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ON THE ORTHOGONALITY MEASURE
OF THE
q-POLLACZEK POLYNOMIALS

by

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Abstract. The q-Pollaczek polynomials $F_n(x)$ depend on four parameters u,v,Δ,q and are given by the recurrence relation $(1-q^{n+1})F_{n+1}(x) = 2[(1-u\Delta q^n)x+vq^n]F_n(x)-(1-\Delta^2q^{n-1})F_{n-1}(x)$, $n \geqslant 1$, and the initial conditions $F_0(x)=1$ $F_1(x)=2[(1-u\Delta)x+v]/1-q$. The measure with respect to which the $F_n(x)$'s are orthogonal is determined when the parameters are subject to the constraints $0<u<\Delta<1$, $\Delta(1-u)>\pm v$, $0<q<1$. This measure turns out to be absolutely continuous with respect to Lebesgue's measure.

§1. Introduction. *The* q-*Rogers polynomials* (Rogers [23], [24]), also called *continuous* q-*ultraspherical polynomials*, are given by the recurrence relation

$$2x(1-\beta q^n)C_n(x;\beta|q) = (1-q^{n+1})C_{n+1}(x;\beta|q) + (1-\beta^2p^{n-1})C_{n-1}(x;\beta|q), \quad n>0 \tag{1.1}$$

(1) This paper contains developments of partial results of L.A. Gomez's M.S. dissertation at the National University of Colombia. The thesis advisor was J.A. Charris.

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and the initial conditions

$$(C_o(x;\beta|q) = 1, \quad C_1(x;\beta|q) = 2x(1-\beta)|(1-q). \tag{1.2}$$

They depend on the two parameters β, q, and for appropriate values of the parameters they form a system of orthogonal polynomials. The q-Rogers polynomials were used by Rogers in the proof of the celebrated Rogers-Ramanujan identities of the theory of partitions; details can be found in [2], [3].

The q-Rogers polynomials generalize the *ultraspherical polinomials* in the sense that

$$\lim_{q\to 1^-} C_n(x;q^\lambda|q) = C_n^\lambda(x) \quad , \tag{1.3}$$

where $C_n^\lambda(x)$ is the n^{th}-ultraspherical polynomial of order λ (Rainville [19], Szego [25]).

Al-Salam, Allaway and Askey [1] set

$$q = s\omega_k, \quad \beta = s^{\lambda k}, \quad \omega_k = \exp(\frac{2\pi i}{k}), \tag{1.4}$$

$$c_n(x;\beta|q) = \frac{(q;q)_n}{(\beta^2;q)_n} C_n(x;\beta|q) \tag{1.5}$$

and

$$c_n^\lambda(x;k) = \lim_{s\to 1} c_n(x;\beta|q). \tag{1.6}$$

They noticed that $\{c_n(x;k)/n \geqslant 0\}$ is a set of orthogonal polynomials and refered to them as *sieved ultraspherical polynomials*. As it turns out, the sieving process of Al-Salam and Askey has a wide scope. Many systems of orthogonal polynomials have been discovered this way. Recent work of Ismail [12], [13], [14], Charris and Ismail [8], [9], Askey and Shukla [6] are a sample of the activity in the field of sieved orthogonal polynomials. Further research of Geronimo and Van Assche [11] has greatly contributed to shed ligth on the underlying mathematical structure.

Only recently the orthogonality measure of the q-Rogers polynomials has been determined by Askey and Wilson [7] and Askey and Ismail [4].

Systems of polynomials related to the q-Rogers polynomials have been introduced by Ismail [12] and Charris and Ismail [9]. Their purpose was to obtain sieved analogues of the Pollaczek polynomials. The q-*Pollaczek polynomials* of Charris and Ismail depend on four parameters u, v, Δ, q. Denoted with $F_n(x;u,v,\Delta;q)$, or $F_n(x)$ for short, they satisfy the recurrence relation

$$(1-q^{n+1})F_{n+1}(x) = 2[(1-u\Delta q^n)x + vq^n]F_n(x) - (1-\Delta^2 q^{n-1})F_{n-1}(x) \tag{1.7}$$

and the initial conditions

$$F_o(x) = 1, \quad F_1(x) = \frac{2[(1-u\Delta)x+v]}{1-q}. \tag{1.8}$$

The polynomials $F_n(x)$ generalize the Pollaczek polynomials in the sense that

$$p_n^\lambda(x;a,b) = \lim_{q\to 1-o} F_n(x;q^a,1-q^b,q^\lambda;q) \tag{1.9}$$

is the n^{th} Pollaczek polynomial (Pollaczek [20], Szegö [25]).

The purpose of this paper is to determine the orthogonality measure of the q-Pollaczek polynomials under appropriate restrictions on the parameters. These constraints are different from those suggested in [9] and allow very neatly for the use of a theorem of Nevai [16], [17]. Even under these conditions we have not been able to determine the measure solely on the basis of Markoff's theorem and the Stieljes inversion formula, two powerful tools which have proved succesful in many other instances. A deeper knowledge than ours of the Heine-Ramanujan q-functions and their transformations seems neccesary for this purpose. This we see as a proof of the depth and power of Nevai's result.

§2. Orthogonal polynomials. If a sequence of polynomials $\{p_n(x)\}$ satisfies a three terms recurrence relation

$$p_{n+1}(x) = (A_n x + B_n) p_n(x) - C_n p_{n-1}(x), \quad n > 0, \tag{2.1}$$

and the initial conditions

$$p_o(x) = 1, \quad p_1(x) = A_o x + B_o, \tag{2.2}$$

then, provided that

$$C_n / A_n A_{n-1} > 0, \quad n > 0, \tag{2.3}$$

$\{p_n(x)\}$ is a system of orthogonal polynomials with respect to a positive measure μ such that

$$\int_{-\infty}^{+\infty} p_n(x) p_m(x) d\mu(x) = \{\frac{A_o}{A_n} \prod_{k=1}^{n} C_k\} \delta_{mn}. \tag{2.4}$$

This is known as *Favard's Theorem* (Szegö [25]). In general, there are several measures for which (2.4) holds true. However, in view of the Weierstrass approximation theorem, conditions securing compact support are sufficient for uniqueness. Such is the case if for some constant $M > 0$

$$\sqrt{C_{n+1}/A_n A_{n+1}} \leq M/3, \quad |B_n/A_n| \leq M/3, \quad n \geqslant 0. \tag{2.5}$$

Then, the support of μ, also called the *spectrum of* $\{p_n(x)\}$, can be shown to be contained in $[-M,M]$ (Chihara [10]).

The system $\{p_n^*(x)\}$ of polynomials satisfying (2.1) and the initial conditions

$$p_o^*(x) = 0, \quad p_1^*(x) = A_o \tag{2.6}$$

is called the *system of polynomials of the second kind*. It is useful in computing the measure μ. For example, under conditions (2.5), it can be shown (Chihara [10]) that

$$\chi(z) := \lim_{n\to\infty} \frac{p_n^*(z)}{p_n(z)} = \int_{-\infty}^{+\infty} \frac{d\mu(t)}{z-t}, \quad z \notin [-M, M], \tag{2.7}$$

i.e.,

$$\chi(z) = -2\pi i \hat{\mu}(z), \tag{2.8}$$

where

$$\hat{\mu}(z) = \frac{1}{2\pi i} \int_{-\infty}^{+\infty} \frac{d\mu(t)}{t-z} \tag{2.9}$$

is the *Cauchy-Stieljes transform of the measure* μ. The *Stieljes inversion formula* (Lang [15]) then applies to give

$$\int_{-\infty}^{+\infty} \phi\, d\mu = \lim_{\varepsilon\to 0+} \frac{1}{2\pi i} \int_{-\infty}^{+\infty} \{\chi(x-i\varepsilon) - \chi(x+i\varepsilon)\}\phi(x)\, dx \tag{2.10}$$

for any continuous function ϕ on $\mathbb{R}$, so that μ can be recorvered from $\chi(z)$.

The function $\chi(z)$, called the *continued fraction*, can be frecuently determined by means of *Darboux's asymptotic method*. This is based on the following theorem (Olver [18]), a consecuence of the *Riemann-Lebesgue Lemma:*

THEOREM 2.1. *Let* $f(z) = \sum_{n=0}^{\infty} a_n z^n$ *be analytic in* $|z| < r$ *with finitely many singulatities on* $|z| = r$. *Assume also that there is a comparison function* $g(z) = \sum_{n=0}^{\infty} b_n z^n$ *such that* $g(z)$ *is analytic in* $|z| < r$ *and* $f(z) - g(z)$ *is continuous on* $|z| \leq r$. *Then*

$$a_n = b_n + o(r^{-n}) \tag{2.11}$$

Under appropriate assumptions, such as for example that $h(\theta) = f(re^{i\theta}) - g(re^{i\theta})$ be C^∞ for $0 \leq \theta \leq 2\pi$, Theorem 2.1 allows to conclude that

$$a_n \sim b_n, \quad n \to \infty$$

(i.e., $\lim_{n\to\infty} a_n/b_n = 1$). The appropriate conditions being grante , if

$$p(z,t) = \sum_{n=o}^{\infty} p_n(z)t^n, \quad p^*(z,t) = \sum_{n=o}^{\infty} p_n^*(z)t^n, \quad z \notin [-M,M] \tag{2.13}$$

are generating functions for the systems $\{p_n(x)\}$ and $p_n^*(x)$, and if

$$\bar{p}(z,t) = \sum_{n=o}^{\infty} \bar{p}_n(z)t^n, \quad \bar{p}^*(z,t) = \sum_{n=o}^{\infty} \bar{p}^*(z)t^n \tag{2.14}$$

are respective comparison functions, then

$$p_n(z) \sim \bar{p}_n(z), \quad p_n^*(z) \sim \bar{p}_n^*(z), \quad n \to \infty \tag{2.15}$$

and $\chi(z)$ can be obtained as

$$\chi(z) = \lim_{n\to\infty} \frac{\bar{p}_n^*(z)}{\bar{p}_n(z)} . \tag{2.16}$$

Usually $\bar{p}(z,t)$ and $\bar{p}^*(z,t)$ can be so chosen that the limit in (2.16) is more easily determined than that in (2.7).

The success of Darboux's method depends on the choice of the comparison functions (2.14). A more direct aproach that is sometimes applicable is provided by the following theorem of Nevai [17], [18]:

THEOREM 2.2. *If the series*

$$\sum_{n=o}^{\infty} \{|\frac{B_n}{A_n}| + |(\frac{C_{n+1}}{A_nA_{n+1}})^{½} - \frac{\gamma}{2}|\} \tag{2.17}$$

is convergent for some $\gamma > 0$, *then*

$$d\mu = \phi dx + d\psi , \tag{2.18}$$

where ϕ *vanishes outside* $[-\gamma,\gamma]$ *and is positive and continuous in* $(-\gamma,\gamma)$, *and* ψ *is a jump function which is constant in* $(-\gamma,\gamma)$ *and its jumps outside this interval are located at the points* $x \in \mathbb{R}$ *such that*

$$\sum_{n=o}^{\infty} \frac{p_n^2(x)}{\lambda_n} < +\infty, \quad \lambda_n = A_o/A_n\ C_1 \dots C_n. \tag{2.19}$$

Futhermore

$$\lim_{n\to\infty} \sup\{\phi(x)\sqrt{\gamma^2 - x^2}\ \frac{p_n^2(x)}{\lambda_n}\} = 2/\pi \tag{2.20}$$

holds for almost all x *in* $[-\gamma,\gamma]$.

§3. The q-Pollaczek polynomials. Now we turn back our attention to the polynomials defined by (1.7) and the initial conditions (1.8). We shall require the parameters Δ, u, v, q to satisfy

$$0 < u < \Delta < 1, \quad 0 < q < 1, \quad \Delta(1-v) > \pm v. \tag{3.1}$$

The following notations will be used

$$(a;q)_n = \begin{cases} 1, & n = 0, \\ \prod_{k=1}^{n} (1-aq^{k-1}), & 1 \leqslant n < \infty, \end{cases} \tag{3.2}$$

$$(a;q)_\infty = \lim_{n\to\infty} (a;q)_n = \prod_{k=o}^{\infty} (1-aq^k). \tag{3.3}$$

Let

$$A_n = \frac{2(1-u\Delta q^n)}{1-q^{n+1}}, \quad B_n = \frac{2vq^n}{1-q^{n+1}}, \quad C_n = \frac{1-\Delta^2 q^{n-1}}{1-q^{n+1}}, \quad n \geqslant 0. \tag{3.4}$$

Then, $\{F_n(x)\}$ satisfies the recurrence relation (2.1) and the initial conditions (2.2). Also

$$\frac{C_{n+1}}{A_n A_{n+1}} = \frac{1}{4}\ \frac{(1-\Delta^2 q^n)(1-q^{n+1})}{(1-u\Delta q^n)(1-u\ q^{n+1})} > 0, \quad n \geqslant 0, \tag{3.5}$$

and the positivy condition (2.3) holds. A simple calculation shows that

$$\lambda_n = \frac{A_o}{A_n} \prod_{k=1}^{n} C_k = \frac{(1-u\Delta)}{(1-u\Delta q^n)} \frac{(\Delta^2;q)_n}{(q;q)_n}, \quad n \geqslant 0. \tag{3.6}$$

Hence

$$\lambda_n \sim (1-u\Delta) \frac{(\Delta^2;q)_\infty}{(q;q)_\infty}, \qquad n \to \infty . \tag{3.7}$$

Straightforward calculations also show that

$$\frac{B_n}{A_n} = \frac{v\ q^n}{1-u\Delta q^n} . \tag{3.8}$$

Notice that $B_n/A_n = 0$ if $v = 0$. If $v \neq 0$, we still have

$$B_n/A_n \sim vq^n , \qquad n \to \infty . \tag{3.9}$$

From (3.5) it follows that

$$C_n/A_nA_{n+1} \sim \frac{1}{4} , \quad n \to \infty, \tag{3.10}$$

which suggest $\gamma = 1$ in (2.17). Then

$$\left(\frac{C_{n+1}}{A_nA_{n+1}}\right)^{\frac{1}{2}} - \frac{1}{2} = \frac{q^n[\Delta^2(q^{n+1}-1) + u\Delta q(1-u\Delta q^n) + u\Delta - q]}{\sqrt{(1-u\Delta q^n)(1-u\Delta q^{n+1})}\ [\sqrt{(1-\Delta^2 q^n)(1-q^{n+1})}+\sqrt{(1-u\Delta q^n)(1-u\Delta q^{n+1})}]}$$

$$\sim \frac{q^n}{2} [\Delta(uq-\Delta) + u\Delta - q], \qquad n \to \infty . \tag{3.11}$$

Since $\Delta uq + \Delta u < q + \Delta^2$ follows from (3.1), (3.9) and (3.11) ensure that, with A_n, B_n, C_n given by (3.4), the series in (2.17) is convergent if $\gamma = 1$.

In order to determine the functions ϕ and ψ in (2.18), knowledge is needed of the asymptotic behaviour of $F_n(x)$ as $n \to \infty$. This can be obtained from the generating function

$$F(x;t) = \sum_{n=o}^{\infty} F_n(x)t^n \tag{3.12}$$

via Darboux's method. To compute (3.12) we need to introduce some special functions (Charris and Ismail [9]). Let

$\sqrt{1+z}$ be the branch of the square root of $1+z$ which is analytic on $\mathbb{C}-(-\infty,-1)$ and positive on $(-1,+\infty)$. Also, let $\sqrt{z-1}$ be analytic in $\mathbb{C}-(-\infty,1]$ and positive on $(1,+\infty)$. It is easily verified that

$$\sqrt{z^2-1} := \sqrt{z+1}\ \sqrt{z-1} \quad (*) \tag{3.13}$$

is analytic in $\mathbb{C}-[-1,1]$. Futhermore

$$\sqrt{z^2-1} = \begin{cases} \sqrt{z^2-1}, & z > 1 \\ i\sqrt{1-z^2}, & -1 < z \leqslant 1, \\ -\sqrt{z^2-1}, & z < -1. \end{cases} \tag{3.14}$$

Then

$$\alpha(z) = z + \sqrt{z^2-1}, \quad \beta(z) = z - \sqrt{z^2-1} \tag{3.15}$$

are the roots of $t^2-2zt+1 = 0$ which are analytic in $\mathbb{C}-[-1,1]$ and satisfy $|\alpha(z)| = |\beta(z)|$ if and only if $-1 \leqslant z \leqslant 1$ and $|\beta(z)| < |\alpha(z)|$ if $z \in \mathbb{C}-[-1,1]$. Also notice that $\beta(x)$ is decreasing in $(-\infty,-1]$ and $[1,+\infty)$. Observe that

$$\alpha(z) + \beta(z) = 2z, \quad \alpha(z)\beta(z) = 1. \tag{3.16}$$

If

$$\xi(z) = \frac{1}{\Delta}\alpha(uz - \frac{v}{\Delta}), \quad \zeta(z) = \frac{1}{\Delta}\beta(uz - \frac{v}{\Delta}). \tag{3.17}$$

Then

$$\xi(z) + \zeta(z) = \frac{2}{\Delta}(uz - \frac{v}{\Delta}), \quad \xi(z)\zeta(z) = \frac{1}{\Delta^2}. \tag{3.18}$$

Hence, $\xi(z)$, $\zeta(z)$ are analytic selections of the roots of

$$\Delta^2 t^2 - 2(u\Delta z - v)t + 1 = 0. \tag{3.19}$$

Note that $|\xi(z)| \geqslant |\zeta(z)|$ and observe that the conditions on the parameters guaranty that

$$-1 < ux - \frac{v}{\Delta} < 1, \quad \text{if} \quad -1 \leqslant x \leqslant 1. \tag{3.20}$$

(*) Observe the difference between the notations $\sqrt{\ }$ and $\sqrt{\ }$.

Hence $\xi(x)$ and $\zeta(x)$ are non-real and $\xi(x) = \overline{\zeta(x)}$ for $-1 < x < 1$. This also makes it impossible to have

$$\zeta(x) = q^k \beta(x), \quad k = 0,1,2,\ldots, \quad -1 < x < 1,$$

as then, from (3.16) and (3.17), we could derive $\Delta q^k x = ux - \frac{v}{\Delta}$, $\sqrt{1-(ux-\frac{v}{\Delta})^2} = \Delta q^k \sqrt{1-x^2}$, so that $1 = \Delta^2 q^{2k}$, a contradiction. Furthermore

LEMMA 3.1. *If* $x \in \mathbb{R}$ *and* $|x| > 1$ *then*

$$\zeta \neq \beta q^k, \quad k = 0,1,2,\ldots$$

Proof. Assume $\zeta(x) = \beta(x) q^k$. Since $\beta(x) q^k$ is real also $\zeta(x)$ is real and therefore $|ux - v/\Delta| \geqslant 1$. If $x > 1$, it follows from (3.1) that $\Delta(1-u)x > -v$, so that $x > ux - v/\Delta$. Since also $ux - v/\Delta \geqslant 1$ and β is decreasing in $[1,+\infty)$ then $\Delta^{-1}\beta(ux-v/\Delta) > \beta(x) > 0$, and therefore $|\zeta(x)| > |\beta(x)|$. It is shown in the same manner that if $x < -1$ then $x < ux - v/\Delta \leqslant -1$, and therefore $\Delta^{-1}\beta(ux-v/\Delta) < \beta(x) < 0$. Hence, also $|\zeta(x)| > |\beta(x)|$, and the assertion follows. ▲

COROLLARY. *If* $x \in \mathbb{R}$ *and* $|x| > 1$ *then*

$$\xi \neq \beta q^k, \quad k = 0,1,2,\ldots$$

Proof. In fact, $|\xi(x)| \geqslant |\zeta(x)| > |\beta(x)|$. ▲

To determine $F(x,t)$ multiply both sides of (1.7) by t^{n+1}, $n = 1,2,\ldots$, and add, to obtain

$$(t^2 - 2xt + 1) F(x;t) = [\Delta^2 t^2 - 2(u\Delta x - v) + 1] F(x;qt).$$

Hence

$$F(x;t) = \frac{(1-t/\xi)(1-t/\zeta)}{(1-t/\alpha)(1-t/\beta)} F(x;qt). \tag{3.21}$$

Taking into account that $F(x;q^n r) \to F_o(x) = 1$ as $n \to \infty$, iteration of (3.21) readily gives

$$F(x;t) = \frac{(t/\xi;q)_\infty(t/\zeta;q)_\infty}{(t/\alpha;q)_\infty(t/\beta;q)_\infty}. \tag{3.22}$$

If $x \notin [-1,1]$ then $|\beta(x)| < |\alpha(x)|$, and therefore

$$|\beta| < |\beta q^{-k}| < |\alpha q^{-k}|, \qquad k = 1,2,\ldots$$

Hence $F(x;t)$ is analytic for $|t| < |\beta(x)|$. A comparison functions is

$$\begin{aligned}\tilde{F}(x;t) &= |\lim_{t\to\beta}(1-t/\beta)F(x;t)|(1-t/\beta)^{-1}\\ &= \frac{(\beta/\xi;q)_\infty(\beta/\zeta;q)_\infty}{(\beta/\alpha;q)_\infty(q;q)_\infty}(1-t/\beta)^{-1}.\end{aligned} \tag{3.23}$$

Darboux's method readily gives

$$F_n(x) \sim \frac{(\beta/\xi;q)_\infty(\beta/\zeta;q)_\infty}{(\beta/\alpha;q)_\infty(q;q)_\infty}\alpha^n. \tag{3.24}$$

From (3.7) it follows that

$$\frac{F_n^2(x)}{\lambda_n} \sim \frac{(\beta/\xi;q)_\infty^2(\beta/\xi;q)_\infty^2}{(1-u\Delta)(\Delta^2;q)_\infty(\beta/\alpha;q)^2(q;q)}\alpha^{2n} \tag{3.25}$$

and Lemma 3.1 and its corollary ensure that the coefficient of α^{2n} on the rigth hand side term never vanishes. Since $|\alpha| > 1$ then

$$\sum_{n=o}^{\infty}\frac{F_n^2(x)}{\lambda_n} = +\infty, \qquad x \in \mathbb{R}-[-1,1] \tag{3.26}$$

and the function ψ has no jumps on $\mathbb{R}-[-1,1]$.

Now let $-1 < x < 1$. Then $|\alpha(x)| = |\beta(x)|$ and both and β are algebraic branch singularities of $F(x;t)$. Recall that in such case $\overline{\alpha(x)} = \beta(x)$ and, since $ux-v/\Delta \in (-1,1)$, also $\overline{\xi(x)} = \zeta(x)$. Darboux's method gives

$$F_n(x) \sim \frac{(\beta/\xi;q)_\infty(\beta/\zeta;q)_\infty}{(\beta/\alpha;q)_\infty(q;q)_\infty}\alpha^n + \frac{(\alpha/\xi;q)_\infty(\alpha/\zeta;q)_\infty}{(\alpha/\beta;q)_\infty(q;q)_\infty}\beta^n$$

Hence $\xi(x)$ and $\zeta(x)$ are non-real and $\xi(x) = \overline{\zeta(x)}$ for $-1 < x < 1$. This also makes it impossible to have

$$\zeta(x) = q^k \beta(x), \quad k = 0,1,2,\ldots, \quad -1 < x < 1,$$

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If $x \notin [-1,1]$ then $|\beta(x)| < |\alpha(x)|$, and therefore

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Hence $F(x;t)$ is analytic for $|t| < |\beta(x)|$. A comparison functions is

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Darboux's method readily gives

$$F_n(x) \sim \frac{(\beta/\xi;q)_\infty(\beta/\zeta;q)_\infty}{(\beta/\alpha;q)_\infty(q;q)_\infty}\alpha^n. \tag{3.24}$$

From (3.7) it follows that

$$\frac{F_n^2(x)}{\lambda_n} \sim \frac{(\beta/\xi;q)_\infty^2(\beta/\xi;q)_\infty^2}{(1-u\Delta)(\Delta^2;q)_\infty(\beta/\alpha;q)^2(q;q)}\alpha^{2n} \tag{3.25}$$

and Lemma 3.1 and its corollary ensure that the coefficient of α^{2n} on the rigth hand side term never vanishes. Since $|\alpha| > 1$ then

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$$F_n(x) \sim \frac{(\beta/\xi;q)_\infty(\beta/\zeta;q)_\infty}{(\beta/\alpha;q)_\infty(q;q)_\infty}\alpha^n + \frac{(\alpha/\xi;q)_\infty(\alpha/\zeta;q)_\infty}{(\alpha/\beta;q)_\infty(q;q)_\infty}\beta^n$$

$$= \frac{(\beta/\xi;q)_\infty(\beta/\zeta;q)_\infty}{(\beta/\alpha;q)_\infty(q;q)_\infty}\alpha^n + \text{conjugate}.$$

Notice that both the coeficients of α^n and β^n are non-vanishg. If we write

$$\frac{(\beta/\xi;q)_\infty(\beta/\zeta;q)_\infty}{(\beta/\alpha;q)_\infty(q;q)_\infty} = A(x)e^{i\phi(x)} \tag{3.28}$$

where $A(x)$ is the absolute value of left hand side and $\phi(x)$ is the argument, then

$$F_n(x) \sim Ae^{i\phi}\,|\alpha|^n\,e^{in\theta} + \text{conjugate},$$

where $\theta(x) = \arg\alpha(x)$. Since $|\alpha| = 1$ it follows that

$$F_n(x) \sim 2A\cos(n\theta+\phi). \tag{3.29}$$

Hence

$$\sqrt{1-x^2}\,\frac{F_n^2(x)}{\lambda_n} \sim \frac{4\sqrt{1-x^2}}{1-u\Delta}\left|\frac{(\beta/\xi;q)_\infty(\beta/\zeta;q)_\infty}{(\beta/\alpha;q)_\infty(q;q)_\infty}\right|^2\frac{(q;q)_\infty}{(\Delta^2;q)_\infty}\cos^2(n\theta+\phi)$$

and, since $\limsup\limits_{n\to\infty}\cos(n\theta+\phi) = 1$, (2.20) gives

$$\phi(x) = \frac{(1-u\Delta)(\Delta^2;q)_\infty}{2\pi\sqrt{1-x^2}(q;q)_\infty}\left|\frac{(q;q)_\infty(\beta/\alpha;q)_\infty}{(\beta/\xi;q)_\infty(\beta/\zeta;q)_\infty}\right|^2 \tag{3.30}$$

for $x \in (-1,1)$. This determines the absolutely continuous part of the measure.

As for ψ, we already know from Nevai's theorem that it is constant in $(-\infty,1)$, $(-1,1)$ and $(1,+\infty)$. The jumps, if any, are then located in $\{-1,1\}$. However,

$$\sum_{n=o}^{\infty}\frac{F_n^2(\pm 1)}{\lambda_n} = +\infty. \tag{3.31}$$

In fact, from (3.22) with $x = \pm 1$, and noticing that $\alpha(\pm 1) = \beta(\pm 1) = \pm 1$, it follows that

$$F(\pm 1,t) = \frac{(t/\xi;q)_\infty(t/\zeta;q)_\infty}{(\pm t;q)_\infty^2}. \tag{3.32}$$

A comparison function is

$$\tilde{F}(\pm 1,t) = \frac{(1/\xi;q)_\infty(1/\zeta;q)_\infty}{(\pm q;q)_\infty}\frac{1}{(1\mp t)^2}, \qquad (3.33)$$

and Darboux's method gives

$$F_n(\pm 1) \sim \frac{(1/\xi;q)_\infty(1/\zeta;q)_\infty}{(\pm q;q)_\infty}(\pm 1)^n(n+1). \qquad (3.34)$$

Since $-1 < u-v/\Delta < 1$, ξ and ζ are non-real, so that $\xi,\zeta \neq q^k$, $k = 0,1,2,\ldots$ Hence

$$\frac{F_n^2(\pm 1)}{\lambda_n} \sim \frac{(1/\xi;q)_\infty^2(1/\zeta;q)_\infty^2(q;q)_\infty}{(1-u\Delta)(\pm q;q)_\infty^2(\Delta^2;q)_\infty}(n+1)^2 = C(n+1)^2 \qquad (3.35)$$

where $C \neq 0$. This proves (3.31). Thus ψ is constant on $\mathbb{R}$ and $d\psi = 0$.

Summing up:

THEOREM 3.1. *The p-Pollaczek polynomials are, when* $0 < u < \Delta < 1$, $\Delta(1-u) > \pm v$, $0 < q < 1$, *a system of orthogonal polynomials with respect to the absolutely continuous measure*

$$d\mu(x) = \chi_{(-1,1)}(x)\,\frac{(1-u\Delta)(\Delta^2;q)_\infty(q;q)_\infty}{\pi\sqrt{1-x^2}}\left|\frac{(\beta/\alpha;q)_\infty}{(\beta/\xi;q)_\infty(\beta/\xi;q)_\infty}\right|^2 dx, \qquad (3.36)$$

where $\chi_{(-1,1)}$ *is the characteristic function of the interval* $(-1,1)$.

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