Revista Colombiana de Matemáticas
Vol. XXI (1987), págs. 285-300

S_λ - ENTROPIES

by

Siegfried WEBER

Dedicado al profesor Henri Yerli,
a quien conocí como gran docente
y amigo.

Resumen. Aquellas entropías H que se caracteriza por la propiedad que $H(P*Q) = H(P)\, S\, H(Q)$ para una operación S de semigrupo arquímediano se expresarán como entropías aditivas. Se considerará especialmente la familia con respecto a las operaciones $S_\lambda(x,y) = x+y+\lambda xy$. Casi todas las extropías conocidas de la literatura resultan casos particulares. Se deriva un teorema de código y se construye entropías condicionales.

Abstract. Those entropies H which are characterized by the property that $H(P*Q) = H(P)\, S\, H(Q)$ for an Archimedean semigroup operation S will be expressed by additive entropies. The family with respect to the operations $S_\lambda(x,y) = x+y+\lambda xy$ will be considered. Almost all entropies known from the literature become special cases. A coding theorem is derived and conditional entropies are constructed.

Introduction. Hartley (1928) was the first to introduce a measure of entropy. The generalization by Shannon (1948) is the entropy most used in many applications. Both entropies H are additive in the sense that

$$H(P*Q) = H(P) + H(Q) \tag{1}$$

where $P*Q$ is the product of the finite probability distributions P and Q. Rényi (1961) introduced a family of additive entropies of order α, including Shannon's and Hartley's entropies for $\alpha \to 1$ and $\alpha \to 0$ respectively. Under additional assumptions it was shown that Rényi's entropies are the only additive ones, see e.g. Daróczy (1964).

Later on other entropies have been suggested, e.g. by Aczél and Daróczy (1963), Kapur (1967), Havrda and Charvát (1967), Daróczy (1970), Arimoto (1971), Rathie (1971a), Sharma and Mittal (1975), and Boekee and Van der Lubbe (1980). All these entropies have on property in common, namely

$$H(P*Q) = H(P) S_\lambda H(Q), \tag{2}$$

where

$$S_\lambda(x,y) = x+y+\lambda xy \tag{3}$$

with the parameter $\lambda \in \mathbb{R}$ depending on the entropy-parameters.

The indices of diversity of Gini (1912), considered also by Simpson (1949), and of Mc Intosh (1967) have property (2) also. Both are non additive.

Every S_λ is a strict Archimedean semigroup operation on a certain interval $[0,M] \subset [0,\infty]$ and therefore, see Aczél (1949), additively generated, i.e.

$$S_\lambda(x,y) = h_\lambda^{-1}(h_\lambda(x) + h_\lambda(y)). \tag{4}$$

It follows immediately that the compositions $h_\lambda \circ H$ are additive entropies.

This is the crucial step: reducing the entropies H to additive entropies $h_\lambda \circ H$, we can obtain results concerning H by means of the corresponding ones for $h_\lambda \circ H$. As an example for this procedure, Campbell's (1965) coding theorem can be rewritten. It is not necessary to give the proof for each of the special entropies mentioned above, as has been done e.g. by Boekee and Van der Lubbe (1980) for one case. We obatin the same result and more rapidly.

Also conditional entropies can be constructed using the underlying structure of the unconditional measure given by S_λ. This approach is quite different from the construction of conditional entropies which are based on some generalized mean value property known from the current literature. In a forthcoming paper the author will discuss and compare these concepts. Some of these results were presented in Weber (1985).

Tha aim of the present paper is not to enlarge the number of "mostly formal generalizations... popping up almost daily in the literature", as has been criticized by Aczél (1984) in his survey. But in contrast to this I will stress the common property (2) and show that it is sufficient to consider only very few (classes of) entropies.

§1. S_λ-entropies. Considering entropies H for finite probability distributions

$$P = (p_1,\ldots,p_n)$$

we will suppose as minimal conditions that

$$0 = H(0,\ldots,0,1,0,\ldots,0) \leqslant H(P) \leqslant H(\tfrac{1}{n},\ldots,\tfrac{1}{n}) =: H_{max} . \qquad (5)$$

Furthermore we suggest to look at entropies with the following property.

1.1. DEFINITION. A function H with property (5) will be called an *additively generated entropy* or briefly *S-entropy* iff for any product $P*Q := (p_i\, q_k)$, $i = 1,\ldots,n$, $k = 1,\ldots,m$, of distributions $P = (p_1,\ldots,p_n)$ and $Q = (q_1,\ldots,q_m)$

$$H(P*Q) = H(P)\, S\, H(Q) \tag{6}$$

where S is a continuous binay operation on $[0,M]$, which is associative, (strictly) increasing in each argument, continuos and has 0 as a unit, and where M has to be chosen with

$$M \geqslant \sup_P H(P)$$

The +-entropies are usually called *additive entropies.*

The properties equired on S are more or less natural except that of "increasing", but this, together with the others, leads to the following representation, essentially due to Aczél (1949), in the modified form given by Ling (1965).

1.2. **THEOREM.** *A binary operation* S *on* $[0,M]$ *has the properties listed in* 1.1. *if and only if there exists an increasing and continuius function* $h:[0,M] \to [0,\infty]$ *with* $h(0) = 0$ *and* $h(M) = \infty$ *so that*

$$S(x,y) = h^{-1}(h(x) + h(y)). \tag{7}$$

Furthermore, h *is unique up to a positive factor.*

This last property justifies the notation "additively generated entropy" used in the definition, where usually h will be called "additive generator" of S.

1.3. **EXAMPLE.** For any $\lambda \in (-1,\infty)$,

$$S_\lambda(x,y) := x+y+\lambda xy \tag{3}$$

gives a binary operation S_λ on $[0,M]$ with the properties listed in 1.1 and where

$$M := \begin{cases} \frac{1}{-\lambda} & \text{if} \quad -1 < \lambda < 0 \\ \infty & \text{if} \quad \lambda \geqslant 0 \end{cases} .$$

The additive generators h_λ of S_λ can be written e.g. in the form

$$h_\lambda(x) = c\,\frac{\log(1+\lambda x)}{\log(1+\lambda)} \quad \text{with any } c > 0, \tag{8}$$

including by continuous extension

$$h_o(x) = c\cdot x\ . \tag{9}$$

In the following we will use the symbol **log** always for the logarithm to the base 2.

1.4. EXAMPLES. **The following are all additive entropies.**

a) $H_o(P) := \log n$ (10)

gives Hartley's (1928) entropy H_o.

b) $H_1(P) := -\sum_{i=1}^{n} p_i \log p_i$ (11)

gives Shannon's (1948) entropy H_1.

c) $H_\alpha(P) := \frac{1}{1-\alpha}\cdot\log\sum_{i=1}^{n} p_i^\alpha\ ,\quad \alpha \neq 1,$ (12)

define the family of Rényi's (1961) entropies H_α of order α, including H_o and, by continuous extension, H_1.

d) $H_{\alpha,\beta}(P) := \frac{1}{1-\alpha}\cdot\log\frac{\sum_{i=1}^{n} p_i^{\alpha+\beta-1}}{\sum_{i=1}^{n} p_i^{\beta}},\quad \alpha \neq 1,$ (13)

define a family of entropies $H_{\alpha,\beta}$, introduced by Aczél and Daróczy (1963) and also considered by Kapur (1967). This family generalizes Rényi's family, i.e.

$$H_{\alpha,1} = H_\alpha\ ,$$

and includes by continuous extension

$$H_{1,\beta}(P) = \frac{-\sum_{i=1}^{n} p_i^\beta\cdot\log p_i}{\sum_{i=1}^{n} p_i^{\beta}}. \tag{14}$$

The representation theorem 1.2 leads immediately to the following characterizacion of all S-entropies.

1.5. **THEOREM.** *Every S-entropy can be written as function*

$$h^{-1} \circ H \tag{15}$$

of an additive entropy H, *where* h *is some additive generator of* S.

Applying this characterization theorem to 1.4 we obtain the following.

1.6. **EXAMPLES.** *Let* h_λ *and* S_λ *as in example* 1.3. *Then the following are all* S_λ*-entropies:*

a) $$h_\lambda^{-1}(H_{\alpha,\beta}(P)) = \frac{1}{\lambda} \cdot \left\{ \left(\frac{\sum_{i=1}^{n} p_i^{\alpha+\beta-1}}{\sum_{i=1}^{n} p_i^{\beta}} \right)^{\frac{\log(1+\lambda)}{c(1-\alpha)}} - 1 \right\} \tag{16}$$

including by continuous extension the cases $\alpha = 1$ or $\lambda = 0$ respectively. Setting in (a) $\beta = 1$ we obtain the entropies:

b) $$h_\lambda^{-1}(H_\alpha(P)) = \frac{1}{\lambda} \cdot \left\{ \left(\sum_{i=1}^{n} p_i^{\alpha} \right)^{\frac{\log(1+\lambda)}{c(1-\alpha)}} - 1 \right\}, \tag{17}$$

introduced by Sharma and Mittal (1975), including

$$h_\lambda^{-1}(H_1(P)) = \frac{1}{\lambda} \cdot \left\{ (1+\lambda)^{-\frac{1}{c}\sum_{i=1}^{n} p_i \log p_i} - 1 \right\}. \tag{18}$$

On the other hand, setting in (a) and (b) respectively $\lambda = 2^{1-\alpha} - 1$, $c = 1$, we are led to the entropies:

c) $$\hat{H}_{\alpha,\beta}(P) := \frac{1}{2^{1-\alpha}-1} \cdot \left\{ \frac{\sum_{i=1}^{n} p_i^{\alpha+\beta-1}}{\sum_{i=1}^{n} p_i^{\beta}} - 1 \right\} \tag{19}$$

introduced by Rathie (1971a), and respectively

$$\text{d)}\quad \hat{H}_\alpha(P) := \frac{1}{2^{1-\alpha}-1}\cdot\left\{\sum_{i=1}^{n} p_i^{\alpha} - 1\right\} \tag{20}$$

introduced by Havrda and Charvat (1967) and also considered by Daróczy (1970).

Both families contain by continuous extension the corresponding additive entropies, i.e.

$$\hat{H}_{1,\beta} = H_{1,\beta}\ ,\quad \hat{H}_1 = H_1 .$$

Furthermore, setting in (b) $\alpha > 0$, $\lambda = 2^{\frac{1-\alpha}{\alpha}} - 1$, $c = 1$, we obtain the entropies:

$$\text{e)}\quad \tilde{H}_\alpha(P) := \frac{1}{2^{\frac{1-\alpha}{\alpha}}-1}\cdot\left\{\left(\sum_{i=1}^{n} p_i^{\alpha}\right)^{\frac{1}{\alpha}} - 1\right\}, \tag{21}$$

considered by Arimoto (1971) and Boekee and Van der Lubbe (1980), which includes

$$\tilde{H}_1 = H_1 .$$

In their papers we find $\tilde{H}_\alpha$ with a factor $\frac{1}{1-\alpha}$ instead of $\frac{1}{2^{\frac{1-\alpha}{\alpha}}-1}$ which corresponds to the choice of $\lambda = \frac{1-\alpha}{\alpha}$, $c = \frac{-\alpha}{1-\alpha}\log\alpha$. It is interesting to note that, except for positive constants, $\hat{H}_2$ is the index of Gini (1912)/Simpson (1949) and $\tilde{H}_2$ is the index of McIntosh (1967), both have been applied as indices of diversity of populations.

1.7. COUNTEREXAMPLE. The modified version of (14), namely $-\sum_{i=1}^{n} p_i^{\beta}\log p_i$ for $\beta \neq 1$, introduced by Rathie (1971b), is not additively generated.

I will finish this paragraph with a remark of caution. In their original paper, Sharma and Mittal presented their entropies in another form than (17), namely

$$\frac{1}{2^{1-\gamma}-1}\cdot\left\{\left(\sum_{i=1}^{n} p_i^{\alpha}\right)^{\frac{1-\gamma}{1-\alpha}} - 1\right\}. \tag{22}$$

For c = 1, the transformation $\lambda = 2^{1-\gamma} - 1$ shows the equivalence between (17) and (22). Sometimes one can find still another form, see Aczél (1978):

$$\frac{1}{2^{\delta(1-\alpha)}-1}\cdot\left\{\left(\sum_{i=1}^{n} p_i^{\alpha}\right)^{\delta} - 1\right\}. \tag{23}$$

We can see that, for c = 1, formulae (17) and (23) are no equivalent, any more, if we include the entropies for $\lambda = 0$, $\delta = 0$, $\alpha = 1$ by continuous extension, as we have always done in this paper. The reason for this is that the application:

$$\begin{aligned} \mathbb{R}^2 &\longrightarrow \mathbb{R}\times(1-,\infty) \\ (\alpha,\delta) &\longmapsto (\alpha,\lambda)\ , \end{aligned}$$

given by $\lambda = 2^{\delta(1-\alpha)} - 1$, is not a biyection. More precisely,

$$(1,\delta) \longmapsto (1,0) \quad \text{for all } \delta.$$

In terms of entropies this means that for $\alpha \to 1$: (17) reduces to $h_\lambda^{-1}(H_1(P)$, but (23) reduces to $H_1(P)$ for all δ.

§2. A coding theorem for S_λ-entropies. Suppose that we have n message letters x_i with probability $p_i := P\{x_i\}$ which start a memoryless source and are encoded, for simplicity, by 2 symbols. Let us denote the resulting codeword length by n_i. Then we can use the caracterization theorem 1.5 to obtain coding theorems for a whole family of entropies provided that we have a coding theorem for one member of that family. As an example for this procedure we look at the special case 1.6 (b) which leads to the following.

2.1. THEOREM. *For each entropy* $H_\alpha^\lambda := h_\lambda^{-1} \circ H_\alpha$ *(in (17)) of a source there exists an uniquely decipherable code such that:*

$$H_\alpha^\lambda \leq L\left(\frac{1-\alpha}{\alpha}, \lambda\right) < (1+\lambda)^{\frac{1}{c}} H_\alpha^\lambda + \frac{1}{\lambda} \cdot \left\{(1+\lambda)^{\frac{1}{c}} - 1\right\} \tag{24}$$

with equality at the left side if and only if

$$2^{-n_i} = p_i^{\alpha} / \left(\sum_{k=1}^{n} p_k^{\alpha}\right) \quad \textit{for all } i,$$

and where

$$L(t,\lambda) := \frac{1}{\lambda} \cdot \left\{ \left(\sum_{i=1}^{n} p_i \cdot 2^{t \cdot n_i}\right)^{\frac{\log(1+\lambda)}{c \cdot t}} - 1 \right\} \tag{25}$$

is an average codeword length.

Proof. Campbell (1965) proved the corresponding result:

$$H_\alpha \leq L\left(\frac{1-\alpha}{\alpha}\right) < H_\alpha + 1 \tag{26}$$

for Rényi's entropies H_α in (12) and the average codeword length:

$$L(t) := \frac{1}{t} \cdot \log\left(\sum_{i=1}^{n} p_i \; 2^{t \cdot n_i}\right). \tag{27}$$

Applying h_λ^{-1} from (8) to (26) and using (25) and (27) we are led to

$$H_\alpha^\lambda \leq L\left(\frac{1-\alpha}{\alpha}, \lambda\right) < h_\lambda^{-1}\left(h_\lambda(H_\alpha^\lambda) + 1\right).$$

The rest follows by (8).

2.2. COROLLARY. *For the entropies* (20) *and* (21) *respectively we obtain the results corresponding to* (24):

a) $$\hat{H}_\alpha \leq \frac{\left(\sum_{i=1}^{n} p_i \; 2^{\frac{1-\alpha}{\alpha} \cdot n_i}\right)^{\alpha} - 1}{2^{1-\alpha} - 1} < 2^{1-\alpha} \hat{H}_\alpha + 1, \tag{28}$$

b) $$\tilde{H}_\alpha \leqslant \frac{\sum_{i=1}^{n} p_i\, 2^{\frac{1-\alpha}{\alpha}\cdot n_i} - 1}{2^{\frac{1-\alpha}{\alpha}} - 1} < 2^{\frac{1-\alpha}{\alpha}}\, \tilde{H}_\alpha + 1\,. \tag{29}$$

The latter formula is essentially theorem 11 of Boekee and Van der Lubbe (1980) except that their entropy differs from $\tilde{H}_\alpha$ by a factor as explained in 1.6.

Let us remark that the average codeword lengths from (25) are strictly monotone and continuous functions of:

$$K(t) := \sum_{i=1}^{n} p_i\ 2^{t\cdot n_i} \tag{30}$$

which can be interpreted as *average exponential coding cost*. More precisely:

$$L(t,\lambda) = h_\lambda^{-1}\left(\frac{\log K(t)}{t}\right)\,, \tag{31}$$

including by continuous extension:

$$L(0,\lambda) = h_\lambda^{-1}\left(\sum_{i=1}^{n} p_i\cdot n_i\right) \tag{32}$$

which is a function of the *average linear coding cost*

$$\sum_{i=1}^{n} p_i\cdot n_i\,. \tag{33}$$

§3. Conditional entropies. The defining property (6) for *S*-entropies can be rewritten as

$$H(P,Q) = H(P)\, S\, H(Q) \quad \text{for independent } P,\ Q. \tag{34}$$

If we want to preserve this structure for the general situation, we are led to the following construction of a conditional entropy I am proposing in this paper.

3.1. **DEFINITION.** Let H be a S-entropy in the sense of 1.1. Then we define a conditional entropy $H(Q/P)$ as solution of:

$$H(P,Q) = H(P)\ S\ H(Q/P). \tag{35}$$

The representation theorem 1.2 leads immediately to

3.2. **THEOREM.** *For all* P *with* $H(P) < M$, *the conditional entropy* $H(Q/P)$ *is uniquely determined by:*

$$H(Q/P) = h^{-1}(h\circ H(P,Q) - h\circ H(P)) \ , \tag{36}$$

where h *is any additive generator of* S. *Furthermore:*

$$H(Q/P) = H(Q) \quad \textit{for independent } P,\ Q. \tag{37}$$

Using the usual notation

$$(P,Q) = (p_{ik}) = (p_i\ q_{k/i}), \quad i=1,\dots,n; \quad k=1,\dots,m \ ,$$

for the joint distribution, we obtain by 1.4 and 1.6 the corresponding special cases. We will point out only a few cases.

3.3. **EXAMPLES.**

a) For Rényi's (additive) entropies H_α from (12) we obtain

$$H_\alpha(Q/P) = \frac{1}{1-\alpha}\log\frac{\sum_{i,k} p_i^\alpha\ q_{k/i}^\alpha}{\sum_j p_j^\alpha}. \tag{38}$$

This formula (38) appears in Rényi (1960, formula (35)). But in general it is used there as conditional entropy an expression where the weights $p_i^\alpha/(\sum_j p_j^\alpha)$ in (38) are replaced by p_i, see Rényi (1960, formula (24)).

b) For the entropies $\hat{H}_\alpha$ from (20) we obtain

$$\hat{H}_\alpha(Q/P) = \frac{1}{2^{1-\alpha}-1}\cdot\left\{\frac{\sum_{i,k} p_i^\alpha \cdot q_{k/i}^\alpha}{\sum_j p_j^\alpha} - 1\right\}. \tag{39}$$

This formula (39) differs from that obtained by the construction given by Daróczy (1970, formula (5.4)), which can be written as

$$\hat{H}_\alpha(Q/P) \sum_j p_j^\alpha .$$

c) For the entropies $\tilde{H}_\alpha$ from (21) we obtain

$$\tilde{H}_\alpha(Q/P) = \frac{1}{2^{\frac{1-\alpha}{\alpha}}-1}\cdot\left\{\left(\frac{\sum_{i,k} p_i^\alpha q_{k/i}^\alpha}{\sum_j p_j^\alpha}\right)^{\frac{1}{\alpha}} - 1\right\}. \tag{40}$$

The formula (40) is different from the two proposed by Boekee and Van der Lubbe (1980) and denoted by

$$'H_\alpha(Q/P) = \frac{\sum_i p_i \left(\sum_k q_{k/i}^\alpha\right)^{\frac{1}{\alpha}} - 1}{\frac{1-\alpha}{\alpha}}, \tag{41}$$

$$''H_\alpha(Q/P) = \frac{\left(\sum_i p_i \sum_k q_{k/i}^\alpha\right)^{\frac{1}{\alpha}} - 1}{\frac{1-\alpha}{\alpha}}. \tag{42}$$

The essential difference is not in the factor, but in the different manner in which the p_i's are used as weights.

Concluding remark. In the present paper the author considered additively generated entropies, i.e. those which have (6) as a basic property. This is reasonable because a lot of entropies is of that type. In view of this property, the proposed definition of a conditional entropy (35) seems to be quite natural, especially since this construction ensures the important property that

$$H(Q|P) = H(Q) \text{ for independent } P,\ Q. \tag{37}$$

For many situations another property is also important, namely:

$$H(Q|P) \leq H(Q) \quad \text{for all } P, Q. \tag{43}$$

This is equivalent to the property

$$h \circ H(P,Q) \leq h \circ H(P) + h \circ H(Q) \quad \text{for all } P, Q. \tag{44}$$

For the entropies H in (17) the composed entropies $h \circ H = H_\alpha$ belong to Rényi's family, for which it is known that (44) holds only for $\alpha = 1$.

On the other hand, other constructions for conditional entropies do not fulfill the first important property (37). This dilemma will be discussed in authors forthcoming paper mentioned in the introduction, using several mean value properties.

Acknowledgement. The author likes to thank the referee for some useful comments.

REFERENCES

[1] Aczél, J., (1949), *Sur les opérations définis pour nombres réels*, Bull. Soc. Math. France **76**, 59-64.

[2] Aczél, J., (1978), *Some recent results on characterizations of measures of information related to coding*, IEEE Trans. on Information Theory **24**, Nº 5, 592-595.

[3] Aczél, J., (1984), *Measuring information beyond communication theory - Why some generalized information measures may be useful, others not*, Aequationes Mathematicae **27**, 1-19.

[4] Aczél, J. and Daróczy, Z., (1963), *Über verallgemeinerte quasilineare Mittelwerte, die mit Gewichtsfunktionen gebildet sind*, Publ. Math. Debrecen **10**, 171-190.

[5] Arimoto, S., (1971), *Information theoretical considerations on estimation problems*, Information and Control **19**, 181-194.

[6] Boekee, D.E. and Van der Lubbe, J.C.A. (1980), *The R-norm information measure*, Information and Control **45**, 136-155.

[7] Campbell, L.L. (1965), *A coding theorem and Rényi's entropy*, Information and Control **8**, 423-429.
[8] Daróczv, Z., (1964), *Über Mittelwerte und Entropien vollständiger Wahrscheinlichkeitsverteilungen*, Acta Math. Acad. Sci. Hung. **15**, 203-210.
[9] Daróczy, Z., (1970), *Generalized information functions*, Information and Control **16**, 36-51.
[10] Gini, C., (1912), *Variabilita e mutabilita*, Studi Economico-Giuridici della Facolta di Giurisprudenza dell Universita di Cagliari, a III, parte II.
[11] Hartley, R.V., (1928), *Transmission of information*, Bell System Techn. J. **7**, 535-563.
[12] Havrda, J., and Charvat, F., (1967), *Quantification method of classification processes, concept of structural α-entropy*, Kybernetika **3**, 30-35.
[13] Kapur, J.N., (1967), *Generalized entropy of order α and type β*, The mathematical Seminar **4**, 78-94.
[14] Ling, C.H., (1965), *Representation of associative functions*, Publ. Math. Debrecen **12**, 189-212.
[15] Mc Intosh, R.P., (1967), *An index of diversity and the relation of certain concepts to diversity*, Ecology **48**, 392-404.
[16] Rathie, P.N., (1971a), *A generalization of the non-additive measures of uncertainty and information and their axiomatic characterizations*, Kybernetika **7**, 125-132.
[17] Rathie, P.N., (1971b), *On some new measures of uncertainty, inaccuracy and information and their characterizations*, Kybernetika **7**, 394-403.
[18] Rényi, A., (1960), *Some fundamental questions of information theory*, MTA III, Oszt. Közl. **10**, 251-282.
[19] Rényi, A., (1961), *On measure of entropy and information*, Proc. Fourth Berkeley Symp., Math. Statist. Prob. Nº 1, 547-561.
[20] Shannon, C.E., (1948), *A mathematical theory of communication*, Bell System Tech. J. **27**, 378-423 and 623-656.
[21] Sharma, B.D. and Mittal, D.P., (1975), *New non-additive measures of entropy for a discrete probability distribution*, J. Math. Sci., Delhi, **10**, 28-40.
[22] Simpson, E.H., (1949), *Measurement of diversity*, Nature **163**, 688.
[23] Weber, S., (1984), *Teoría de probabilidad e información-Temas especiales con aplicaciones*, Universidad de los Andes, Departamento de Matemáticas, Bogotá, Colombia.
[24] Weber, S., (1985), *Conditional entropies - a discussion of different concepts*, in: B. Bouchon (Ed.), Journees Remoises "Analyse de problemes decisionnels dans un environnement incertain et imprecis", Reims, France, 165-180.

* *

Fachbereich Mathematik
Johannes Gutenberg-Universität
D-6500 Mainz
Alemania Federal

(Recibido en mayo de 1985, la versión revisada en febrero de 1987).