Revista Colombiana de Matemáticas
Vol. XXII (1988), págs. 29-36

FIXED POINT THEOREMS ON CERTAIN CLASS OF CONTRACTIVE MAPPINGS

by

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Abstract. The objective of this paper is to introduce the concepts of a H-contractive mapping and a r-ultimately contractive mapping on cartesian product of two metric spaces, and to show that, under certain restrictions, such mappings have a fixed point.

§1. Introduction. According to Nadler [2], on their own words, a number of mathematicians have investigated the problem of determining what kinds of mappings defined on the cartesian product of two spaces have fixed points (for an historical survey see [6]).

In the present paper we introduce the concepts of H-contrative mapping and r-ultimately contractive mapping and show that, with certain restrictions, such mappings have fixed points on the cartesian product of two metric spaces. Throughout this paper (X,d_1) and (Y,d_2) will denote metric spaces and P_1 and P_2 will be the natural projection of cartesian product $X\times Y$ onto X and Y, respectively. Further, "$\mathcal{D}$" will denote the metric product on $X\times Y$ given by

$$\mathcal{D}((x,y),(\hat{x},\hat{y})) = (d_1^2(x,\hat{x})+d_2^2(y,\hat{y}))^{\frac{1}{2}} \text{ for all } (x,y),(\hat{x},\hat{y}) \text{ in } X\times Y$$

For the sake of completeness, we give the following definitions.

DEFINITION 1. Let (X,d_1) and (Y,d_2) be metric spaces and let f be a mapping from $X\times Y$ into $X\times Y$. We say that f is a *H-contrative mapping on* $X\times Y$ if there exists a function $H:\mathbf{R}_+ \to \mathbf{R}_+$ such that $H(0) = 0$ and

(I) $D(f(x,y),f(\hat{x},\hat{y})) \leqslant d_1(x,\hat{x})+H(d_2(y,\hat{y}))$ for all $(x,y),(\hat{x},\hat{y})$ in $X\times Y$.

DEFINITION 2. [5]. A mapping $h:X \to Y$ is said to be a *nonexpansive mapping* if and only if $d_2(h(x),h(x)) \leqslant d_1(x,\hat{x})$ for all $x,\hat{x}$ in X.

DEFINITION 3. [4]. A subset K of a metric space (M,d) is said to have *the fixed point property for nonexpansive mappings* if, whenever $h:K \to K$ is nonexpansive mapping, then h has a fixed point in K.

REMARK 1. There are interesting results about the fixed point property for nonexpansive mapping (see, for example, [1],[3] or [4]).

§2. Main results.

First we shall consider the proof of the following fixed point theorem for H-contractive mapping:

THEOREM 1. *Let (X,d_1) be a metric space with the fixed point property for nonexpansive self-maps. Let (Y,d_2) be a metric space and suppose that there exists a function θ mapping Y in the nonnegative real numbers, such that $\theta(y) \geqslant \theta(y_0)$ for some $y_0 \in Y$ and all $y \in Y$. Suppose also that $f:X\times Y \to X\times Y$ is a H-contractive mapping satisfying the following condition:*

(i) *If $\bar{x} \in X$, $y_1,y_2 \in Y$ with $y_1 \neq y_2$ and $f(\bar{x},y_1) = (\bar{x},y_2)$, then $\theta(y_2) < \theta(y_1)$.*

Then f has a fixed point in $X\times Y$.

Proof. If $y \in Y$, then let $f_y:X \to X$ be defined by

$f_y(x) = P_1 \circ f(x,y)$ for all x in X. We clain that f_y is nonexpansive mapping for each y in Y. In fact, let y be an arbitrary element of Y and let x and $\hat{x}$ be any two points of X. Since P_1 is a nonexpansive mapping and by assumption f is H-contractive we conclude that

$$d_1(f_y(x), f_y(\hat{x})) = d_1(P_1 \circ f(x,y), P_1 \circ f(\hat{x},y))$$

$$\leq D(f(x,y), f(\hat{x},y)) \leq d_1(x,\hat{x}) + H(d(y,y)) = d_1(x,\hat{x}),$$

so that f_y is indeed a nonexpansive mapping. Since (X,d_1) has the fixed point property for nonexpansive self maps, we conclude that given $y \in Y$, there exists x in X such that $f_y(x) = x$. We now define a multi-valued mapping on Y as follows: For y in Y we define $F(y) = \{x \in X: f_y(x) = x\}$. Let us define other multi-valued mapping on Y as follows:

$$G(y) = \{P_2 \circ f(x,y): x \in F(y)\} \text{ for all } y \text{ in } Y.$$

By assumption (Y,d_2) is a metric space and $\theta: Y \to \mathbf{R}_+$ a function such that $\theta(y) \geq \theta(y_o)$ for some y_o in Y and all y in Y. We can observe that $G(y)$ is not empty for all y in Y, once the same occur with $F(y)$. Consequently, there exists y_2 in $G(y_o)$ and $\bar{x}$ in $F(y_o)$ such that $y_2 = P_2 \circ f(\bar{x},y_o)$. But $\bar{x} \in F(y_o)$ implies that $\bar{x} = f_{y_o}(\bar{x}) = P_1 \circ f(\bar{x},y_o)$ and so we conclude that $f(\bar{x},y_o) = (\bar{x},y_2)$. We can observe that, by the assumption above, we have $\theta(y_2) \geq \theta(y_o)$. Now, let's suppose that $y_o \neq y_2$. Then by condition (i) of the hypothesis we obtain $\theta(y_2) < \theta(y_o)$. This however leaves to a contradiction. Hence $y_o = y_2$ and since $f(\bar{x},y_o) = (\bar{x},y_o)$ we obtain the affirmation of the theorem. Q.E.D.

The following examples serve to illustrate Theorem 1.

a) Let $m \in \mathbf{R}$, $p \in \mathbf{R}$ with $0 < m < p$ and let n be an integer number $n \geq 2$. We consider $X = Y = I^n$, where I is the closed interval $[m,p]$ and we suppose $f: X \times Y \to X \times Y$ be the mapping defined by

$f(x,y) = (y_0, x)$ for all (x,y) in $X \times Y$, where

$$y_0 = (m,m,\ldots,m) \in X.$$

Let $\theta : Y \to \mathbf{R}_+$ be defined by $\theta(y) = \max_n \{y_1, y_2, \ldots, y_n\}$ for all $y = (y_1, y_2, \ldots, y_n)$. Since I^n is bounded closed convex subset of a Hilbert space, we conclude by Browder's theorem [3] that $X = I^n$ has the fixed point property for nonexpansive mappings. Furthermore the above function θ has a minimum value on the compact I^n, which is given by $\theta(y_0) = m$. We have also that the condition (i) of hypothesis is satisfied. In fact, let's suppose that $f(x,y) = (x,\hat{y})$ where $x = (x_1, x_2, \ldots, x_n) \in I^n$, $y = (y_1, y_2, \ldots, y_n) \in I^n$, with $y \neq \hat{y} = (\hat{y}_1, \ldots, \hat{y}_n)$. Then by definition of f we get $y_0 = x = \hat{y} \neq y$. Hence, there exists an integer number , $1 \leqslant j \leqslant n$, such that $\hat{y}_j \neq y_j$. But $y_0 = \hat{y}$ implies that $\hat{y}_j = m$ and so $m \neq y_j$. Thus, since $m \leqslant y_j \leqslant p$ we have $m < y_j \leqslant p$. On the other hand, $\theta(\hat{y}) = \theta(y_0) = m$ and $\theta(y) \geqslant y_j$. Consequently we obtain $\theta(\hat{y}) < \theta(y)$. We observe also that f is a H-contractive mapping on $X \times Y$ since for all functions $H : \mathbf{R}_+ \to \mathbf{R}_+$ satisfying $H(0) = 0$, we have $D(f(x,y), f(\hat{x},\hat{y})) \leqslant d_1(x,\hat{x}) + H(d_2(y,\hat{y}))$ for all (x,y) and $(\hat{x},\hat{y})$ in $X \times Y$. Therefore we have showed that all assumptions of Theorem 1 are fulfilled. Finally, we observe that $P = (y_0, y_0)$ is the unique fixed point of f.

b) Let (X, d_1) be an arbitrary metric space with the fixed point property for nonexpansive self-maps and let (Y, d_2) be the real interval $[0,1]$ with the usual metric. We define $f : X \times Y \to X \times Y$ by $f(x,y) = (x, ry)$ for some $r \in \mathbf{R}$, $0 < r < 1$ and all (x,y) in $X \times Y$. Putting $H(t) = r\,t$ for all $t \in \mathbf{R}_+$, it is easy to see that f is H-contractive mapping. We consider $\theta : [0,1] \to \mathbf{R}_+$ an arbitrary increasing and continous function. Then $\theta(0) \leqslant \theta(y)$ for all y in $[0,1]$. Furthermore, we observe that, if $\bar{x} \in X$, $y_1, y_2 \in [0,1]$ with $y_1 \neq y_2$. and $f(\bar{x}, y_1) = (\bar{x}, y_2)$, then $ry_1 = y_2 \neq y_1$ and so $0 < y_2 < y_1$.

Hence, since θ is an increasing function we have $\theta(y_2) < \theta(y_1)$.

Consequently, all assumption of Theorem 1 are fulfilled. We observe also that $(x,0)$ is a fixed point of f for all $x \in X$.

Our next theorem is closely related to the following result of Nadler's ([2],p.584).

PROPOSITION 1. *Let* (X,d_1) *be a complete metric space, let* (Y,d_2) *be a metric space with the fixed point property and let* f *be a function from* $X\times Y$ *into* $X\times Y$.
(1) *if* f *is uniformly continuous on* $X\times Y$ *and a contraction mapping in the first variable then* f *has a fixed point.*

REMARK 2. We recall that a function $f:X\times Y \to X\times Y$ is said to be a *contraction mapping in the first variable* if and only if for each y in Y, there is a real number $a(y)$, $0 \leqslant a(y) < 1$ such that

$$D(f(x,y),f(\hat{x},y)) \leqslant a(y).\ D((x,y),(\hat{x},y)) \quad \text{for all } x,\hat{x} \text{ in } X. \quad [2]$$

Our objective here is to show that the conditions on f in Proposition 1 can be relaxed if we strengthen the condition on Y. Before proceeding to the detailed discussion of our next result, we introduce a special class of H-contractive mappings.

DEFINITION 4. Let (X,d_1) and (Y,d_2) be a metric spaces. A mapping $f:X\times Y \to X\times Y$ is called *r-ultimately contractive* on $X\times Y$ if there is a positive real number $r < 1$ such that for all (x,y), $(\hat{x},\hat{y})$ in $X\times Y$.

$$D(f(x,y),f(\hat{x},\hat{y})) \leqslant r d_1(x,\hat{x})+(1-r)d_2(y,\hat{y}).$$

REMARK 3. Nonexpansive mappings contain all r-ultimately contractive mappings as a proper subclass. Furthermore, if f is a r-ultimately contractive mapping, then f is H-contractive mapping with H given by $H(t) = (1-r)t$ for all $t \in \mathbb{R}_+$.

THEOREM 2. *Let* (X,d_1) *and* (Y,d_2) *be metric spaces and let* $f:X\times Y \to X\times Y$ *be a mapping such that* f *is* r*-ultimately contractive on* $X\times Y$. *Assume that* (X,d_1) *is complete and let* (Y,d_2) *be a space with the fixed point property for nonexpansive mappings. Then* f *has a unique fixed point.*

Proof. If $y \in Y$, then let $f_y:X \to X$ be the mapping defined by $f_y(x) = P_1\circ f(x,y)$ for all $x \in X$.

Since f is r-ultimately contractive mapping on $X\times Y$ we conclude that $D(f(x,y),f(\hat{x},y)) \leqslant rD((\hat{x},y),(x,y))$ for all x, $\hat{x} \in X$ and $y \in Y$. It follows that for each $y \in Y$, f_y is a contraction mapping of X into X. Since (X,d_1) is complete it follows by the wellknown Banach's contraction mapping principle that for each $y \in Y$, the mapping f_y has one and only one fixed point in X.

Let $F:Y \to X$ be given by $F(y)$ is the unique fixed point of f_y. First it will be shown that the mapping F is nonexpansive. Let y_1 and y_2 be arbitrary elements in Y and let $F(y_1) = x_1$ and $F(y_2) = x_2$. Then, $F(y_1) = x_1 = f_{y1}(x_1) = P_1\circ f(x_1,y_1)$ and $F(y_2) = x_2 = f_{y2}(x_2) = P_1\circ f(x_2,y_2)$.

Consequently we have

$$d_1(F(y_1),F(y_2)) = d_1(P_1\circ f(x_1,y_1),P_1\circ f(x_2,y_2)) \leqslant D(f(x_1,y_1),f(x_2,y_2))$$

so that $d_1(F(y_1),F(y_2)) \leqslant rd_1(x_1,x_2)+(1-r)d_2(y_1,y_2)$, because f is r-ultimately contractive. Hence using $x_1 = F(y_1)$ and $x_2 = F(y_2)$ we obtain $d_1F(y_1),F(y_2)) \leqslant d_2(y_1,y_2)$, that is, F is a nonexpansive mapping. Now let $G:Y \to Y$ be the mapping defined by $G(y) = P_2\circ f(F(y),y)$ for each $y \in Y$. Next, it will be shown that G is also a nonexpansive mapping. To this end, we observe that, if y_1 and y_2 are arbitrary elements in Y, then we have

$$d_2(G(y_1),G(y_2)) = d_2(P_2\circ f(F(y_1),y_1),P_2\circ f(F(y_2)y_2))$$

$$\leqslant D(d(F(y_1),y_1),f(F(y_2),y_2))$$

and since f is r-ultimately contractive and F is nonexpansive, we get

$$d_2(G(y_1),G(y_2)) \leqslant r d_1(F(y_1),F(y_2))+(1-r)d_2(y_1,y_2)$$

$$\leqslant r d_1(y_1,y_2)+(1-r)d_2(y_1,y_2) = d_1(y_1,y_2),$$

that is, G is nonexpansive. Consequently, since Y has the fixed point property for nonexpansive mapping, there is a point W in Y such that $G(W) = W = P_2 \circ f(F(W),W)$. But $f_W(F(W)) = F(W) = P_1 \circ f(F(W),W)$ and so it follows that $f(P) = P$, where $P = (F(W),W)$.

Now we shall complete the proof demonstrating that the fixed point of f is unique. Let $P = (x,y)$ and $Q = (\hat{x},\hat{y})$ be two fixed points of f. Since f is r-ultimately contractive, $d_1(x,\hat{x}) = d_2(y,\hat{y})$ and $\sqrt{2}d_1(x,\hat{x}) \leqslant d_1(x,\hat{x})$. Consequently, if $P \neq Q$, the, we have a contradiction. This completes the proof of the theorem. Q.E.D.

In the following examples all assumptions of Theorem 2 are fulfilled.

EXAMPLE 1. Let X be a Hilbert space and Y a convex closed and bounded subset of X, with $0 \in Y$. Let $S(x) = r \cdot x$ for some $r \in \mathbf{R}$, $0 < r < 1$ and all $x \in X$. Let $T(y) = (1-r)y$ for all $y \in Y$. Putting $f(x,y) = (S(x),T(y))$ for all $(x,y) \in X \times Y$ one can easily show that f is r-ultimately contractive. We observe that $P = (0,0)$ is the unique fixed point of f.

EXAMPLE 2. Let X be a nonempty convex, closed and bounded subset of H, where H is a real Hilbert space. Let $r \in \mathbf{R}$, $0 < r < 1$ and let u be a fixed element of X. Let $f(x,y) = (u, rx+(1-r)y)$ for all $(x,y) \in X \times X$.

It is easy to see that f is r-ultimately contractive and $P = (u,u)$ is the unique fixed point of f.

Finally we observe that in Theorem 1, if f is a r-ultimately contractive mapping, then the fixed point of f is unique.

Acknowledgement. I wish to thank the Referee for his helpful comments.

I also wish to thank for its hospitality the Department of Mathematics of the "University of California, Santa Barbara", where a part of the present paper was completed.

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(Recibido en Agosto de 1985).