}(D), & \text{for } (x_1,x_2,\dots,x_n) \in D ; \\ 0 , & \text{otherwise} ; \end{cases} \quad (3)$$

where

$$\mathrm{vol}(D) = \int_D \cdots \int dt_1 \dots dt_n . \quad (4)$$

DEFINITION 2. The random variable E is EXPS(p)-*distributed* (p-th *order symmetrical exponential distribution*) if it has the following density function

$$f(x) = [1/(2.p^{1/p-1}\ \Gamma(1/p)]\ \exp(-|x|^p/p) \quad (5)$$

where

$$\Gamma(a) = \int_0^\infty t^{a-1}\exp(-t)\,dt \quad ; \quad a > 0. \quad (6)$$

DEFINITION 3. The random variable E is EXPN(p)-*distributed* (p-th *order non symmetrical exponential distribution*) if it has the following density function

$$f(x) = \begin{cases} [1/(p^{1/p-1}\Gamma(1/p)]\exp(-x^p/p), & \text{for } x > 0 \\ 0 , & \text{otherwise.} \end{cases} \quad (7)$$

DEFINITION 4. The random variable G is *gamma distributed with shape parameter* a, a > 0 if it has the following density function

$$f(x) = \begin{cases} [x^{a-1}\exp(-x)]/\Gamma(a), & \text{for } x > 0 ; \\ 0 , & \text{otherwise} . \end{cases} \quad (8)$$

REMARK 1. Random variables having the normal distribution with mean 0 and variance 1 are EXPS(2)-random variables. The exponential random variables may be regarded as

EXPN(1)-random variables or as gamma random variables with shape parameter 1.

We shall formulate an algorithm for generating uniformly distributed points over the domain S(n,p).

§2. Theoretical results.

THEOREM 1. *If* $Y_1, Y_2, \ldots, Y_n, Y_{n+1}$ *are independent random variables,* $Y_1, Y_2, \ldots, Y_n$ *have* EXPS(p) *distribution,* Y_{n+1} *has the density function*

$$f_{n+1}(y) = \begin{cases} y^{p-1}\exp(-y^p/p), & \text{if } y > 0 \\ 0, & \text{if } y \leq 0 \end{cases} \tag{9}$$

and

$$X_i = Y_i/(|Y_1|^p + |Y_2|^p + \ldots + |Y_n|^p + Y_{n+1}^p)^{1/p}; \quad 1 \leq i \leq n, \tag{10}$$

then the random vector $(X_1, X_2, \ldots, X_n)$ *is uniformly distributed over the domain* S(n,p).

Proof. For any $y_{n+1} > 0$ ($y_{n+1} \in R_+$), the transformation

$$\begin{aligned} x_j &= y_j/(|y_1|^p + |y_2|^p + \ldots + |y_n|^p + y_{n+1}^p)^{1/p}, \quad 1 \leq j \leq n, \\ x_{n+1} &= (|y_1|^p + |y_2|^p + \ldots + |y_n|^p + y_{n+1}^p)^{1/p} \end{aligned} \tag{11}$$

is one to one between $\mathbf{R}^n \times \mathbf{R}_+$ and $S(n,p) \times \mathbf{R}_+$. From (11), we obtain

$$\begin{aligned} y_i &= x_i x_{n+1}, \quad 1 \leq i \leq n \\ y_{n+1} &= x_{n+1}(1 - |x_1|^p - |x_2|^p - \ldots - |x_n|^p)^{1/p}. \end{aligned} \tag{12}$$

Let $J = D(y_1, y_2, \ldots, y_n, y_{n+1})/D(x_1, x_2, \ldots, x_n, x_{n+1})$ be the determinant of the matrix $(\partial y_i/\partial x_j)_{1 \leq i, j \leq n+1}$

$$J = \det((\partial y_i/\partial x_j)_{1\leq i,j\leq n+1}) \qquad (13)$$

From (12), we obtain

$$J = (-x_{n+1}^{n}/r_x^{p-1})F(n,x_1,x_2,\ldots,x_n,-r_x^p,p) \qquad (14)$$

where

$$r_x = (1-|x_1|^p - |x_2|^p - \ldots - |x_n|^p)^{1/p}$$

$$F(n,x_1,x_2,\ldots,x_n,w,p) = \det\begin{pmatrix} 1 & 0 & 0 & \ldots & 0 & x_1 \\ 0 & 1 & 0 & \ldots & 0 & x_2 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & 0 & \ldots & 1 & x_n \\ d_1 & d_2 & d_3 & \ldots & d_n & w \end{pmatrix} \qquad (15)$$

and $d_i = \mathrm{sign}\,(x_i)\,|x_i|^{p-1}$, $1 \leq i \leq n$, sign (x) being the sign of the real number x. Expanding the determinant (15) by the first column we obtain the following recurrence relationship

$$F(x_1,x_2,\ldots,x_n,w,p) = F(n-1,x_2,x_3,\ldots,x_n,w,p) + \\ + (-1)^{n+2}\mathrm{sign}(x_1)|x_1|^{p-1}\det(A) \qquad (16)$$

where A is the matrix

$$A = \begin{pmatrix} 0 & 0 & 0 & \ldots & 0 & 0 & x_1 \\ 1 & 0 & 0 & \ldots. & 0 & 0 & x_2 \\ 0 & 1 & 0 & \ldots. & 0 & 0 & x_3 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & 0 & \ldots. & 1 & 0 & x_{n-1} \\ 0 & 0 & 0 & \ldots. & 0 & 1 & x_n \end{pmatrix}. \qquad (17)$$

From the matrix A one can obtain a diagonal matrix by moving the first row to the last place. Therefore

$$\det(A) = (-1)^{n-1}x_1 \qquad (18)$$

and hence the relation (16) may be written in the form

$$F(n,x_1,x_2,x_3,\ldots,x_n,w,p) = F(n-1,x_2,x_3,\ldots,x_n,w,p)-|x_1|^p . \tag{19}$$

By n sequencial applications of equality (19) and from (14) and (15) we obtain

$$J = x_{n+1}^n(1-|x_1|^p-|x_2|^p-\ldots-|x_n|^p)^{1/p-1} . \tag{20}$$

The random variables $Y_1,Y_2,\ldots,Y_n,Y_{n+1}$ being independent, it follows that the random vector $(Y_1,Y_2,\ldots,Y_n,Y_{n+1})$ has the density function

$$g_Y(y_1,y_2,\ldots,y_n,y_{n+1}) = \begin{cases} y_{n+1}^{p-1}\left[\prod_{j=1}^{n+1}\exp(-y_j^p/p)\right]\Big/\left[2p^{1/p-1}\Gamma(1/p)\right]^n , \\ \qquad \text{if } y_{n+1}>0 ; \\ 0 , \quad \text{if } y_{n+1}\leqslant 0 . \end{cases} \tag{21}$$

Let $(X_1,X_2,\ldots,X_n,X_{n+1})$ be the random vector obtained from the random vector $(Y_1,Y_2,\ldots,Y_n,Y_{n+1})$ by the transformation (11). If $g_X(x_1,x_2,\ldots,x_n,x_{n+1})$ is the density function of the random vector $(X_1,X_2,\ldots,X_n,X_{n+1})$, then

$$g_X(x_1,\ldots,x_n,x_{n+1}) = |J|g_Y(y_1(x_1,\ldots,x_{n+1}),\ldots,y_{n+1}(x_1,\ldots,x_{n+1})) \tag{22}$$

where the Jacobian J of the transformation (11) is given by (20). Now from (12) it follows that

$$g_X(x_1,\ldots,x_n,x_{n+1}) = \begin{cases} \left[x_{n+1}^{n+p-1}\exp(-x_{n+1}^p/p\right]\Big/\left[2p^{1/p-1}\Gamma(1/p)\right]^n \\ \text{if } x_{n+1}>0,\ (x_1,x_2,\ldots,x_n)\in S(n,p) ; \\ 0 , \text{ otherwise.} \end{cases} \tag{23}$$

If $g(x_1,x_2,\ldots,x_n)$ is the density function of the random vector $(X_1,X_2,\ldots,X_n)$ given by (10), then

$$g(x_1,x_2,\ldots,x_n) = \int_0^\infty g_X(x_1,x_2,\ldots,x_n,x_{n+1})dx_{n+1} = \text{constant} \qquad (24)$$

which proves the theorem.

REMARK 2. Making the substitution $t = x_{n+1}^p/p$ in the integral (24) we obtain

$$g(x_1,\ldots,x_n) = [p^n\Gamma(n/p+1)]/[2\Gamma(1/p)]^n = 1/\text{vol}(S(n,p)).$$

COROLLARY 1. (Stefănescu [12]). *If $Z_1,Z_2,\ldots,Z_n,Z_{n+1}$ are independent random variables, Z_i, $1 \leq i \leq n$, having a normal distribution with mean 0 and variance 1, and Z_{n+1} having the density function*

$$f(z) = \begin{cases} z\exp(-z^2/2), & \text{if } z > 0; \\ 0, & \text{if } z < 0; \end{cases}$$

and

$$X_i = Z_i/(Z_1^2 + Z_2^2 + \ldots + Z_n^2 + Z_{n+1}^2)^{1/2}, \qquad 1 \leq i \leq n,$$

then the random vector $(X_1,X_2,\ldots,X_n)$ is uniformly distributed in the sphere S(n,2) (of Euclidean norm).

The proof of this statement results from Remark 1 and Theorem 1 for p = 2.

§3. Generating algorithm. Since the density function (5) is obtained by symmetry from the density function (7), we have:

PROPOSITION 1. *If W is a discrete random variable that may take the values -1 and 1 each with probability 1/2 and E is a EXPN(p)-distributed random variable then W.E is a EXPS(p) distributed random variable.*

REMARK 3. This proposition can be used for computer generation of EXPS(p) random variables starting from EXPN(p) random variables. Stefănescu and Vaduva [13] (extending the method suggest by Kinderman and Monahan [5], [6]) indicate efficient algorithms for generating EXPN(p) random variables.

The EXPN(p) random variables may be generated using:

PROPOSITION 2. *If* G *is a gamma random variable with shape parameter* 1/p *then the random variable* $(p.G)^{1/p}$ *is* EXPN(p) *distributed.*

PROPOSITION 3. *If* U *is a uniform random variable on the interval* (0,1), *then the random variable* $(-p\cdot\ell n(U))^{1/p}$ *has the density function given by formula* (9).

Proof. The random variable Z with density function (9), has for $z \geq 0$, the distribution function

$$F(z) = \int_0^z t^{p-1}\exp(-t^p/p)dt = 1-\exp(-z^p/p).$$

Since for a random variable U uniformly distributed on (0,1), the random variable $F^{-1}(1-U)$ has the distribution function F, the proposition is established.

THEOREM 2. *If* $W_1,W_2,\ldots,W_n,G_1,G_2,\ldots,G_n,U$ *are independent random variables,* $W_1,W_2,\ldots,W_n$ *discrete random variables that may take only the values* -1 *and* 1 *each with probability* 1/2, $G_1,G_2,\ldots,G_n$ *random variables gamma distributed with shape parameter* 1/p, U *uniformly distributed on* (0,1), *and*

$$X_i = (W_i G_i^{1/p})/(G_1+G_2+\ldots+G_n-\ell n(U))^{1/p}, \quad 1 \leq i \leq n,$$

then the random vector $(X_1,X_2,X_3,\ldots,X_n)$ *is uniformly distributed on the domain* S(n,p).

The proof of Theorem 2 results from Theorem 1 and Proposition 1, 2 and 3.

The Theorem 2 leads to the UNIFS generating algorithm of points $P(x_1,x_2,\dots,x_n)$ uniformly distributed on the domain $S(n,p)$.

Algorithm UNIFS (UNIFormly distribited points inside the Sphere $S(n,p)$).

Step 0. Read n,p.

Step 1. Generate $U_1,U_2,\dots,U_n,U$ independent random variables uniformly distributed on $(0,1)$.

Step 2. Generate $G_1,G_2,\dots,G_n$ independent random variables having a gamma distribution with shape parameter $1/p$.

Step 3. $S \leftarrow (G_1+G_2+\dots+G_n-\ell n(U))^{1/p}$.

Step 4. If $U_i < 1/2$ then $W_i \leftarrow -1$; else $W_i \leftarrow 1$; (for $i = 1,2,3,4,\dots,n$).

Step 5. $x_i \leftarrow W_i G_i^{1/p}/S$; $1 \leqslant i \leqslant n$.

Step 6. Write the point $P(x_1,x_2,x_3,\dots,x_n)$. STOP.

REMARK 4. The UNIFS algorithm is very fast. Comparing the results obtained by Deák [3] and Stefanescu [12], the UNIFS algorithm (the case $p = 2$) is proved to be the fastest algorithm for generating uniformly distributed points inside the sphere of Euclidean norm.

For computer generating gamma and uniformly distributted random variables, a subroutine library is used, the RAVAGE (Văduva [16]). It may be consulted the references [7],[9], [10],[11] or more precisely [1],[2],[5],[6],[14].

§4. Other domains. Further we shall generate sequences of uniformly distributed points on other bounded domains D,

$D \subset \mathbb{R}^n$. Let:

$$S_+(n,p) = \{(x_1,x_2,\dots,x_n) \mid (x_1,x_2,\dots,x_n) \in S(n,p),\ x_i > 0,\ 1 \leq i \leq n\}$$

$$S_o(n,p,h,\bar{c},\bar{b}) = \{(x_1,x_2,\dots,x_n) \in \mathbb{R}^n \mid \|(x_1-b_1,\dots,x_n-b_n)\|_p^{(\bar{c})} < h\}$$

$$S_{o+}(n,p,h,\bar{c},\bar{b}) = \{(x_1,x_2,\dots,x_n) \in S_o(n,p,h,\bar{c},\bar{b}) \mid x_i > b_i,\ 1 \leq i \leq n\}$$

$$SS(n) = \{(x_1,x_2,\dots,x_n) \in \mathbb{R}^n \mid x_1^2 + x_2^2 + \dots + x_n^2 = 1\}$$

where $\bar{c} = (c_1,c_2,\dots,c_n)$, $\bar{b} = (b_1,b_2,\dots,b_n)$ with $c_i > 0$, $b_i \in R$, $1 \leq i \leq n$, $h > 0$, and for $(y_1,y_2,\dots,y_n) \in R^n$ we have

$$\|(y_1,y_2,\dots,y_n)\|_p^{(\bar{c})} = (c_1|x_1|^p + c_2|x_2|^p + \dots + c_n|x_n|^p)^{1/p}.$$

4.1. Uniformly distributed random points on $S_+(n,p)$. The proof of the following theorem is similar to that of Theorem 1.

THEOREM 3. *If $Y_1,Y_2,\dots,Y_n,Y_{n+1}$ are independent random variables, $Y_1,Y_2,\dots,Y_n$ have EXPN(p) distribution, Y_{n+1} has the density function (9), and*

$$X_i = Y_i/(Y_1^p + Y_2^p + \dots + Y_n^p + Y_{n+1}^p)^{1/p}, \quad 1 \leq i \leq n,$$

then the random vector $(X_1,X_2,\dots,X_n)$ is uniformly distributed on the domain $S_+(n,p)$.

From Theorem 3, for $p = 1$, we obtain

COROLLARY 2. (Feller [4], p.76). *If E_i, $1 \leq i \leq n+1$, are independent exponential random variables and*

$$X_i = E_i/(E_1 + E_2 + \dots + E_n + E_{n+1}), \quad 1 \leq i \leq n,$$

then the random vector $(X_1,X_2,\dots,X_n)$ is uniformly distributed in the "unity" n-simplex $S_+(n,1)$.

4.2. Uniformly distributed random points on $S_o(n,p,h,\bar{c},\bar{b})$ and $S_{o+}(n,p,h,\bar{c},\bar{b})$. Starting from the points $P(x_1,x_2,\dots,x_n)$, uniformly distributed on the domain $S(n,p)$, the following theorem enables the generation of points $Q(y_1,y_2,\dots,y_n)$ uniformly distributed inside the sphere $S_o(n,p,h,\bar{c},\bar{b})$.

THEOREM 4. *If the random vector* $(X_1,X_2,\dots,X_n)$ *is uniformly distributed on the domain* $S(n,p)$ *and*

$$Y_i = (h\,X_i)/c_i + b_i, \quad 1 \leqslant i \leqslant n,$$

then the random vector $(Y_1,Y_2,\dots,Y_n)$ *is uniformly distributed on the domain* $S_o(n,p,h,\bar{c},\bar{b})$.

The proof follows from Definition 1 and using the transformation

$$y_i = (h\,x_i)/c_i + b_i, \quad 1 \leqslant i \leqslant n, \tag{25}$$

that is one to one between $S(n,p)$ and $S_o(n,p,h,\bar{c},\bar{b})$ and whose Jacobian is constant:

$$J = D(y_1,y_2,\dots,y_n)/D(x_1,x_2,\dots,x_n) = h^n/(c_1.c_2.c_3\dots c_n).$$

REMARK 5. One can obtain a similar result for domains $S_{o+}(n,p,h,\bar{c},\bar{b})$ considering the transformation (25) with $(x_1,x_2,\dots,x_n) \in S_+(n,p)$.

4.3. Uniformly distributed random points on a bounded domain. For generating uniformly distributed random points on a bounded domain D, $D \subset R^n$, $\mathrm{vol}(D) \neq 0$, we can use a composition-rejection procedure.

We consider a partition $\{D_j\}_{1\leqslant j\leqslant m}$ of the domain D:

$$D = D_1 \cup D_2 \cup \dots \cup D_m, \quad D_i \cap D_j = \emptyset, \quad \mathrm{vol}(D_j) \neq 0, \quad 1 \leqslant i < j \leqslant m.$$

In this case the generation of a uniformly random point

$P(x_1, x_2, \ldots, x_n)$ in the domain D can be made by a random choice of a domain D_j, $1 \leq j \leq m$ (depending on $vol(D_j)$), and then generating the uniformly distributed point on D_j.

We can obtain uniformly distributed points on the domain D_j by a rejection procedure:

- find a domain $S_o(n,p,h,\bar{c},\bar{b})$ (or $S_{o+}(n,p,h,\bar{c},\bar{b})$) that includes the domain D_j;
- generate uniformly distributed points $P(x_1, x_2, \ldots, x_n)$ on the domain $S_o(n,p,h,\bar{c},\bar{b})$ (Theorem 4 and Algorithm UNIFS);
- reject these points $P(x_1, x_2, \ldots, x_n)$ such that $P \notin D_j$.

The accepting probability P_{ac} of points into the domain D_j is equal to

$$P_{ac} = vol(D_j)/vol(S_o(n,p,h,\bar{c},\bar{b})) \leq 1.$$

We can increase the accepting probability value P_{ac} (accelerating the speed of rejection procedure) by finding the values $p_o, h_o, \bar{c}_o, \bar{b}_o$ of the parameters $p, h, \bar{c}, \bar{b}$ so that $S_o(n, p_o, h_o, \bar{c}_o, \bar{b}_o)$, with $S_o(n, p_o, h_o, \bar{c}_o, \bar{b}_o) \supseteq D_j$, will have the minimum volume.

4.4. <u>Uniformly distributed points on</u> SS(n). The following result is well known

PROPOSITION 4. (Deák [3], Knuth [7]). *If the random vector* $(X_1, X_2, \ldots, X_n)$ *is uniformly distributed on the domain* $S(n,2)-\{0\}$,

$$Y_i = X_i/(X_1^2 + X_2^2 + X_3^2 + \ldots + X_n^2)^{1/2}, \quad 1 \leq i \leq n,$$

then the random vector $(Y_1, Y_2, \ldots, Y_n)$ *is uniformly distributed on the domain* SS(n).

Using Proposition 4 and Corollary 1, one obtains Muller's result.

COROLLARY 3. (Muller [8]). *If* $Z_1, Z_2, Z_3, \ldots, Z_n$ *are independent random variables, normally distributed with mean* 0 *and variance* 1, *and*

$$Y_i = Z_i/(Z_1^2 + Z_2^2 + \ldots + Z_{n-1}^2 + Z_n^2)^{1/2}, \quad 1 \leq i \leq n,$$

then the random vector $(Y_1, Y_2, \ldots, Y_n)$ *is uniformly distributed on the domain* SS(n).

REMARK 6. In the case of uniformly distributed random points $Q(y_1, y_2, \ldots, y_n)$ on SS(n), the Definition 1 is not used (since vol(SS(n)) = 0).

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University of Bucharest
Computing Center
Bucharest, Rumania

University of Bucharest
Faculty of Mathematics
Bucharest, Rumania

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