INTRINSIC STOCHASTIC PROCESSES

by

Sergio FAJARDO

In this paper we continue the study of stochastic processes which live on adapted probability spaces. We do it from the point of view introduced by Hoover and Keisler in their paper "*Adapted Probability Distributions*" ([Hok]). We introduce a new way of relating two stochastic processes which are defined on the same adapted space. The new concept is that of a process Y being intrinsic with respect to a process X (see section III for details). Informally, this means that the process Y is "definable" from X by mean of the adapted functions of Hoover and Keisler. This new notion is related to that of intrinsic stochastic process, due to Hoover and Keisler, but which has never been published before. In section III we present all these new concepts and develop their basic properties. As a main application of our study of intrinsic processes we can within this framework, analyze the operation of stopping a stochastic process by mean of a stopping (random) time. As it is well known (see [DM1] and [DM2]) this operation is fundamental in the general theory of stochastic processes. We single out the class of intrinsic stopping times and prove among other things that for these stopping times the operation is preserved for processes with the

same distribution (see Theorem III.10).

The contents of this paper are detailed as follows. Section I is a brief introduction to adapted distributions. The reader can find there the basic definitions and some of the most important theorems. This section makes the paper basically self-contained but nevertheless [HoK] is strongly recommended. In section II we present some of the ideas that have motivated the study carried out in this paper. Section III contains the new results, develops some of the questions posed in section II and finishes with examples, comments and suggestions for further applications of the concepts and results presented in this paper.

§1. **Introduction.** In the General Theory of Processes, stochastic processes "live" on the so called adapted probability spaces. These are structures that, intuitively, allow us to model the evolution on time of a random phenomenen and its relationship with the information, which increases with time, about the world where the process takes place. Formally this is done as follows.

DEFINITION 1.1. (a) An *Adapted Probability Space* is a structure of the form $\underline{A} = (A, (F_t), P)$, $t \in [0,1]$ where (A, F_1, P) a probability space and (F_t), $t \in [0,1]$, is an increasing family of σ-algebras, called a *Filtration*, satisfying the following (usual) conditions:

(i) Right continuity: For each $t < 1$, $F_t = \bigcap_{s>t} F_s$.

(ii) Completeness: F_0 is P-complete.

(b) A *Stochastic Process* on A is a collection $X = (X_t)$, $t \in [0,1]$ of random variables defined on A and taking values in $\mathbf{R}$. We say that X is *Measurable* if viewed, in the natural way, as a function $X: A \times [0,1] \to \mathbf{R}$ it is measurable with respect to $F_1 \times \text{Leb}([0,1])$, where $\text{Leb}([0,1])$ is the Lebesgue σ-algebra on $[0,1]$. Observe that for simplicity we are using $[0,1]$ as a time parameter set instead by $\mathbf{R}^+ \cup \{\infty\}$ which

is more common, but there are no substancial differences
(1 plays the role of ∞).

Going back to the intuitive motivation given in the
first paragraph, we can think of the σ-algebra F_t as contain-
ning the information up to time t and the variable X_t as
telling us about the random phenomenon at that time. The
connection between the process and the filtration is made
through the conditional expectation operator. For example,
if the evolution of the process up to time t is known, in
our model this can be expressed by $E[X_t \mid F_t] = X_t$ a.s..
Processes which satisfy this condition for every t are cal-
led *Adapted*. Familiar concepts such us Markov processes and
martingales are among the many notions studied in the gener-
al theory of processes. A very comprehensive account of this
theory can be found in [DM1], [DM2] and [DM3].

Once we start working within this context a natural
question comes up: when do two processes living on adapted
spaces "share the same probabilistic properties"? Clearly
the old notion of finite dimensional distribution does not
work, since it does not take into account the role played by
the filtration. A new concept was needed and one was proposed
by Hoover and Keisler in [HoK]. In order to make this paper
as self-contained as possible we present it here. Before giv-
ing the definition let's fix some notation. When we consider
a process together with the adapted space where it is defined
we write $(\underline{A},X)$ or $\underline{X}$ if there is no confusion about $\underline{A}$.

DEFINITION 1.2. (a) The Class AF of *Adapted Functions*
in $\mathbb{R}$ is defined inductively as follows:
(i) (Basis step) if $\Phi:\mathbb{R} \to \mathbb{R}$ is a bounded continuous function
    and $t \in [0,1]$, then the expression (Φ,t) belongs to AF.
(ii) (Composition step) if $f_1,\ldots,f_n$ belong to AF and
    $\emptyset:\mathbb{R}^n \to \mathbb{R}$ is bounded and continuous, then $\emptyset(f_1,\ldots,f_n)$
    belongs to AF.
(iii) (Conditional Expectation step) if g belongs to AF and

$t \in [0,1]$ then the expression $E[g \mid t]$ belongs to AF.

(b) The *Value* $f(\underline{X})$ of f is the random variable defined inductively by:

i) $(\Phi,t)(\underline{X}) = \Phi(X_t)$.

ii) $\emptyset(f_1,\ldots,f_n)(\underline{X}) = \emptyset(f_1(\underline{X}),\ldots,f_n(\underline{X}))$.

iii) $E[g \mid t](\underline{X}) = E[g(\underline{X}) \mid F_t]$.

Sometimes we write $f^A(\underline{X})$ when we want to emphasize the adapted structure $\underline{A}$ we are working with.

(c) The *Rank* of f is defined by:

i) $\text{Rank}((\Phi,t)) = 0$.

ii) $\text{Rank}(\emptyset(f_1,\ldots,f_n)) = \text{Max}(\text{Rank}(f_1),\ldots,\text{Rank}(f_n))$.

iii) $\text{Rank}(E[g \mid t]) = \text{Rank}(g) + 1$.

(d) The class of adapted functions f with $\text{rank}(f) \leqslant n$ is denoted by AF^n.

(e) Let X and Y be stochastic processes defined on adapted structures $\underline{A}$ and $\underline{B} = (B,(G_t),Q)$ respectively. $\underline{X}$ and $\underline{Y}$ have the same *Adapted Distribution* if for every $f \in AF$ $E(f(\underline{X})) = E(f(\underline{Y}))$. In symbols we write $\underline{X} \equiv \underline{Y}$ or $(\underline{A},\underline{X}) \equiv (\underline{B},Y)$.

If for every $f \in AF^n$ we have $E(f(\underline{X})) = E(f(\underline{Y}))$ we say that $\underline{X}$ and $\underline{Y}$ have *the same Adapted Distribution up to Rank* n and denote this relation by $\underline{X} \overset{n}{\equiv} \underline{Y}$.

COMMENTS. (a) The concepts just introduced can be extended in a natural way to the setting where we are interested in studying more than one stochastic process at a time. For example, let's consider the case where X and X' are processes defined on $\underline{A}$ and Y and Y' are processes defined on $\underline{B}$. We say that (X,X') and (Y,Y') have the *Same Adapted Distribution* and denote this relation by $(\underline{X},\underline{X}') \equiv (\underline{Y},\underline{Y}')$ if for every $f \in AF$, $E(f(\underline{X},\underline{X}')) = E(f(\underline{Y},\underline{Y}'))$. The only thing new is that we have to have clauses (a)-i) and (b)-i) for both X and X', then when we iterate the other two rules we capture the interaction between X, X' and the filtration (F_t). Formally this can be done as follows.

Replace clause (a)-i) by:

    If $\Phi:\mathbb{R} \to \mathbb{R}$ is a bounded continuous function and
    $t \in [0,1]$ then the expressions (Φ_1,t) and (Φ_2,t)
    belong to AF.

Replace clause (b)-i) by:

    $(\Phi_1,t)(X,X') = \Phi(X_t).$
    $(\Phi_2,t)(X,X') = \Phi(X'_t).$

Another way of extending the definition to this case is to consider the pair (X,X') as an $\mathbb{R}^2$-valued stochastic process. This is the way it was done in [HoK] and [H1]. In this paper we will follow the approached sketched above.

    (b) In [HoK] after introducing the above concepts it is argued that if $\underline{X}$ and $\underline{Y}$ are processes such that $\underline{X} \equiv \underline{Y}$ then "they share almost the same probabilistic properties". Of course this statement cannot be proved formally, but we can look at particular cases and see if for those it is correct. Many important cases were examined in [HoK] and several important related topics have been studied in [H1], [H2], [K1], [K2], [K3], [HP], [F1], and [F2].

    (c) A crucial feature of the results obtained in the mentioned articles has been a novel application of methods and ideas coming from three different fields: Probability theory, Nonstandard Analysis and Logic (through Model Theory). A result that perfectly illustrates the interaction of these three fields is the saturation theorem due Hoover and Keisler ([HoK]). We present it here as an example of the sort of results which can be obtained along this new line of research. We first need to introduce some notation. Let $\underline{L}$ be the usual hyperfinite adapted space of Nonstandard Analysis (see for example [AFHL], [SB], [K4] or [C]). The reader who does not know Nonstandard Analysis should not be worried since we will not use this result in this paper.

    SATURATION THEOREM. *Let A be an arbitrary adapted space. Suppose X and Y are stochastic processes defined on $\underline{L}$ and $\underline{A}$ respectively and $\underline{X} \equiv \underline{Y}$. If Z is another process on $\underline{A}$ then there is a process W defined on $\underline{L}$ such that $(\underline{X},\underline{Z}) \equiv (\underline{Y},\underline{W})$.*

COROLLARY. (Universality Theorem). *Let A and L be as in the above theorem. If X is a process defined on A then we can find a process Y on L such that X ≡ Y.*

The saturation property should look familiar to Logicians. As a matter of fact, there is a logic, called *Adapted Probability Logic* which is adequate for the study of stochastic processes. Its models are precisley the adapted spaces and the relation ≡ is its elementary equivalence relation. Purely model theoretic aspects of this logic have been recently studied. For references see [F2] and the survey article [K1] of Keisler (the father of this newly born area of research). An important observation has to be made. All these model theoretic results can be phrased and proved in a way that makes perfect sense to Probabilists (see for example [HoK]) and therefore can be understood by all those who study stochastic processes. We believe this is a good example of a new trend in logic where ideas that used to be exclusive of this subject are now made available to other branches of mathematics.

The implications of the saturation theorem have been striking. In particular it allows us to say: "whatever happens in an arbitrary adapted space, can be replicated inside a hyperfinite adapted space, at least for those properties that are capturated under ≡". Therefore, the more properties we can handle with ≡, the stronger the above claim is. In this paper we are going to explore some ideas that grew out of the previous observations. We assume the reader has some familiartity with the basic concepts of the theory of stochastic processes, but not very much is needed (see the first chapters of [E]).

§2. **Some natural questions.** One of the most important operations usually carried out within the general theory of sto-

chastic processes is that of stopping a stochastic process-
es with a stopping time. The details are as follows.

DEFINITION 2.1. (a) Let $\underline{A}$ be an adapted space. An
$\underline{A}$-*Stopping time* is a function $S:A \rightarrow [0,1]$ such that for every
$t \in [0,1]$,

$$\{w \in A: S(w) \leqslant t\} \in F_t.$$

(b) Let X be a process defined on $\underline{A}$, S an $\underline{A}$-stopping time.
The *Stochastic Process* X *stopped at* S is the new stochastic
process X defined by $X^S(w,t) = X(w,S(w) \wedge t)$ (Here, $S(w) \wedge t$
is the minimum between $S(w)$ and t).

Concepts such as Local Martingale, Optional and Pre-
dictable projections, Class (D) process, Semimartingale and
many others have the above operation involved in their def-
initions (see [DM2]). If we want to lend support to the claim
that the relation $\equiv$ in fact captures the probabilistic prop-
erties shared by two equivalent stochastic processes living
on adapted spaces, then the following question seems natural
to ask:

QUESTION 1. Let X and Y be stochastic processes defined
on adapted spaces $\underline{A}$ and $\underline{B}$ respectively. Suppose $X \equiv Y$ and
S is an $\underline{A}$-stopping time. Does there exist a $\underline{B}$-stopping time
T such that $(X,X^S) \equiv (Y,Y^T)$?.

Now let's move in another direction. Recently we are
becoming more familiar with the type of results that can be
obtained in stochastic analysis using nonstandard analysis methods.
Starting with the work of Anderson and Keisler, the use of
hyperfinite adapted spaces has shown how useful these new
techniques can be in providing new simpler and more natural
proofs of known results and also as powerful tools in order
to obtain new standard results. This situation has prompted
the following question:

QUESTION 2. How can we use the properties of hyperfi-
nite spaces within the general theory of stochastic processes?
The saturation property tells us that at least under the
relation ≡ we loose nothing by working with these sapces;
even more, some results suggest that there is a lot to be
gained (see the recent article [HeK]).

But suppose that we are interested in solving a prob-
lem in a specific adapted space which is not hyperfinite.
Is there any use for hyperfinite adapted spaces in this case?
One obvious answer is that if we show that the problem can-
not be solved within the hyperfinite space then it cannot be
solved in the original space. But, what if it can be solved?
It may be that we are making essential use of the saturation
property and according to Henson and Keisler's article this
could mean that the result cannot be proved in the original
space. Given this situation we would like to identify those
problems for which the following informal method works:

"Take your problem, translate it to a hyperfinite adapted
space making use of the saturation property, solve it inside
this space using its nice combinatorial properties and then
go back to your original space where you have a solution".
We call this the "Come-Back problem".
      In the following section we present some results
which can be seen as first steps in answering the questions
just posed.

§3. **Intrinsic processes.** In a preliminary version of "Adapt-
ed Distributions", Hoover and Keisler introduced the con-
cepts of intrinsic filtration and intrinsic stochastic proc-
ess and proved some basic results about these notions. In
the final version of their paper they never included that
material because it was unrelated to the main thrust of the
article. We are going to present some natural extensions of
their ideas and use them in order to deal with the observa-

tions made in the previous section. Their results are includ-
ed here with their kind permission.

DEFINITION 3.1. Let $\underline{A}$ and X be fixed.

(a) Let $n \in N$. The n-*intrinsic* σ-*algebra* $I_n^{\underline{X}}$ of $\underline{X}$ is
the σ-algebra generated by N (the set of P-null sets) and
the random variables $f(\underline{X})$ where $f \in AF^n$. The ∞-*intrinsic*
σ-*algebra* is the σ-algebra generated by N and the random
variables $f(\underline{X})$ where $f \in AF$.

(b) Let $n \in N \cup \{\infty\}$. The n-*intrinsic measure algebra*
of $\underline{X}$ is the measure algebra $(A, I_n^{\underline{X}}, P)/N$.

(c) The n-*intrinsic adapted space of* $\underline{X}$ is the struc-
ture $(A, (I_n^{\underline{X}} \cap F_t), P)$. We denote it by $\underline{A}_n^{\underline{X}}$.

(d) X is said to be n-*intrinsic with respect to* $\underline{A}$ if
$\underline{A} = \underline{A}_n^{\underline{X}}$.

We can now present some elementary properties. Hoover
and Keisler only considered the ∞-case, but the generaliza-
tion to $n \in N$ is straightforward.

PROPOSITION 3.2. (a) *Let* $n \in N \cup \{\infty\}$ *then* $(\underline{A}, X) \stackrel{n}{\equiv} (\underline{A}_n^{\underline{X}}, X)$.
(b) *If* X' *is a version of* X *then* $I_\infty^{\underline{X}} = I_\infty^{\underline{X}'}$ *and* $(\underline{A}, X) \equiv (\underline{A}, X')$.
(c) *For all* $t \in [0,1]$ *and* $n \geq 1$, $I_n^{\underline{X}} \cap F_t$ *is the* σ-*algebra gen-
erated by* N *and the random variables* $f(\underline{X})$ *where* f *is of the
form* $E[g \mid t]$ *with* rank$(g) \leq n-1$.
(d) *If* X *is Markov then* $I^{\underline{X}}$ *is the* σ-*algebra generated
by* N *and the variables* X_s *with* $s \in [0,1]$. $I_\infty^{\underline{X}} \cap F_t$ *is the*
σ-*algebra generated by* N *and* X_u *with* $u \leq t$.

Proof. (a) We show that for every $f \in AF^n$, $\vec{s} \in [0,1]$

$$f^{\underline{A}}(\underline{X})(\vec{s}) = f^{\underline{A}_n^{\underline{X}}}(\underline{X})(\vec{s}) \, a.s. \tag{*}$$

This is done by induction on the complexity of f. It is tri-
vial for the basis and composition steps. Let f be of the
form $E[g \mid t]$ with rank$(g) \leq n-1$ and assume (*) holds for g.
$f^{\underline{A}}(\underline{X})(\vec{s}, t) = E[g^{\underline{A}}(\underline{X})(\vec{s}) \mid F_t]$ and
$f^{\underline{A}_n^{\underline{X}}}(\underline{X})(\vec{s}, t) = E[g^{\underline{A}_n^{\underline{X}}}(\underline{X})(\vec{s}) \mid I_n^{\underline{X}} \cap F_t]$. Observe that $f^{\underline{A}}(\underline{X})(\vec{s}, t)$

is $I_n^X \cap F_t$-measurable by the definition of I_n^X. Then we have:

$$f^A_-(\underline{X})(\vec{s},t) = E[f^A_-(\underline{X})(\vec{s}) \mid I_n^X \cap F_t] = E[E[g^A_-(\underline{X})(\vec{s}) \mid F_t] \mid I_n^X \cap F_t]$$

$$= E[g^A_-(\underline{X}) \mid I_n^X \cap F_t] = E[g^{A_n}_-(\underline{X}) \mid I_n^X \cap F_t] = f^{A_n}_-(\underline{X}).$$

(b) and (c) are left as exercises.

(d) Follows from [HoK] (theorem 2.8) and (b).

NOTE. Parts (a), (b) and (c) of the above proposition can be extended in a natural way to the case where there is more than one stochastic process in the adapted structures.

The original motivation Hoover and Keisler had when they introduced the above notions was to give a character-ization of $\equiv$ in terms of a measure algebra isomorphism. We can now present their result in a form that covers the more general case we introduced before. A definition is first needed.

DEFINITION 3.3. Let X and X' (Y and Y') be stochastic processes defined on $\underline{A}$ ($\underline{B}$). A function $h:(\underline{A},X,X')/_N \to (\underline{B},Y,Y')/_N$ is an n-*adapted measure isomorphism* from $(\underline{A},X,X')$ to $(\underline{B},Y,Y')$ if it satisfies the following conditions:

(i) h is a measure algebra isomorphism from $(A, I_n^{XX'}, P)/_N$ onto $(\underline{B}, I_n^{YY'}, Q)/_N$.

(ii) For all t, h maps $(I_n^{XX'} \cap F_t)/_N$ onto $(I_n^{YY'} \cap G_t)/_N$.

(iii) For all $t, r \in \mathbb{R}$ and $\emptyset:\mathbb{R} \to \mathbb{R}$ bounded and continuous:

a) $h(\{\emptyset(X_t) \geqslant r\}/_N) = \{\emptyset(Y_t) \geqslant r\}/_N$ and

b) $h(\{\emptyset(X'_t) \geqslant r\}/_N) = \{\emptyset(Y'_t) \geqslant r\}/_N$.

Observe that this last definition is given for the case where we consider two stochastic processes at a time. The other possible situations are treated similarly.

Here is the characterization of equivalence up to rank n by means of n-adapted measure isomorphism. The proof is the original of Hoover and Keisler, we just extended it in

the obvious way to $n \in \mathbb{N}$ and to the case where there is more than one stochastic process considered per structure.

THEOREM 3.4. *Let* X *and* X' (Y *and* Y') *be stochastic processes defined on* $\underline{A}$ ($\underline{B}$) *respectively, then:* $(X,X') \overset{\mathrm{D}}{=} (Y,Y')$ *if and only if there exists a unique* n-*adapted measure isomorphism* h *from* $(\underline{A},X,X')$ *to* $(\underline{B},Y,Y')$.

The crucial part of the proof is contained in the following result.

LEMMA 3.5. *Suppose* h *is an* n-*adapted measure isomorphism from* $(\underline{A},X,X')$ *to* $(\underline{B},Y,Y')$. *Then for each* $f \in AF^n$, $\vec{s} \in [0,1]$ *and* $r \in R$:

$$h(\{f(X,X')(\vec{s}) > r\}/_N) = \{f(Y,Y')(\vec{s}) > r\}/_N. \qquad (*)$$

Proof. The proof is by induction on the complexity of f. The basis case is given in the definition of n-adapted measure isomorphism. Let $\emptyset : R^n \to \mathbb{R}$ be continuous and bounded. On any compact subset of $\mathbf{R}^n$, $\emptyset$ may be uniformly approximated by a linear combination of functions of the form $\emptyset'(x_1,\ldots, x_n) = u_1(x_1)\ldots u_n(x_u)$, where each $u_i \in C(\mathbf{R}^n,R)$. Using this fact, together with the induction hypothesis the composition case follows.

Now suppose f is of the form $E[g(\vec{s}) \mid t]$ with rank(g) $\leqslant n-1$ and the lemma holds for g. Since h preserves measure, for each set $U \in I_n^{XX'} \cap F_t$ we have

$$\int_U g(X,X')(\vec{s})dP = \int_{h(U)} g(Y,Y')(\vec{s})dQ \qquad (1)$$

where h(U) is a representative of h(U/N), so that $h(U) \in I_n^{YY'} \cap B_t$. We claim that for each r,

$$h(\{E[g(X,X')(\vec{s}) \mid F_t] > r\}/N) = \{E[g(Y,Y')(\vec{s}) \mid G_t] > r\}/N. \qquad (2)$$

Suppose (2) fails. By 3.2.c). $E[g(X,X')(\vec{s}) \mid F_t]$ is $I_n^{XX'} \cap F_t$-

measurable and $E\big[g(Y,Y')(\bar{s}) \mid G_t\big]$ is $I_n^{YY'} \cap G_t$-measurable.
Let Z be a representative of $h(\{E[g(X,X')(\bar{s}) \mid F_t] > r\}/N)$.
Then $Z \in I_n^{YY'} \cap G_t$, and one of the sets

$$Z - \{E\big[g(Y,Y')(\bar{s}) \mid G_t\big] > r\} = V,$$

$$\{E\big[g(Y,Y')(\bar{s}) \mid G_t\big] > r\} - Z = W$$

has positive measure, say V. We have $V \in I_n^{YY'} \cap G_t$ and $V/N = h(U/N)$ for some $U \in I_n^{XX'} \cap F_t$. Since $V \in Z$, then
$U \in \{E[g(X,X')(s) \mid F_t] > r\}$, a.s. Also, $P(U) = Q(V) > 0$.
Therefore

$$\int_U g(X,X')(\bar{s})\,dP = \int_U E\big[g(X,X')(\bar{s}) \mid F_t\big]\,dP > rP(U).$$

However $V \in \{E\big[g(Y,Y')(\bar{s}) \mid G_t\big] < r\}$, so

$$\int_V g(Y,Y')(\bar{s})\,dQ = \int_V E\big[g(Y,Y')(\bar{s}) \mid G_t\big]\,dQ < rQ(V) = rP(U).$$

Then

$$\int_U g(X,X')(\bar{s})\,dP = \int_{V(U)} g(Y,Y')(\bar{s})\,dQ$$

and this a contradiction to (2).

Proof of Theorem 3.4. Assume $(X,X') \equiv (Y,Y')$. Then for
each $f \in AF^n$, $\bar{s}$ and r we have $P\{f(X,X')(\bar{s}) > r\} = Q\{f(Y,Y')(\bar{s}) > r\}$. Since adapted functions are closed under
composition, then there is a measure algebra isomorphism
such that $h(\{f(X,X')(\bar{s}) > r\}/N) = \{f(Y,Y')(\bar{s}) > r\}/N$. In par-
ticular,

$$h(\{E[f(X,X')(\bar{s}) \mid F_t] > r\}/N) = \{E[f(Y,Y')(\bar{s}) \mid G_t]\}/N.$$

Then by 3.3.c), h maps $I_n^{XX'} \cap F_t$ onto $I_n^{YY'} \cap G_t$. This shows
that h is an adapted isomorphism. Uniqueness follows easily
from the previous lemma. Now assume h is an n-adapted iso-
morphism from $(\underline{A},X,X')$ to $(\underline{B},Y,Y')$. Then by the lemma we
have

$P\{f(X,X')(\vec{s}) > r\} = Q\ f(Y,Y')(\vec{s}) > r\}$ for each $f \in AF^n$, $\vec{s}$ and r.

Therefore $E[f(X,X')(\vec{s})] = E[f(Y,Y')(\vec{s})]$, and so $(X,X') \stackrel{n}{\equiv} (Y,Y')$.

Now we introduce a concept that will play a fundamental role in what follows.

DEFINITION 3.6. Let X and X' be stochastic processes defined on the same adapted structure $\underset{\sim}{A}$. X' is n-*intrinsic with respect to* X if $I_{\tilde{n}}^{X'} \subseteq I_{\tilde{n}}^{X}$.

Again, this notion can be extended to the cases where there are several processes involved without any difficulties. Observe that if X is n-intrinsic with respecto to $\underset{\sim}{A}$ then *any* other stochastic process we define on $\underset{\sim}{A}$ becomes n-intrinsic with respecto to $\underset{\sim}{X}$.

A very important source of examples of intrinsic notions is the class of hitting times of a stochastic process. Those readers familiar with the general theory of processes now that these are precisely the most important examples of stopping times. The definition is as follows.

DEFINITION 3.7. Let X be a progresively measurable (see [DM1]) stochastic process defined on $\underset{\sim}{A}$. If B is a Borel subset of $\mathbb{R}$ then

$$S_B(w) = \text{Inf}\{t \in [0,1) : X(w,t) \in B\}$$

is called *first hitting time of* B.

It is well known (see [DM1]) that S_B is an $\underset{\sim}{A}$-stopping time.

PROPOSITION 3.8. S_B *is* 1-*intrinsic with respecto to* X.
Proof. We leave it to the reader. Just look at the proof that hitting times are stopping times.

We are now ready to state and prove our main theorem.

THEOREM 3.9. *Let* X *and* Y *be stochastic processes defined on* $\underline{A}$ *and* $\underline{B}$ *respectively. If* $\underline{X} \overset{n}{\equiv} \underline{Y}$ *and* Z *is a process defined on* $\underline{A}$ *which is* n-*intrinsic with respect to* $\underline{X}$ *then there exists a process* W *defined on* $\underline{B}$ *such that:*

(i)   W *is* n-*intrinsic with respect to* $\underline{Y}$.

(ii) $(\underline{A}, X, Z) \overset{n}{\equiv} (\underline{B}, Y, W)$.

Proof. Let's assume Z is r.c.1.1. (i.e. for each w and $t \in [0,1]$ $Z_t(w) = \lim_{s \downarrow t} Z_s(w)$ and $\lim_{s \uparrow t} Z_s(w)$ always exists). Let h be the unique n-adapted isomorphism between $(\underline{A}, X)$ and $(\underline{B}, Y)$ given by theorem 3.4. For each $t \in [0,1]$, let m_t be the least $s \in [0,1]$ such that Z_s is $I_n^X \cap F_s$-measurable. Observe that $\{s \in [0,1] : Z_s$ is $I_n^X \cap F_s$-measurable$\}$ is $\neq \emptyset$ since Z_t is I_n^X-measurable and by hypothesis $I_n^X \in I_n^X$, we then have that Z is I_n^X-measurable and recall that by definition $I_n^X \subset F_1$. This fact together with the right continuity of the filtration (I_n^X F_t) gives us the existence of m_t.

Fix $q \in Q \cap [0,1]$. Z_q is $I_n^X \cap F_{m_q}$-measurable. Then, as it is well known from elementary measure theory, there exists a sequence $(S_m^{(q)})$ $m \in N$ of $I_n^X \cap F_{m_q}$-simple functions such that

$$\lim_{m \to \infty} S_m^{(q)}(w) = Z_q(w). \text{ a.s.}$$

For each m, write $S_m^{(q)} = \sum_{i=1}^{n_q} s_i X_{A_i}$ where $A_i = \{w \in A : S_m^{(q)}(w) = s_i\}$ $\in I_n^X \cap F_{m_q}$. Clearly the A_i's partition A. For each A_i, let $\hat{A}_i \in I_n^X \cap G_{m_q}$ be such that $h(A_i/N) = \hat{A}_i/N$. By the definition of h, the $\hat{A}_i$'s partition B (modulo a null set) and for each i, $P(A_i) = Q(\hat{A}_i)$. Now for each m, let $\hat{S}_m^{(q)} = \sum_{i=1}^{n_q} s_i X_{\hat{A}_i}$. Define W as follows:

If $q \in Q \cap [0,1]$,

$$W_q(w) = \begin{cases} \text{Lim}_{m \to \infty} \hat{S}_m^{(q)}(w) & \text{if it exists} \\ \\ 0 & \text{otherwise.} \end{cases}$$

If $t \in [0,1]$

$$W_t(w) = \begin{cases} \lim_{q \downarrow t} W_q(w) & \text{if it exists} \\ 0 & \text{otherwise.} \end{cases}$$

We now claim that $(\underline{A}, X, Z) \overset{n}{\equiv} (\underline{B}, Y, W)$. We use theorem 3.4 again. Observe the following facts:

(i) $I_{\frac{n}{n}}^{XZ} = I_{\frac{n}{n}}^{X}$. Since $I_{\frac{n}{n}}^{Z} \subseteq I_{\frac{n}{n}}^{X}$, observe that the genetarors of $I_{\frac{n}{n}}^{XZ}$ belong to $I_{\frac{n}{n}}^{X}$.

(ii) By the way we defined W it is clear that $I_{\frac{n}{n}}^{YW} \subseteq I_{\frac{n}{n}}^{Y}$ (i.e. W is n-intrinsic with respect to Y) and as in (i) $I_{\frac{n}{n}}^{YW} = I_{\frac{n}{n}}^{Y}$.

With the above conditions, our original h can be seen as a mapping from $(A, I_{\frac{n}{n}}^{XZ}, P)/N$ to $(B, I_{\frac{n}{n}}^{YW}, Q)/N$. It satisfies all the conditions required for it to be an n-adapted measure isomorphism. We just verify the third condition, the others follow inmediately from the way W was defined from Z and the properties of h as an n-adapted measure isomorphism from $(\underline{A}, X)$ to $(\underline{B}, Y)$. We have to show for each $t \in [0,1]$, $\emptyset \in C(\mathbb{R}, \mathbb{R})$ and $r \in \mathbb{R}$ that

$$h(\{\emptyset(Z_t) > r\}/N) = \{\emptyset(W_t) > r\}/N.$$

By right continuity of Z and $\emptyset$, it is enough to check this for Z_q with $q \in Q \cap [0,1]$. By the continuity of $\emptyset$ and the fact that $S_m^{(q)} \to Z_q$ it is enough to see that $h(\{\emptyset(S_m^{(q)}) > r\}/N) = \{\emptyset(S_m^{(q)}) > r\}/N$ and this is inmediate given the way we defined W from Z. The case of Z arbitrary we leave it to the reader (Hint: use Proposition 2.24 in [HoK]).

The following result is an important example that shows some of the possible uses of the above theorem. In fact, we were interested first in the study of stopping times and after obtaining this theorem we were lead to the more general result obtained in theorem 3.9. Observe that the key point in the above proof is the fact that once the process Z is n-intrinsic with respect to X then we can use the same adapted

isomorphism that characterized $X \overset{n}{\equiv} Y$. We can think of the notion of intrinsic process as a kind of probabilistic way of "definable". Notice that in general if we are given a structure $(\underline{A}, X)$, an $\underline{A}$-stopping time may have nothing to do with the stochastic process X and consequently with $I\frac{X}{n}$.

THEOREM 3.10. *Let X and Y be right (left) continuous stochastic processes defined on $\underline{A}$ and $\underline{B}$ respectively. If S is an n-intrinsic $\underline{A}$-stopping time with respect to $\underline{X}$ then there exists a $\underline{B}$-stopping time T n-intrinsic with respect to $\underline{Y}$ such that $(X, X^S) \overset{n}{\equiv} (Y, Y^T)$.*

Proof. We just indicate the steps needed in the demostration.

a) First show the following:

If S is a simple stopping time then there exists a simple n-intrinsic stopping time T such that $(X, S) \overset{n}{\equiv} (Y, T)$. The argument is similar to the treatment of simple functions in the proof of the above theorem.

b) With S and T as in (a) prove that

If $(X, S) \overset{n}{\equiv} (Y, T)$ then $(X, X^S) \equiv (Y, Y^T)$. For this case observe that if S is n-intrinsic with respect to X then X^S is n-intrinsic with respect to X.

c) Given S arbitrary then proceed as follows:

First choose a decreasing sequence (S_m) of simple $\underline{A}$-stopping times n-intrinsic with respect to X, such that $\lim_{m \to \infty} S_m = S$ a.s. The fact that the sequence (S_m) can be chosen n-intrinsic with respect to X follows from the way it is defined from S (see [DM1]).

For each S_m of the sequence find a T_m n-intrinsic with respect to Y as indicated in (a). We then have $(X, X^{S_m}) \overset{n}{\equiv} (Y, Y^{T_m})$. Using the right continuity of X and Y and the fact $\overset{n}{\equiv}$ is preserved under pointwise convergence we have:

$$(X, \lim X^{S_m}) \overset{n}{\equiv} (Y, \lim Y^{T_m}) \quad \text{and} \quad \lim X^{S_m} = X^S.$$

Let T be $\lim_{m \to \infty} T_m$ and our theorem follows.

The following example of Hoover shows that the conditions of the above theorem cannot be relaxed.

EXAMPLE 3.11. Let $A = B = [0,1]$, for every t, $F_t = F_1$ = Borel subsets of $[0,1]$ and P = Lebesgue $([0,1])$.

$$X_t(w) = Y_t(w) = 0 \quad \text{if } t < w,$$
$$X_t(w) = 2 \qquad\qquad \text{if } t = w,$$
$$X_t(w) = 1 \qquad\qquad \text{if } t > w,$$
$$Y_t(w) = 1 \qquad\qquad \text{if } t \geqslant w.$$

Since $X_t = Y_t$ a.s. for every t, $X \equiv Y$, but S is the stopping time $S(w) = w$, and clearly there is no stopping time T such that $(X,X^S) \equiv (Y,Y^T)$.

COMMENT. Before the beginning of the general theory of processes, for many probabilistic problems it was common to say "the finite dimensional distributions of a stochastic process is all what matters". If for our pourposes the adapted distribution is all what matters, then with the above results we have a very satisfactory answer to the first question we posed in §2. This is because propòsition 3.2.a) tells us that with respect to the relation $\overset{n}{\equiv}$ we can restrict ourselves to the study of n-intrinsic processes, and we remarked before that if X is n-ntrinsic in $\underline{A}$ then any stopping time defined on $\underline{A}$ is n-intrinsic with respect to X and therefore, theorem 3.10 applies. If for some reason we cannot assume that the process X is n-intrinsic then, for example proposition 3.8, still provides us with a very important source of applications for the above theorem.

Another type of situation where our results may be useful is in the study of preservation of properties under the relations $\overset{n}{\equiv}$. The general form of the problem is as follows: Suppose X and Y are stochastic processes such that $X \overset{n}{\equiv} Y$ and P is a property of stochastic processes (for example, to be

a local martingale). Suppose X has property P, does Y have it?

The answer depends, obvioulsy, in the nature of the property P. The first study along this lines was done by Aldous [A] who introduced the synonimity relation between stochastic processes (in our notation $\overset{1}{\equiv}$) and proved that some basic properties, like the martingale property are preserved under $\overset{1}{\equiv}$. Later Hoover and Keisler [HoK] studied extensively the relation $\equiv$ and making use of the saturation property of the hyperfinite spaces proved the preservation of some properties under $\equiv$ which were then used to study properties of solutions of stochastic integral equations. In [F1] we can also find results of this nature. Hoover in [H1] and [H2] has done very interesting work in this direction. We can use one of his results in [H1], which viewed under our theorem, illustrates a new approach to the preservation problem.

EXAMPLE 3.12. Suppose X $\overset{1}{\equiv}$ Y and X is a local martingale. Then Y is also a local martingale.

Proposition 3.2.a) allows us to draw the following picture:

$$(\underline{A},X) \overset{1}{\equiv} (\underline{B},Y)$$

$$1 \; ||| \qquad\qquad ||| \; 1$$

$$(\underline{A}_{\overline{1}}^X, X) \overset{1}{\equiv} (\underline{B}_{\overline{1}}^Y, Y)$$

It is easy to show, using Theorem 3.10, that if $(\underline{A}_{\overline{1}}^X, X)$ is a local martingale and $(\underline{A}_{\overline{1}}^X, X) \overset{1}{\equiv} (\underline{B}_{\overline{1}}^Y, Y)$ then $(\underline{B}_{\overline{1}}^Y, Y)$ is also a local martingale. Therefore, if we go back to the above picture, we can conclude that in order to prove that the property of being a local martingale is preserved under $\overset{1}{\equiv}$ it is enough to show that it is preserved in the particular case $(\underline{A},X) \overset{1}{\equiv} (\underline{A}_{\overline{1}}^X, X)$. In principle this should be easier given the way these two structures are closely related. In a sense this is what Hoover did in his proof.

An observation that can be useful in finding "Come

back" problems as defined in §2, is the similarity that
exists between theorem 3.9 and the Saturarion Theorem in
§2. Theorem 3.9 basically tells us that with respect to in-
trinsic notions adapted structures are saturated for the
relations $\overset{n}{\equiv}$. With this observation in mind one can do things
like the following. Suppose that you want to prove a Skoro-
hod Embedding type of result (for a very simple example, see
[B]). Do it first for Anderson's Brownian motion (see [AFHL]
or [SB]) using the nice combinatorial properties of the
hyperfinite setting and observe that the stopping times that
you obtain are intrinsic. Once you have this, then you have
a proof that is valid for all Brownian motions. This is be-
cause by the saturation property of the hyperfinite spaces
every Brownian motion is $\equiv$ equivalent to Anderson's Brow-
nian motion (see [K4]) and this fact can be put together
with teorem 3.10. We believe that ideas like this one can
be used in handling problems along the lines of [BC] and more
general Skorohod type of problems. This project would require
to continue the development of the Nonstandard theory of the
general theory of processes (see [S]), adapted probability
distributions and finding more intrinsic notions.

One more idea coming out of these results and proper-
ties of model theory has lead us to develop a notion of game
relations between adapted structures. These results will be
presented elsewhere (see [F3]).

REFERENCES

[HoK] Hoover, D. and Keisler, H.J., *Adapted probability dis-*

tributions. TAMS 286 (1984).

[DM1] Dellacherie, C. and Meyer, P.A., *Probabilities et potentiel I*. Hermann 1975.

[DM2] Dellacherie, C. and Meyer, P.A., *Probabilities et potentiel II*. Hermann 1980.

[DM3] Dellacherie, C. and Meyer, P.A., *Probabilities et potentiel III*. Hermann 1983.

[H1] Hoover, D., *Synonymity, generalized martingales and subfiltrations*. Annals of Probability 12 (1984).

[H2] Hoover, D., *A characterization of adapted distribution*. To appear. Annals of Probability.

[K1] Keisler, H.J., *Probability Quantifiers*. Chapter 14 in: J. Barwise and S. Feferman, eds. Model theoretic Logics. Springer. 1985.

[K2] Keisler, H.J., *A completeness proof for adapted probability Logic*. Ann. of Pure and Applied Logic 31 (1986).

[K3] Keisler, H.J., *Hyperfinite Models of Adapted probability Logic*. Ann. of Pure and Applied Logic 31 (1986).

[K4] Keisler, H.J., *An infinitesimal approach to stochastic Analysis*. AMS. Memoirs 397 (1984).

[F1] Fajardo, S., *Probability Logic with conditional Expectation*. Ann. of Pure and Applied Logic, 28 (1985).

[F2] Fajardo, S., *Completeness theorems for the general theory of processes*. Lecture notes in Mathematics #1030. 1985.

[HP] Hoover, D. and Perkins, E., *Nonstrandard construction of the stochastic integral and applications to stochastic differential equations, I. II.* TAMS 275 (1983).

[AFHL] Alveberio, S., Fenstad, J.E. Hoegh-Krohn, R. and Lindstrom, T., *Nonstandard methods in stochastic analysis and Mathematical Physics*. Academic Press (1986).

[C] Cutland, N., *Non-standard measure theory and its applications*. Bull. London. Math. Society 15 (1983).

[SB] Stroyan, K. and Bayod, J., *Foundations of Infinitesimal stochastic Analysis*. North-Holland. (1986).

[S] Stroyan, K., *Previsible sets for hyperfinite filtrations*. Probability theory and Rel. fields (1986).

[B] Billingsley, P., *Probability and measure*. Wiley (1980).

[A] Aldous, D., *Weak convergence and the general theory of processes*. Unpublished manuscript. (1980).

[BC] Baxter, J. and Chacon, R., *Compactness of stopping times*. ZWVG. 40 (1977).

[E] Elliot, R.J., *Stochastic Calculus and Applications*. Springer. (1982).

[HeK] Henson, C.W. and Keisler, H.J., *The strength of nonstandard Analysis*. Journal of Symbolic Logic (1986).

*

Departamento de Matemáticas
Universidad de los Andes
Apartado Aéreo 4976
Bogotá, Colombia.