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SOME NUMBER THEORETICAL PRODUCTS

by

Emil GROSSWALD

§1. Introduction. While a different topic was investigated, it became necessary to know asymptotic values of products of the form

$$\prod_{\substack{p\equiv a(\mathrm{mod}\ m)\\ p\leq x}} (1-p^{-1}) ,$$

with co-prime integers m and a, and where p stands for rational primes.

The result

$$\prod_{p\leq x} (1-p^{-1}) \simeq e^{-\gamma}/\log x \quad (\gamma = \text{Euler's constant})$$

is classical (see, e.g., [2]), but the asymptotic value of the first mentioned products could not be located in the literature. Here the symbol $\simeq$ stands for asymptotic equality.

Here is an account of the results obtained. The main theorem is valid for all integers m, but at present the methods used, all classical, are completely successful for the numerical determination of some relevant constants only for those moduli m all of whose characters are real, i.e., only for m|24. These methods are also effective in some related problems. It is likely that the results can be extended to the general case without too much trouble. In what follows, $\phi(m)$ stands for Euler's ϕ-function. For the values of m under consideration, we recall that $\phi(3) = \phi(4) = \phi(6) = 2$, $\phi(8) = \phi(12) = 4$, and $\phi(24) = 8$.

§2. **Main Result**. For integers a and m, with $(a,m) = 1$, the following theorem holds.

THEOREM 1. $\prod_{\substack{p\equiv a(\text{mod } m)\\ p\leqslant x}} (1-p^{-1}) = k_a/(\log x)^{1/\phi(m)}$, *where* k_a *is a positive constant.*

Proof. In this section all congruences are understood modulo m, unless a different modulus is stated. We have

$$\log \prod_{\substack{p\equiv a\\ p\leqslant x}} (1-\tfrac{1}{p}) = \sum_{\substack{p\equiv a\\ p\leqslant x}} \log(1-\tfrac{1}{p}) = -\prod_{\substack{p\equiv a\\ p\leqslant x}} \frac{1}{p} - U(x),$$

where, for $x > a$,

$$0 < U(x) = \sum_{\substack{p\equiv a\\ p\leqslant x}} \left(\frac{1}{2p^2} + \frac{1}{3p^3} + \ldots\right) < \frac{1}{2}\sum_{p\leqslant x} \frac{1}{p(p-1)} < \frac{1}{2}\sum_{n=1}^{\infty} \frac{1}{n(n-1)} = \frac{1}{2}.$$

Also, $U(x)$ increases monotonically with x, so that $0 < \lim_{x\to\infty} U(x) = U < \frac{1}{2}$ and, for $x \to \infty$,

$$\log \prod_{\substack{p\equiv a\\ p\leqslant x}} (1-\tfrac{1}{p}) = -\sum_{\substack{p\equiv a\\ p\leqslant x}} \frac{1}{p} - U - o(1). \tag{1}$$

Next, let

$$a_n = \begin{cases} 1 & \text{if } \; n = p \equiv a(\text{mod } m), \\ 0 & \text{otherwise}, \end{cases}$$

and set $A_u = \sum_{n\leqslant u} a_n$ for $u \geqslant a$, $A_u = 0$ for $u < a$. By partial summation with $x \in \mathbb{Z}$ (this is done for simplicity of presentation and leads to no loss of generality), we obtain

$$\sum_{\substack{p\equiv a\\ p<x}} \frac{1}{p} = \sum_{a\leq n\leq x} \frac{a_n}{n} = \sum_{a\leq n\leq x} \frac{A_n - A_{n-1}}{n} = \sum_{a\leq n\leq x} A_n\left(\frac{1}{n} - \frac{1}{n+1}\right) + \frac{A_x}{x+1} - \frac{A_{a-1}}{a}.$$

Here $A_{a-1} = 0$ and, by the Siegel-Walfisz-Page theorem,

$$A_x = \frac{x}{\phi(m)\log x} + O\left(\frac{x}{\log^2 x}\right). \tag{2}$$

It follows that, for $x \to \infty$,

$$\sum_{\substack{p\equiv a\\ p\leq x}} \frac{1}{p} = \sum_{a\leq n\leq x} A_n \int_n^{n+1} \frac{du}{u^2} + o(1) = \int_a^{x+1} \frac{A_u}{u^2}\, du + o(1).$$

Here the integral can be computed by use of (2):

$$\int_{a^-}^{n+1} \frac{A_u}{u^2}\, du = \frac{1}{\phi(m)} \int_a^{x+1} \frac{du}{u\log u} + \int_a^{x+1} \left(\frac{A_u}{u^2} - \frac{1}{\phi(m) u\log u} \right) du$$

$$= \frac{1}{\phi(m)} (\log\log x - \log\log a) + \int_a^{n+1} h(u)\, du.$$

Here $h(u) = O\left(\frac{1}{u\log^2 u}\right)$, so that the integral converges to a finite value, say B, as $x \to \infty$ and we obtain

$$\sum_{\substack{p\equiv a\\ p\leq x}} \frac{1}{p} = \frac{1}{\phi(m)} \log\log x + B + o(1). \tag{3}$$

We now substitute (3) in (1) and obtain

$$\log \prod_{\substack{p\equiv a\\ p\leq x}} (1-p^{-1}) = -\frac{1}{\phi(m)} \log\log x - U - B - o(1),$$

so that

$$\prod_{\substack{p\equiv a\\ p\leq x}} \left(1 - \frac{1}{p}\right) = \frac{e^{-U-B}(1+o(1))}{(\log x)^{1/\phi(m)}} \simeq \frac{k_a}{(\log x)^{1/\phi(m)}}$$

where $k_a = e^{-U-B}$. The proof of the theorem is complete.

3. Determination of the constants k_a. The computational difficulties of the determination of the constants k_a increases with $\phi(m)$. For simplicity of exposition we shall consider for $\phi(m) = 2$ the case $m = 4$ in detail and indicate the modifications needed for $m = 6$. For $\phi(m) = 4$, we shall discuss $m = 8$, the case $m = 12$, is very similar. For $m = 24$ one can use the same method, but the computations become cumbersome and are suppressed. Finally, we consider the related problem to determine the constants k_1 and k_{-1}, for which

$$\prod_{\substack{(p/m)=1 \\ p\leq x}} (1-p^{-1}) \simeq k_1/\sqrt{\log x} \quad \text{and} \quad \prod_{\substack{(p/m)=-1 \\ p\leq x}} (1-p^{-1}) \quad k_{-1}/\sqrt{\log x}$$

holds.

§4. The Case m = 4. In this section all congruence are understood modulo 4. From

$$\frac{k_1}{\sqrt{\log x}} \frac{k_3}{\sqrt{\log x}} \simeq \prod_{\substack{p\equiv 1 \\ p\leq x}} (1-\frac{1}{p}) \prod_{\substack{p\equiv 3 \\ p\leq x}} (1-\frac{1}{p}) = \frac{1}{1-\frac{1}{2}} \prod_{p\leq x} (1-\frac{1}{p}) \simeq \frac{2\,e^{-\gamma}}{\log x}$$

it follows that it is sufficient to determine, say, k_1 as then $k_3 = 2e^{-\gamma}/k_1$. Let $\chi_1(n)$ be the principal character modulo 4 and $\chi_2(n)$ the non-principal character. The corresponding Dirchlet L-series are $L(s,\chi_1) = \sum_{n \text{ odd}} n^{-s} = \prod_{p>3} (1-p^{-s})^{-1}$ and $L(s,\chi_2) = \sum_{n=1}^{\infty} \chi_2(n) n^{-s} = \sum_p (1-\chi_2(p)p^{-s})^{-1}$, both valid for $\mathrm{Re}\, s > 1$. $L(s,\chi_2)$ can be continued as an entire function into the whole complex plane (see [5]) and $L(1,\chi_2) = 1 - \frac{1}{3} + \frac{1}{5} - \ldots = \pi/4$. It follows that

$$L(s,\chi_1)L(s,\chi_2) = \prod_{p\equiv 1} (1-p^{-s})^{-2} \prod_{p\equiv 3} (1-p^{-2s})^{-1}$$

and

$$L(s,\chi_1)/L(s,\chi_2) = \prod_{p\equiv 3} (1-p^{-2s})(1-p^{-s})^{-2}.$$

If we divide these two equations we obtain

$$L^2(s,\chi_2) = \prod_{p\equiv 3}(1-p^{-2s})^{-2}\{\prod_{p\equiv 3}(1-p^{-s})/\prod_{p\equiv 1}(1-p^{-s})\}^2,$$

so that

$$\frac{\prod_{p\equiv 3}(1-p^{-s})}{\prod_{p\equiv 1}(1-p^{-s})} = L(s,\chi_2)\prod_{p\equiv 3}(1-p^{-2s}).$$

For $s \to 1^+$ the right hand side has the limit

$$L(1,\chi_2)\prod_{p\equiv 3}(1-p^{-2}) = (\pi/4)C,$$

say. It follows that the left hand side has the same limit and we shall show that this limit equals k_3/k_1. The proof can be modeled on that in [2], pp.351-353. Here we give only a short sketch. For $\delta > 0$, set

$$F(\delta) = \prod_{p\equiv 3}(1-p^{-1-\delta})/\prod_{p\equiv 1}(1-p^{-1-\delta});$$

then

$$F(\delta) = L(1+\delta,\chi_2)\prod_{p\equiv 3}(1-p^{-2-2\delta}).$$

As $\delta \to 0$, the right hand side approaches the finite constant $(\pi/4)C$. Also, the function is continuous for $\delta = 0$ and the convergence of $F(\delta)$ to $F(0)$ is uniform; hance, $F(0) = \lim_{\delta\to\infty} F(\delta) = (\pi/4)C$. But we also have

$$F(0) = \lim_{x\to\infty}\{\prod_{\substack{p\equiv 3\\ p\leqslant x}}(1-p^{-1})/\prod_{\substack{p\equiv 1\\ p\leqslant x}}(1-p^{-1})\}$$

On the other hand, by Theorem 1, the last limit equals asymptotically $\frac{k_3/\sqrt{\log x}}{k_1/\sqrt{\log x}} = \frac{k_3}{k_1}$, as claimed. From $k_1k_3 = 2e^{-\gamma}$ and $k_1k_3 = 2e^{-\gamma}$ and $k_3/k_1 = (\pi/4)C$ immediately follows that $k_1^2 = 8e^{-\gamma}/\pi C$ and $k_3^2 = \pi Ce^{-\gamma}/2$.

The constant C occurs in some work of Landau (see [4]);

its value is about .856...

§5. The Case m = 6. In this section all congruence are understood modulo 6. By Theorem 1 we have, for appropriate constants k_1 and k_5

$$\prod_{\substack{p\equiv 1\\ p\leqslant x}}\left(1-\frac{1}{p}\right) \simeq \frac{k_1}{\sqrt{\log x}} \quad\text{and}\quad \prod_{\substack{p\equiv 5\\ p<x}}\left(1-\frac{1}{p}\right) \simeq \frac{k_5}{\sqrt{\log x}}.$$

Also,

$$\frac{1}{1-\frac{1}{2}}\,\frac{1}{1-\frac{1}{3}}\prod_{p\leqslant x}\left(1-\frac{1}{p}\right) \simeq \frac{3e^{-\gamma}}{\log x},$$

so that $k_1k_5 = 3e^{-\gamma}$. To obtain the ratio k_5/k_1, we consider the two Dirichlet series formed with the principal and the non-principal characthers modulo 6, respectively:

$$L(s,\chi_1) = \prod_{p\geqslant 5}(1-p^{-s})^{-1} \quad\text{and}\quad L(s,\chi_2) = \prod_{p\equiv 1}(1-p^{-s})^{-1}\prod_{p\equiv 5}(1+p^{-s})^{-1}.$$

It follows as before that

$$L(s,\chi_1)L(s,\chi_2) = \prod_{p\equiv 1}(1-p^{-s})^{-2}\prod_{p\equiv 5}(1-p^{-2s})^{-1}$$

and

$$\frac{L(s,\chi_1)}{L(s,\chi_2)} = \frac{\prod_{p\equiv 5}(1+p^{-s})}{\prod_{p\equiv 5}(1-p^{-s})} = \frac{\prod_{p\equiv 5}(1-p^{-2s})}{\prod_{p\equiv 5}(1-p^{-s})^2}.$$

From these two relations it follows that

$$\frac{\prod_{p\equiv 5}(1-p^{-s})}{\prod_{p\equiv 1}(1-p^{-s})} = L(s,\chi_2)\prod_{p\equiv 5}(1-p^{-2s}).$$

If we let $s\to 1^+$ and denote the product $\prod_{p\equiv 5}(1-p^{-2})$ by C_5, then the second member becomes $L(1,\chi_2)\prod_{p\equiv 5}(1-p^{-2}) = \frac{\sqrt{3}\pi}{6}C_5$. For $s \to 1^+$ the first member has the limit k_5/k_1. The two equations $k_1k_5 = 3e^{-\gamma}$ and $k_5/k_1 = \frac{\sqrt{3}\pi}{6}C_5$ now yield the values $k_1^2 = \frac{6\sqrt{3}\,e^{-\gamma}}{\pi C_5}$ and $k_5^2 = \frac{\sqrt{3}\,e^{-\gamma}\pi}{2}C_5$. No closed form expression for C_5 seems to be known; its value is about .93...

§6. **The Case m = 8.** In this section, congruences are understood modulo 8. This case is more difficult, because now $\phi(8) = 4$. By Theorem 1,

$$\prod_{\substack{p\equiv a\\ p\leq x}} (1-p^{-1}) \simeq k_a(\log x)^{-1/4} \qquad (a = 1,3,5,7)\ .$$

Also, $\prod_{a=1,3,5,7} k_a = 2e^{-\gamma}$. To obtain the ratios of the k_j's, we consider the four characters modulo 8 and the corresponding L-series. In all cases $\chi(n) = 0$ for even n.

$$\chi_1(n) = 1 \quad \text{for all odd } n$$

and

$$L(s,\chi_1) = \prod_{p>3}(1-p^{-s})^{-1} = (1-2^{-s})\zeta(s)\ ;$$

$$\chi_2(n) = \begin{cases} 1 & \text{for } n\equiv 1,5 \\ -1 & \text{for } n\equiv 3,7 \end{cases}$$

$$\chi_3(n) = \begin{cases} 1 & \text{for } n\equiv \pm 1 \\ -1 & \text{for } n\equiv \pm 3\ , \end{cases}$$

and

$$L(s,\chi_j) = \prod_{\chi_j(p)=1}(1-p^{-s})^{-1}\prod_{\chi_j(p)=-1}(1+p^{-s})^{-1};$$

$$\chi_4(n) = \begin{cases} 1 & \text{for } n\equiv 1,3 \\ -1 & \text{for } n\equiv 5,7 \end{cases} \qquad \text{for } j = 2,3,4.$$

It follows that for $\operatorname{Re} s > 1$,

$$L(s,\chi_1)L(s,\chi_j) = \prod_{\chi_j(p)=1} (1-p^{-s})^{-2} \prod_{\chi_j(p)=-1} (1-p^{-2s})^{-1}$$

and

$$\frac{L(s,\chi_j)}{L(s,\chi_1)} = \frac{\prod_{\chi_j(p)=-1} (1-p^{-s})}{\prod_{\chi_j(p)=-1} (1+p^{-s})} = \prod_{\chi_j(p)=-1} \frac{(1-p^{-s})^2}{(1-p^{-2s})} .$$

We multiply these equations and obtain

$$L^2(s,\chi_j) = \prod_{\chi_j(p)=1} (1-p^{-s})^{-2} \prod_{\chi_j(p)=-1} (1-p^{-2s})^{-2} \prod_{\chi_j(p)=-1} (1-p^{-s})^2 .$$

For $j = 2$ in particular, we obtain

$$\prod_{p\equiv 3,7} (1-p^{-s}) \prod_{p\equiv 1,5} (1-p^{-s})^{-1} = L(s,\chi_2) \prod_{p\equiv 3} (1-p^{-2s})^2 \prod_{p\equiv 7} (1-p^{-2s})^2 .$$

If we pass to the limit for $s \to 1^+$ as before and set $C_j = \prod_{p\equiv j} (1-p^{-2})$, we obtain

$$\frac{k_3k_7}{k_1k_5} = L(1,\chi_2)C_3C_7 \tag{4}$$

and similarly

$$\frac{k_5k_3}{k_1k_7} = L(1,\chi_3)C_5C_3 \tag{4'}$$

and

$$\frac{k_7k_5}{k_1k_3} = L(1,\chi_4)C_7C_5 \tag{4''}$$

For brevity we set $C_3C_5C_7 = D$ and multiply (4), (4'), and (4"); we obtain

$$k_1^{-3}k_3k_5k_7 = D^2 \prod_{j=2}^{4} L(1,\chi_j) \tag{5}$$

If we multiply (5) successively by (4), (4') and (4") the result is

$$\left(\frac{k_3k_7}{k_1^2}\right)^2 = D^2 \prod_{j=2}^{4} L(1,\chi_j)L(1,\chi_2)C_3C_7,$$

or

$$k_3^2k_7^2 = k_1^4D^2L^2L(1,\chi_2)C_3C_7 \,, \tag{6}$$

where $L^2 = \prod_{j=2}^{4} L(1,\chi_j)$, and similarly

$$k_5^2k_3^2 = k_1^4D^2L^2L(1,\chi_3)C_5C_3 \tag{6'}$$

and

$$k_7^2k_5^2 = k_1^4D^2L^2L(1,\chi_4)C_7C_5. \tag{6''}$$

By dividing successively each of the equations (4), (4'), and (4") by the succeeding one, we obtain the desired ratios

$$\left(\frac{k_7}{k_5}\right)^2 = \frac{L(1,\chi_2)}{L(1,\chi_3)}\,\frac{C_7}{C_5} \tag{7}$$

$$\left(\frac{k_3}{k_7}\right)^2 = \frac{L(1,\chi_3)}{L(1,\chi_4)}\,\frac{C_3}{C_7} \tag{7'}$$

$$\left(\frac{k_5}{k_3}\right)^2 = \frac{L(1,\chi_4)}{L(1,\chi_2)}\,\frac{C_5}{C_3}. \tag{7''}$$

We now multuply (6) by (7'), (6') by (7"), (6") by (7) and obtain $k_3^4 = k_1^4D^2L^2(1,\chi_2)L^2(1,\chi_3)C_3^2$, or $k_3^2 = k_1^2DL(1,\chi_2)$. $L(1,\chi_3)C_3$ and similarly for k_5^2 and k_7^2. So we have proved the relations

$$\begin{aligned} k_3^2 &= k_1^2\, D\,L(1,\chi_2)L(1,\chi_3)C_3 \\ k_5^2 &= k_1^2\, D\,L(1,\chi_3)L(1,\chi_4)C_5 \\ K_7^2 &= k_1^2\, D\,L(1,\chi_4)L(1,\chi_2)C_7. \end{aligned} \tag{8}$$

We now recall that $(k_1k_3k_5k_7)^2 = 4e^{-2\gamma}$, substitute the values of the k_j's from (8) in its first member and solve for

k_1^2. This leads to $k_1^2 = \surd(2e^{-\gamma})/DL$. If we use this value in (8), we obtain explicit expressions for all k_j,s, namely

$$k_1^2 = \surd(2e^{-\gamma})/DL$$

$$k_3^2 = \surd(2e^{-\gamma})LC_3/L(1,\chi_4)$$

$$k_5^2 = \surd(2e^{-\gamma})LC_5/L(1,\chi_2)$$

$$k_7^2 = (2e^{-\gamma})LC_7/L(1,\chi_3).$$

Classical methods (see, e.g., [3],[4], or [1]) permit us to obtain the values of $L(1,\chi_j)$ in closed form, namely $L(1,\chi_2) = \pi/4 \simeq .785398\ldots$; $L(1,\chi_3) = \frac{\sqrt{2}}{4}\log(3+2\surd 2) \simeq .6232252\ldots$; $L(1,\chi_4) = \pi\surd 2/4 \simeq 1.110720\ldots$, which lead to $L = \frac{\pi\surd 2}{8}(\log\cdot(3+2\surd 2))^{\frac{1}{2}} \simeq .73734\ldots$ Also, $\surd(2e^{-\gamma}) \simeq 1.059678709\ldots$ On the other hand, apparently no closed formulae are known for C_3, C_5, or C_7. Their approximate vales are $C_3 \simeq .877\ldots$, $C_5 \simeq .951\ldots$, and $C_7 \simeq .975\ldots$, which lead to $D \simeq .81\ldots$ This completes the discussion of the case $m = 8$.

§7. **A special result.** Previous results hold only for moduli m with $m|24$. In applications we sometimes need the products

$$\prod_{\substack{(\frac{p}{m})=1 \\ p\leq x}} (1-\frac{1}{p}) \quad \text{and} \quad \prod_{\substack{(\frac{p}{m})=-1 \\ p\leq x}} (1-\frac{1}{p}),$$

for m not necessarily a divisor of 24. If $m|24$, then these products can be obtained from Theorem 1, but they can be determined directly for arbitrary integer m.

THEOREM 2. *For arbitrary integer* m *and* x,

$$\prod_{\substack{(\frac{p}{m})=1 \\ p\leq x}} (1-\frac{1}{p}) \simeq \frac{k_1}{\surd\log x} \quad \textit{and} \quad \prod_{\substack{(\frac{p}{m})=-1 \\ p\leq x}} (1-\frac{1}{p}) \simeq \frac{k_{-1}}{\surd\log x}, \tag{9}$$

where

$$k_1^2 = \prod_{p|m}(1-p^{-1})^{-1} \prod_{(\frac{p}{m})=-1}(1-p^{-2})^{-1} e^{-\gamma} L(1,\chi)^{-1}$$

$$k_{-1}^2 = \prod_{p|m}(1-p^{-1})^{-1} \prod_{(\frac{p}{m})=-1}(1-p^{-2}) e^{-\gamma} L(1,\chi);$$

here $\chi(n) = (\frac{n}{m})$ *and* $L(1,\chi) = \sum_{n=1}^{\infty} \chi(n) n^{-s}$ *is the corresponding L-series and* γ *stands for the Euler-Mascheroni constant.*

COROLLARY. *If* $(\frac{2}{m}) = 1$, *then* $\prod_{\substack{(\frac{p}{m})=1 \\ 3\leq p\leq x}} (1-p^{-1}) \simeq 2k_1/\sqrt{\log x}$

and if $(\frac{2}{m}) = -1$, *then* $\prod_{\substack{(\frac{p}{m})=-1 \\ 3\leq p\leq x}} (1-p^{-1}) \simeq 2k_{-1}/\sqrt{\log x}$.

Proof. The corollary follows immediately from Theorem 2 and it is sufficient to prove the theorem. We consider again $L(s,\chi_o) = \prod_{p\nmid m}(1-p^{-s})^{-1}$, where $\chi_o(n)$ is the principal character modulo m and

$$L(s,\chi) = \sum_{n=1}^{\infty} (\frac{n}{m}) n^{-s} = \prod_{\chi(p)=1} (1-p^{-s})^{-1} \prod_{\chi(p)=-1} (1+p^{-s})^{-1}$$

Then formulae (9) are proved exactly as in Theorem 1. Also, for $x \to \infty$ (hence, $x > m$), we have

$$\prod_{\substack{(\frac{p}{m})=1 \\ p\leq x}} (1-\frac{1}{p}) \prod_{\substack{(\frac{p}{m})=-1 \\ p\leq x}} (1-\frac{1}{p}) \prod_{p|m} (1-\frac{1}{p}) \simeq \frac{e^{-\gamma}}{\log x},$$

so that $k_1 k_{-1} = e^{-\gamma} \prod_{p|m} (1-p^{-1})^{-1}$. Next

$$\frac{L(x,\chi)}{L(x,\chi_o)} = \prod_{\chi(p)=-1} \frac{(1-p^{-s})^2}{(1-p^{-2s})}$$

and

$$L(s,\chi)L(s,\chi_o) = \prod_{\chi(p)=1} (1-p^{-s})^{-2} \prod_{\chi(p)=-1} (1-p^{-2s})^{-1}.$$

From here the proof continues exactly like the correspondin one in Section 2 and leads to

$$\lim_{s\to 1^+} \{ \prod_{\chi(p)=-1} (1-p^{-s}) / \prod_{\chi(p)=1} (1-p^{-s}) \} = L(s,\chi) \prod_{\chi(p)=-1} (1-p^{-2s}) ,$$

whence

$$\lim_{x\to\infty} \{ \prod_{\substack{\chi(p)=-1 \\ p\leq x}} (1-p^{-1}) / \prod_{\substack{\chi(p)=1 \\ p\leq x}} (1-p^{-1}) \} = k_{-1}/k_1 = L(1,\chi) \prod_{\chi(p)=-1} (1-p^{-2}).$$

From $k_1 k_{-1} = e^{-\gamma} \prod_{p|m} (1-p^{-1})^{-1}$ and $k_{-1}/k_1 = L(1,\chi) \prod_{\chi(p)=-1} (1-p^{-2})$ the indicated values of k_1^2 and k_{-1}^2 immediately follow.

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Department of Mathematics
College of Arts and Sciences
Temple University,
Philadelphia, Pennsylvania 19122
U. S. A.

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