In this paper we continue the study of stochastic processes which live on adapted probability spaces. We do it from the point of view introduced by Hoover and Keisler in their paper "Adapted Probability Distributions" ([Hok]). We introduce a new way of relating two stochastic processes which are defined on the same adapted space. The new concept is that of a process $Y$ being intrinsic with respect to a process $X$ (see section III for details). Informally, this means that the process $Y$ is "definable" from $X$ by mean of the adapted functions of Hoover and Keisler. This new notion is related to that of intrinsic stochastic process, due to Hoover and Keisler, but which has never been published before. In section III we present all these new concepts and develop their basic properties. As a main application of our study of intrinsic processes we can within this framework, analyze the operation of stopping a stochastic process by mean of a stopping (random) time. As it is well known (see [DM1] and [DM2]) this operation is fundamental in the general theory of stochastic processes. We single out the class of intrinsic stopping times and prove among other things that for these stopping times the operation is preserved for processes with the
same distribution (see Theorem III.10).

The contents of this paper are detailed as follows. Section I is a brief introduction to adapted distributions. The reader can find there the basic definitions and some of the most important theorems. This section makes the paper basically self-contained but nevertheless [HoK] is strongly recommended. In section II we present some of the ideas that have motivated the study carried out in this paper. Section III contains the new results, develops some of the questions posed in section II and finishes with examples, comments and suggestions for further applications of the concepts and results presented in this paper.

§1. Introduction. In the General Theory of Processes, stochastic processes "live" on the so called adapted probability spaces. These are structures that, intuitively, allow us to model the evolution on time of a random phenomenon and its relationship with the information, which increases with time, about the world where the process takes place. Formally this is done as follows.

DEFINITION 1.1. (a) An Adapted Probability Space is a structure of the form $\mathcal{A} = (A, (F_t), P)$, $t \in [0,1]$ where $(A, F_1, P)$ a probability space and $(F_t)$, $t \in [0,1]$, is an increasing family of $\sigma$-algebras, called a Filtration, satisfying the following (usual) conditions:

(i) Right continuity: For each $t < 1$, $F_t = \bigcap_{s>t} F_s$.

(ii) Completeness: $F_0$ is $P$-complete.

(b) A Stochastic Process on $A$ is a collection $X = (X_t)$, $t \in [0,1]$ of random variables defined on $A$ and taking values in $\mathbb{R}$. We say that $X$ is Measurable if viewed, in the natural way, as a function $X: A \times [0,1] \rightarrow \mathbb{R}$ it is measurable with respect to $F_1 \times \text{Leb}([0,1])$, where $\text{Leb}([0,1])$ is the Lebesgue $\sigma$-algebra on $[0,1]$. Observe that for simplicity we are using $[0,1]$ as a time parameter set instead by $\mathbb{R}^+ \cup \{\infty\}$ which
Going back to the intuitive motivation given in the first paragraph, we can think of the $\sigma$-algebra $F_t$ as containing the information up to time $t$ and the variable $X_t$ as telling us about the random phenomenon at that time. The connection between the process and the filtration is made through the conditional expectation operator. For example, if the evolution of the process up to time $t$ is known, in our model this can be expressed by $E[X_t | F_t] = X_t$ a.s.. Processes which satisfy this condition for every $t$ are called Adapted. Familiar concepts such us Markov processes and martingales are among the many notions studied in the general theory of processes. A very comprehensive account of this theory can be found in [DM1], [DM2] and [DM3].

Once we start working within this context a natural question comes up: when do two processes living on adapted spaces "share the same probabilistic properties"? Clearly the old notion of finite dimensional distribution does not work, since it does not take into account the role played by the filtration. A new concept was needed and one was proposed by Hoover and Keisler in [HoK]. In order to make this paper as self-contained as possible we present it here. Before giving the definition let's fix some notation. When we consider a process together with the adapted space where it is defined we write $(A,X)$ or $X$ if there is no confusion about $A$.

**DEFINITION 1.2.** (a) The Class $\text{AF}$ of Adapted Functions in $\mathbb{R}$ is defined inductively as follows:

(i) (Basis step) if $f: \mathbb{R} \rightarrow \mathbb{R}$ is a bounded continuous function and $t \in [0,1]$, then the expression $(f,t)$ belongs to $\text{AF}$.

(ii) (Composition step) if $f_1, \ldots, f_n$ belong to $\text{AF}$ and $g: \mathbb{R}^n \rightarrow \mathbb{R}$ is bounded and continuous, then $g(f_1, \ldots, f_n)$ belongs to $\text{AF}$.

(iii) (Conditional Expectation step) if $g$ belongs to $\text{AF}$ and
(b) The value $f(X)$ of $f$ is the random variable defined inductively by:

i) $(\phi, t)(X) = \phi(X_t)$.

ii) $\emptyset(f_1, \ldots, f_n)(X) = \emptyset(f_1(X), \ldots, f_n(X))$.

iii) $E[g \mid t](X) = E[g(X) \mid F_t]$.

Sometimes we write $f^A(X)$ when we want to emphasize the adapted structure $A$ we are working with.

(c) The rank of $f$ is defined by:

i) $\text{Rank}((\phi, t)) = 0$.

ii) $\text{Rank}(\emptyset(f_1, \ldots, f_n)) = \max(\text{Rank}(f_1), \ldots, \text{Rank}(f_n))$.

iii) $\text{Rank}(E[g \mid t]) = \text{Rank}(g) + 1$.

(d) The class of adapted functions $f$ with rank($f$) $\leq n$ is denoted by $A^F_n$.

(e) Let $X$ and $Y$ be stochastic processes defined on adapted structures $A$ and $B = (B, (G_t), Q)$ respectively. $X$ and $Y$ have the same adapted distribution if for every $f \in A^F$ $E(f(X)) = E(f(Y))$. In symbols we write $X \equiv Y$ or $(A, X) \equiv (B, Y)$.

If for every $f \in A^F_n$ we have $E(f(X)) = E(f(Y))$ we say that $X$ and $Y$ have the same adapted distribution up to rank $n$ and denote this relation by $X \equiv_n Y$.

COMMENTS. (a) The concepts just introduced can be extended in a natural way to the setting where we are interested in studying more than one stochastic process at a time. For example, let's consider the case where $X$ and $X'$ are processes defined on $A$ and $Y$ and $Y'$ are processes defined on $B$. We say that $(X, X')$ and $(Y, Y')$ have the same adapted distribution and denote this relation by $(X, X') \equiv (Y, Y')$ if for every $f \in A^F$, $E(f(X, X')) = E(f(Y, Y'))$. The only thing new is that we have to have clauses (a)-i) and (b)-i) for both $X$ and $X'$, then when we iterate the other two rules we capture the interaction between $X$, $X'$ and the filtration $(F_t)$. Formally this can be done as follows.
Replace clause (a)-i) by:

If \( \phi : \mathbb{R} \rightarrow \mathbb{R} \) is a bounded continuous function and \( t \in [0,1] \) then the expressions \((\phi_1, t)\) and \((\phi_2, t)\) belong to AF.

Replace clause (b)-i) by:

\[
(\phi_1, t)(X, x') = \phi(X_t' t) \\
(\phi_2, t)(X, x') = \phi(X_t' t).
\]

Another way of extending the definition to this case is to consider the pair \((X, X')\) as an \( \mathbb{R}^2 \)-valued stochastic process. This is the way it was done in [HoK] and [H1]. In this paper we will follow the approached sketched above.

(b) In [HoK] after introducing the above concepts it is argued that if \( X \) and \( Y \) are processes such that \( X \equiv Y \) then "they share almost the same probabilistic properties". Of course this statement cannot be proved formally, but we can look at particular cases and see if for those it is correct. Many important cases were examined in [HoK] and several important related topics have been studied in [H1], [H2], [K1], [K2], [K3], [HP], [F1], and [F2].

(c) A crucial feature of the results obtained in the mentioned articles has been a novel application of methods and ideas coming from three different fields: Probability theory, Nonstandard Analysis and Logic (through Model Theory). A result that perfectly illustrates the interaction of these three fields is the saturation theorem due Hoover and Keisler ([HoK]). We present it here as an example of the sort of results which can be obtained along this new line of research. We first need to introduce some notation. Let \( L \) be the usual hyperfinite adapted space of Nonstandard Analysis (see for example [AFHL], [SB], [K4] or [C]). The reader who does not know Nonstandard Analysis should not be worried since we will not use this result in this paper.

**SATURATION THEOREM.** Let \( A \) be an arbitrary adapted space. Suppose \( X \) and \( Y \) are stochastic processes defined on \( L \) and \( A \) respectively and \( X \equiv Y \). If \( Z \) is another process on \( A \) then there is a process \( W \) defined on \( L \) such that \((X, Z) \equiv (Y, W)\).
COROLLARY. (Universality Theorem). Let $A$ and $L$ be as in the above theorem. If $X$ is a process defined on $A$ then we can find a process $Y$ on $L$ such that $X \equiv Y$.

The saturation property should look familiar to Logicians. As a matter of fact, there is a logic, called Adapted Probability Logic which is adequate for the study of stochastic processes. Its models are precisely the adapted spaces and the relation $\equiv$ is its elementary equivalence relation. Purely model theoretic aspects of this logic have been recently studied. For references see [F2] and the survey article [K1] of Keisler (the father of this newly born area of research). An important observation has to be made. All these model theoretic results can be phrased and proved in a way that makes perfect sense to Probabilists (see for example [HoK]) and therefore can be understood by all those who study stochastic processes. We believe this is a good example of a new trend in logic where ideas that used to be exclusive of this subject are now made available to other branches of mathematics.

The implications of the saturation theorem have been striking. In particular it allows us to say: "whatever happens in an arbitrary adapted space, can be replicated inside a hyperfinite adapted space, at least for those properties that are captured under $\equiv$". Therefore, the more properties we can handle with $\equiv$, the stronger the above claim is. In this paper we are going to explore some ideas that grew out of the previous observations. We assume the reader has some familiarity with the basic concepts of the theory of stochastic processes, but not very much is needed (see the first chapters of [E]).

§2. Some natural questions. One of the most important operations usually carried out within the general theory of sto-
stochastic processes is that of stopping a stochastic processes with a stopping time. The details are as follows.

**DEFINITION 2.1.** (a) Let \( \mathcal{A} \) be an adapted space. An \( \mathcal{A} \)-Stopping time is a function \( S: \mathcal{A} \to [0,1] \) such that for every \( t \in [0,1] \),

\[
\{ w \in \mathcal{A} : S(w) \leq t \} \in \mathcal{F}_t.
\]

(b) Let \( X \) be a process defined on \( \mathcal{A} \), \( S \) an \( \mathcal{A} \)-stopping time. The Stochastic Process \( X \) stopped at \( S \) is the new stochastic process \( X \) defined by \( X^S(w,t) = X(w, S(w) \wedge t) \) (Here, \( S(w) \wedge t \) is the minimum between \( S(w) \) and \( t \)).

Concepts such as Local Martingale, Optional and Predictable projections, Class (D) process, Semimartingale and many others have the above operation involved in their definitions (see [DM2]). If we want to lend support to the claim that the relation \( \equiv \) in fact captures the probabilistic properties shared by two equivalent stochastic processes living on adapted spaces, then the following question seems natural to ask:

**QUESTION 1.** Let \( X \) and \( Y \) be stochastic processes defined on adapted spaces \( \mathcal{A} \) and \( \mathcal{B} \) respectively. Suppose \( X \equiv Y \) and \( S \) is an \( \mathcal{A} \)-stopping time. Does there exist a \( \mathcal{B} \)-stopping time \( T \) such that \( (X, X^S) \equiv (Y, Y^T) \)?

Now let's move in another direction. Recently we are becoming more familiar with the type of results that can be obtained in stochastic analysis using nonstandard analysis methods. Starting with the work of Anderson and Keisler, the use of hyperfinite adapted spaces has shown how useful these new techniques can be in providing new simpler and more natural proofs of known results and also as powerful tools in order to obtain new standard results. This situation has prompted the following question:
QUESTION 2. How can we use the properties of hyperfinite spaces within the general theory of stochastic processes? The saturation property tells us that at least under the relation $\equiv$ we loose nothing by working with these spaces; even more, some results suggest that there is a lot to be gained (see the recent article [HeK]).

But suppose that we are interested in solving a problem in a specific adapted space which is not hyperfinite. Is there any use for hyperfinite adapted spaces in this case? One obvious answer is that if we show that the problem cannot be solved within the hyperfinite space then it cannot be solved in the original space. But, what if it can be solved? It may be that we are making essential use of the saturation property and according to Henson and Keisler's article this could mean that the result cannot be proved in the original space. Given this situation we would like to identify those problems for which the following informal method works:

"Take your problem, translate it to a hyperfinite adapted space making use of the saturation property, solve it inside this space using its nice combinatorial properties and then go back to your original space where you have a solution". We call this the "Come-Back problem".

In the following section we present some results which can be seen as first steps in answering the questions just posed.

§3. Intrinsic processes. In a preliminary version of "Adapted Distributions", Hoover and Keisler introduced the concepts of intrinsic filtration and intrinsic stochastic process and proved some basic results about these notions. In the final version of their paper they never included that material because it was unrelated to the main thrust of the article. We are going to present some natural extensions of their ideas and use them in order to deal with the observa-
tions made in the previous section. Their results are includ-
ed here with their kind permission.

DEFINITION 3.1. Let $A$ and $X$ be fixed.

(a) Let $n \in \mathbb{N}$. The $n$-intrinsic $\sigma$-algebra $I^X_n$ of $X$ is the $\sigma$-algebra generated by $N$ (the set of $P$-null sets) and the random variables $f(X)$ where $f \in AF^n$. The $\infty$-intrinsic $\sigma$-algebra is the $\sigma$-algebra generated by $N$ and the random variables $f(X)$ where $f \in AF$.

(b) Let $n \in \mathbb{N} \cup \{\infty\}$. The $n$-intrinsic measure algebra of $X$ is the measure algebra $(A, I^X_n, P)/N$.

(c) The $n$-intrinsic adapted space of $X$ is the structure $(A, (I^X_n \cap F_t), P)$. We denote it by $A^X_n$.

(d) $X$ is said to be $n$-intrinsic with respect to $A$ if $A = A^X_n$.

We can now present some elementary properties. Hoover and Keisler only considered the $\infty$-case, but the generalization to $n \in \mathbb{N}$ is straightforward.

PROPOSITION 3.2. (a) Let $n \in \mathbb{N} \cup \{\infty\}$ then $(A, X) \overset{n}{=} (A^X_n, X)$.

(b) If $X'$ is a version of $X$ then $I^X_n = I^X_{\infty}$ and $(A, X) \equiv (A, X')$.

(c) For all $t \in [0,1]$ and $n \geq 1$, $I^X_n \cap F_t$ is the $\sigma$-algebra generated by $N$ and the random variables $f(X)$ where $f$ is of the form $E[g | t]$ with rank$(g) \leq n - 1$.

(d) If $X$ is Markov then $I^X_n$ is the $\sigma$-algebra generated by $N$ and the variables $X_s$ with $s \in [0,1]$. $I^X_n \cap F_t$ is the $\sigma$-algebra generated by $N$ and $X_u$ with $u \leq t$.

Proof. (a) We show that for every $f \in AF^n$, $\tilde{s} \in [0,1]$,

\[f^A(X)(\tilde{s}) = f^{A^n}(X)(\tilde{s}) \text{a.s.} \quad (*)\]

This is done by induction on the complexity of $f$. It is trivial for the basis and composition steps. Let $f$ be of the form $E[g | t]$ with rank$(g) \leq n - 1$ and assume (*) holds for $g$. $f^A(X)(\tilde{s}, t) = E[g^A(X)(\tilde{s}) | F_t]$ and $f^{A^n}(X)(\tilde{s}, t) = E[g^{A^n}(X)(\tilde{s}) | I^n X \cap F_t]$. Observe that $f^A(X)(\tilde{s}, t)$
is \( I^X_n \cap F_t \)-measurable by the definition of \( I^X_n \). Then we have:

\[
E[t^A(X)(s) I^X_n \cap F_t] = E[E[g^A(X)(\tilde{s}) | F_t] | I^X_n \cap F_t] = E[g^A_n(X) | I^X_n \cap F_t] = f^A_n(X).
\]

(b) and (c) are left as exercises.

(d) Follows from [HoK] (theorem 2.8) and (b).

**NOTE.** Parts (a), (b) and (c) of the above proposition can be extended in a natural way to the case where there is more than one stochastic process in the adapted structures.

The original motivation Hoover and Keisler had when they introduced the above notions was to give a characterization of \( \equiv \) in terms of a measure algebra isomorphism. We can now present their result in a form that covers the more general case we introduced before. A definition is first needed.

**DEFINITION 3.3.** Let \( X \) and \( X' \) (\( Y \) and \( Y' \)) be stochastic processes defined on \( A \) (\( B \)). A function \( h: (A, X, X')_N \rightarrow (B, Y, Y')_N \) is an \( n \)-adapted measure isomorphism from \( (A, X, X') \) to \( (B, Y, Y') \) if it satisfies the following conditions:

(i) \( h \) is a measure algebra isomorphism from \( (A, I^X_n, P)_N \) onto \( (B, I^{YY'}_n, Q)_N \).

(ii) For all \( t, h \) maps \( (I^X_n \cap F_t)_N \) onto \( (I^{YY'}_n \cap G_t)_N \).

(iii) For all \( t, r \in \mathbb{R} \) and \( \emptyset: \mathbb{R} \rightarrow \mathbb{R} \) bounded and continuous:

a) \( h(\{0(X_t) \geq r\}_N) = \{0(Y_t) \geq r\}_N \) and

b) \( h(\{0(X'_t) \geq r\}_N) = \{0(Y'_t) \geq r\}_N \).

Observe that this last definition is given for the case where we consider two stochastic processes at a time. The other possible situations are treated similarly.

Here is the characterization of equivalence up to rank \( n \) by means of \( n \)-adapted measure isomorphism. The proof is the original of Hoover and Keisler, we just extended it in
The obvious way to \( n \in \mathbb{N} \) and to the case where there is more than one stochastic process considered per structure.

**Theorem 3.4.** Let \( X \) and \( X' \) (\( Y \) and \( Y' \)) be stochastic processes defined on \( \mathcal{A}(\mathcal{B}) \) respectively, then: \( (X, X') \overset{D}{=} (Y, Y') \) if and only if there exists a unique \( n \)-adapted measure isomorphism \( h \) from \( (\mathcal{A}, X, X') \) to \( (\mathcal{B}, Y, Y') \).

The crucial part of the proof is contained in the following result.

**Lemma 3.5.** Suppose \( h \) is an \( n \)-adapted measure isomorphism from \( (\mathcal{A}, X, X') \) to \( (\mathcal{B}, Y, Y') \). Then for each \( f \in \mathcal{A}^{\mathbb{R}^n}, \ s \in [0,1] \) and \( r \in \mathbb{R} 

\[
h\left(\{f(X, X')(s) > r\} \cap \mathcal{N}\right) = \{f(Y, Y')(s) > r\} \cap \mathcal{N}.
\]

*Proof.* The proof is by induction on the complexity of \( f \). The basis case is given in the definition of \( n \)-adapted measure isomorphism. Let \( \varnothing: \mathbb{R}^n \to \mathbb{R} \) be continuous and bounded. On any compact subset of \( \mathbb{R}^n \), \( \varnothing \) may be uniformly approximated by a linear combination of functions of the form \( \varnothing'(x_1, \ldots, x_n) = u_1(x_1) \cdots u_n(x_n) \), where each \( u_i \in C(\mathbb{R}^n, \mathbb{R}) \). Using this fact, together with the induction hypothesis the composition case follows.

Now suppose \( f \) is of the form \( E[g(\tilde{s}) | t] \) with \( \text{rank}(g) \leq n - 1 \) and the lemma holds for \( g \). Since \( h \) preserves measure, for each set \( U \in I^n_{\mathbb{X}}' \cap \mathcal{F}_t \) we have

\[
\int_{U} g(X, X')(\tilde{s})dP = \int_{h(U)} g(Y, Y')(\tilde{s})dQ \tag{1}
\]

where \( h(U) \) is a representative of \( h(U/\mathcal{N}) \), so that \( h(U) \in I^n_{\mathbb{Y}}' \cap \mathcal{B}_t \). We claim that for each \( r \),

\[
h\left(\{E[g(X, X')(\tilde{s}) | F_t] > r\} \cap \mathcal{N}\right) = \{E[g(Y, Y')(\tilde{s}) | G_t] > r\} \cap \mathcal{N}. \tag{2}
\]

Suppose (2) fails. By 3.2.c). \( E[g(X, X')(\tilde{s}) | F_t] \) is \( I^n_{\mathbb{X}}' \cap \mathcal{F}_t \).
measurable and $E[g(Y,Y')(\tilde{s}) \mid G_t]$ is $I_{n}^{YY'} \cap G_t$-measurable.

Let $Z$ be a representative of $h\{(E[g(X,X')(\tilde{s}) \mid F_t] > r)/N\}$. Then $Z \subseteq I_{n}^{XX'} \cap G_t$, and one of the sets

$$Z \cap \{E[g(Y,Y')(\tilde{s}) \mid G_t] > r\} = V,$$

$$\{E[g(Y,Y')(\tilde{s}) \mid G_t] > r\} - Z = W$$

has positive measure, say $V$. We have $V \subseteq I_{n}^{YY'} \cap G_t$ and $V/N = h(U/N)$ for some $U \subseteq I_{n}^{XX'} \cap F_t$. Since $V \subseteq Z$, then $U \subseteq \{E[g(X,X')(s) \mid F_t] > r\}$, a.s. Also, $P(U) = Q(V) > 0$. Therefore

$$\int_{U} g(X,X')(\tilde{s})dP = \int_{U} E[g(X,X')(\tilde{s}) \mid F_t]dP > rP(U).$$

However $V \subseteq \{E[g(Y,Y')(\tilde{s}) \mid G_t] < r\}$, so

$$\int_{V} g(Y,Y')(\tilde{s})dQ = \int_{V} E[g(Y,Y')(\tilde{s}) \mid G_t]dQ < rQ(V) = rP(U).$$

Then

$$\int_{U} g(X,X')(\tilde{s})dP = \int_{V(U)} g(Y,Y')(\tilde{s})dQ$$

and this a contradiction to (2).

**Proof of Theorem 3.4.** Assume $(X,X') \equiv (Y,Y')$. Then for each $f \in AF_n$, $\tilde{s}$ and $r$ we have $P\{f(X,X')(\tilde{s}) > r\} = Q\{f(Y,Y')(\tilde{s}) > r\}$. Since adapted functions are closed under composition, then there is a measure algebra isomorphism such that $h\{(f(X,X')(\tilde{s}) > r)/N\} = \{f(Y,Y')(\tilde{s}) > r\}/N$. In particular,

$$h\{(E[f(X,X')(\tilde{s}) \mid F_t] > r)/N\} = \{E[f(Y,Y')(\tilde{s}) \mid G_t]/N\}.$$

Then by 3.3.c), $h$ maps $I_{n}^{XX'} \cap F_t$ onto $I_{n}^{YY'} \cap G_t$. This shows that $h$ is an adapted isomorphism. Uniqueness follows easily from the previous lemma. Now assume $h$ is an $n$-adapted isomorphism from $(A,X,X')$ to $(B,Y,Y')$. Then by the lemma we have
\[ P\{f(X,X')(\tilde{s}) > r\} = Q\{f(Y,Y')(\tilde{s}) > r\} \text{ for each } f \in AF^n, \]
\[ \tilde{s} \text{ and } r. \]

Therefore \( E[f(X,X')(\tilde{s})] = E[f(Y,Y')(\tilde{s})] \), and so \((X,X') \equiv (Y,Y')\).

Now we introduce a concept that will play a fundamental role in what follows.

**DEFINITION 3.6.** Let \( X \) and \( X' \) be stochastic processes defined on the same adapted structure \( \mathcal{A} \). \( X' \) is \textit{n-intrinsic with respect to} \( X \) if \( \mathcal{I}^X_n \subseteq \mathcal{I}^X_n \).

Again, this notion can be extended to the cases where there are several processes involved without any difficulties. Observe that if \( X \) is \( n \)-intrinsic with respecto to \( \mathcal{A} \) then any other stochastic process we define on \( \mathcal{A} \) becomes \( n \)-intrinsic with respecto to \( X \).

A very important source of examples of intrinsic notions is the class of hitting times of a stochastic process. Those readers familiar with the general theory of processes now that these are precisely the most important examples of stopping times. The definition is as follows.

**DEFINITION 3.7.** Let \( X \) be a progressively measurable (see [DM1]) stochastic process defined on \( \mathcal{A} \). If \( B \) is a Borel subset of \( \mathbb{R} \) then

\[ S_B(w) = \operatorname{Inf}\{t \in [0,1): X(w,t) \in B\} \]

is called \textit{first hitting time} of \( B \).

It is well known (see [DM1]) that \( S_B \) is an \( \mathcal{A} \)-stopping time.

**PROPOSITION 3.8.** \( S_B \) is \( 1 \)-intrinsic with respecto to \( X \).

\textit{Proof.} We leave it to the reader. Just look at the proof that hitting times are stopping times.
We are now ready to state and prove our main theorem.

**THEOREM 3.9.** Let $X$ and $Y$ be stochastic processes defined on $A$ and $B$ respectively. If $X^n = Y$ and $Z$ is a process defined on $A$ which is n-intrinsic with respect to $X$ then there exists a process $W$ defined on $B$ such that:

(i) $W$ is n-intrinsic with respect to $Y$.

(ii) $(A,X,Z)^n \equiv (B,Y,W)$.

**Proof.** Let's assume $Z$ is r.c.1.1. (i.e. for each $w$ and $t \in [0,1]$, $Z^t(w) = \lim_{s \uparrow t} Z^s(w)$ and $\lim_{s \uparrow t} Z^s(w)$ always exists).

Let $h$ be the unique n-adapted isomorphism between $(A,X)$ and $(B,Y)$ given by theorem 3.4. For each $t \in [0,1]$, let $m_t$ be the least $s \in [0,1]$ such that $Z$ is $I^X_n \cap F_s$-measurable. Observe that $\{s \in [0,1] : Z$ is $I^X_n \cap F_s$-measurable$\}$ is $\emptyset$ since $Z^t$ is $I^X_n$-measurable and by hypothesis $I^X_n \equiv I^X_n$, we then have that $Z$ is $I^X_n$-measurable and recall that by definition $I^X_n \subseteq F_1$. This fact together with the right continuity of the filtration $(I^X_n, F_t)$ gives us the existence of $m_t$.

Fix $q \in Q \cap [0,1]$. $Z^q$ is $I^X_n \cap F_{mq}$-measurable. Then, as it is well known from elementary measure theory, there exists a sequence $(S_m(q))_{m \in N}$ of $I^X_n \cap F_{mq}$-simple functions such that

$$\lim_{m \to \infty} S_m(q)(w) = Z^q(w). \text{ a.s.}$$

For each $m$, write $S_m(q) = \sum_{i=1}^{n_q} s_i X^{A_i}$ where $A_i = \{w \in A : S_m(q)(w) = s_i\} \subseteq I^X_n \cap F_{mq}$. Clearly the $A_i$'s partition $A$. For each $A_i$, let $\hat{A}_i \subseteq I^X_n \cap G_{mq}$ be such that $h(A_i/N) = \hat{A}_i/N$. By the definition of $h$, the $\hat{A}_i$'s partition $B$ (modulo a null set) and for each $i$, $P(A_i) = Q(\hat{A}_i)$. Now for each $m$, let $\hat{S}_m(q) = \sum_{i=1}^{n_q} s_i X^{\hat{A}_i}$.

Define $W$ as follows:

If $q \in Q \cap [0,1]$,

$$W^q(w) = \begin{cases} \lim_{m \to \infty} \hat{S}_m(q)(w) & \text{if it exists} \\ 0 & \text{otherwise.} \end{cases}$$
If \( t \in [0,1] \)

\[
W_t(w) = \begin{cases} 
\lim_{q \uparrow t} W_q(w) & \text{if it exists} \\
0 & \text{otherwise}.
\end{cases}
\]

We now claim that \((A,X,Z)_n \equiv (B,Y,W)_n\). We use theorem 3.4 again. Observe the following facts:

(i) \( I_{n}^{XZ} = I_{n}^{X} \). Since \( I_{n}^{Z} \subseteq I_{n}^{X} \), observe that the genetarors of \( I_{n}^{XZ} \) belong to \( I_{n}^{X} \).

(ii) By the way we defined \( W \) it is clear that \( I_{n}^{YW} \subseteq I_{n}^{Y} \) (i.e. \( W \) is \( n \)-intrinsic with respect to \( Y \)) and as in

(i) \( I_{n}^{YW} = I_{n}^{Y} \).

With the above conditions, our original \( h \) can be seen as a mapping from \((A,I_{n}^{XZ}, P)/N\) to \((B,I_{n}^{YW}, Q)/N\). It satisfies all the conditions required for it to be an \( n \)-adapted measure isomorphism. We just verify the third condition, the others follow immediately from the way \( W \) was defined from \( Z \) and the properties of \( h \) as an \( n \)-adapted measure isomorphism from \((A,X)\) to \((B,Y)\). We have to show for each \( t \in [0,1], 0 \in C(\mathbb{R},\mathbb{R}) \) and \( r \in \mathbb{R} \) that

\[
h(\{\emptyset(Z_t) > r\}/N) = \{\emptyset(W_t) > r\}/N.
\]

By right continuity of \( Z \) and \( \emptyset \), it is enough to check this for \( Z_q \) with \( q \in Q \cap [0,1] \). By the continuity of \( \emptyset \) and the fact that \( S_m(q) \to Z_q \) it is enough to see that \( h(\{\emptyset(S_m(q)) > r\}/N) = \{\emptyset(S_m(q)) > r\}/N \) and this is immediate given the way we defined \( W \) from \( Z \). The case of \( Z \) arbitrary we leave it to the reader (Hint: use Proposition 2.24 in [HoK]).

The following result is an important example that shows some of the possible uses of the above theorem. In fact, we were interested first in the study of stopping times and after obtaining this theorem we were lead to the more general result obtained in theorem 3.9. Observe that the key point in the above proof is the fact that once the process \( Z \) is \( n \)-intrinsic with respect to \( X \) then we can use the same adapted
isomorphism that characterized $X \cong Y$. We can think of the notion of intrinsic process as a kind of probabilistic way of "definable". Notice that in general if we are given a structure $(A, X)$, an $A$-stopping time may have nothing to do with the stochastic process $X$ and consequently with $I^n_X$.

**THEOREM 3.10.** Let $X$ and $Y$ be right (left) continuous stochastic processes defined on $A$ and $B$ respectively. If $S$ is an n-intrinsic $A$-stopping time with respect to $X$ then there exists a $B$-stopping time $T$ n-intrinsic with respect to $Y$ such that $(X, X^S) \cong (Y, Y^T)$.

**Proof.** We just indicate the steps needed in the demonstration.

a) First show the following:

If $S$ is a simple stopping time then there exists a simple n-intrinsic stopping time $T$ such that $(X, S) \cong (Y, T)$. The argument is similar to the treatment of simple functions in the proof of the above theorem.

b) With $S$ and $T$ as in (a) prove that

If $(X, S) \cong (Y, T)$ then $(X, X^S) \cong (Y, Y^T)$. For this case observe that if $S$ is n-intrinsic with respect to $X$ then $X^S$ is n-intrinsic with respect to $X$.

c) Given $S$ arbitrary then proceed as follows:

First choose a decreasing sequence $(S_m)$ of simple $A$-stopping times n-intrinsic with respect to $X$, such that $\lim_{m \to \infty} S_m = S$ a.s. The fact that the sequence $(S_m)$ can be chosen n-intrinsic with respect to $X$ follows from the way it is defined from $S$ (see [DM1]).

For each $S_m$ of the sequence find a $T_m$ n-intrinsic with respect to $Y$ as indicated in (a). We then have $(X, X^{S_m}) \cong (Y, Y^{T_m})$. Using the right continuity of $X$ and $Y$ and the fact $\cong$ is preserved under pointwise convergence we have:

$$(X, \lim X^{S_m}) \cong (Y, \lim Y^{T_m}) \text{ and } \lim X^{S_m} = X^S.$$
Let $T$ be $\lim_{m \to \infty} T_m$ and our theorem follows.

The following example of Hoover shows that the conditions of the above theorem cannot be relaxed.

**EXAMPLE 3.11.** Let $A = B = [0,1]$, for every $t$, $F_t = F_1 = \text{Borel subsets of } [0,1]$ and $P = \text{Lebesgue } ([0,1])$.

$$X_t(w) = Y_t(w) = 0 \text{ if } t < w,$$

$$X_t(w) = 2 \text{ if } t = w,$$

$$X_t(w) = 1 \text{ if } t > w,$$

$$Y_t(w) = 1 \text{ if } t \geq w.$$

Since $X_t = Y_t$ a.s. for every $t$, $X \equiv Y$, but $S$ is the stopping time $S(w) = w$, and clearly there is no stopping time $T$ such that $(X, X^S) \equiv (Y, Y^T)$.

**COMMENT.** Before the beginning of the general theory of processes, for many probabilistic problems it was common to say "the finite dimensional distributions of a stochastic process is all what matters". If for our purposes the adapted distribution is all what matters, then with the above results we have a very satisfactory answer to the first question we posed in §2. This is because proposition 3.2.a) tells us that with respect to the relation $\equiv$, we can restrict ourselves to the study of $n$-intrinsic processes, and we remarked before that if $X$ is $n$-intrinsic in $A$ then any stopping time defined on $A$ is $n$-intrinsic with respect to $X$ and therefore, theorem 3.10 applies. If for some reason we cannot assume that the process $X$ is $n$-intrinsic then, for example proposition 3.8, still provides us with a very important source of applications for the above theorem.

Another type of situation where our results may be useful is in the study of preservation of properties under the relations $\equiv$. The general form of the problem is as follows: Suppose $X$ and $Y$ are stochastic processes such that $X \equiv Y$ and $P$ is a property of stochastic processes (for example, to be
a local martingale). Suppose X has property P, does Y have it?

The answer depends, obviously, in the nature of the property P. The first study along this lines was done by Aldous [A] who introduced the synonymity relation between stochastic processes (in our notation \( \equiv \)) and proved that some basic properties, like the martingale property are preserved under \( \equiv \). Later Hoover and Keisler [HoK] studied extensively the relation \( \equiv \) and making use of the saturation property of the hyperfinite spaces proved the preservation of some properties under \( \equiv \) which were then used to study properties of solutions of stochastic integral equations. In [F1] we can also find results of this nature. Hoover in [H1] and [H2] has done very interesting work in this direction. We can use one of his results in [H1], which viewed under our theorem, illustrates a new approach to the preservation problem.

**EXAMPLE 3.12.** Suppose \( X \equiv Y \) and \( X \) is a local martingale. Then \( Y \) is also a local martingale.

Proposition 3.2.a) allows us to draw the following picture:

\[
(A, X) \equiv (B, Y)
\]

\[
(A_1^X, X) \equiv (B_1^Y, Y)
\]

It is easy to show, using Theorem 3.10, that if \((A_1^X, X)\) is a local martingale and \((A_1^X, X) \equiv (B_1^Y, Y)\) then \((B_1^Y, Y)\) is also a local martingale. Therefore, if we go back to the above picture, we can conclude that in order to prove that the property of being a local martingale is preserved under \( \equiv \) it is enough to show that it is preserved in the particular case \((A, X) \equiv (A_1^X, X)\). In principle this should be easier given the way these two structures are closely related. In a sense this is what Hoover did in his proof.

An observation that can be useful in finding "Come
back" problems as defined in §2, is the similarity that exists between theorem 3.9 and the Saturarion Theorem in §2. Theorem 3.9 basically tells us that with respect to intrinsic notions adapted structures are saturated for the relations $\equiv$. With this observation in mind one can do things like the following. Suppose that you want to prove a Skorohod Embedding type of result (for a very simple example, see [B]). Do it first for Anderson's Brownian motion (see [AFHL] or [SB]) using the nice combinatorial properties of the hyperfinite setting and observe that the stopping times that you obtain are intrinsic. Once you have this, then you have a proof that is valid for all Brownian motions. This is because by the saturation property of the hyperfinite spaces every Brownian motion is $\equiv$ equivalent to Anderson's Brownian motion (see [K4]) and this fact can be put together with theorem 3.10. We believe that ideas like this one can be used in handling problems along the lines of [BC] and more general Skorohod type of problems. This project would require to continue the development of the Nonstandard theory of the general theory of processes (see [S]), adapted probability distributions and finding more intrinsic notions.

One more idea coming out of these results and properties of model theory has lead us to develop a notion of game relations between adapted structures. These results will be presented elsewhere (see [F3]).

Acknowledgments. Some of the results contained in this paper have been presented at the University of Colorado-Boulder and at meetings in the Universidad Catolica de Chile and the University of Hull, England. We wish to thank Professors Malitz, Bertossi, Chuaqui and Cutland for their hospitality. We also want to thank Professors Keisler and Hoover for their useful comments. This research has been partially supported by the Universidad de los Andes.

REFERENCES

[HoK] Hoover, D. and Keisler, H.J., Adapted probability dis-


* 

Departamento de Matemáticas
Universidad de los Andes
Apartado Aéreo 4976
Bogotá, Colombia.

(Recibido en abril de 1987, versión revisada en junio 1987).