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MULTIPLICITY THEORY OF PROJECTIONS IN ABELIAN VON NEUMANN ALGEBRAS

by

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Abstract. The spectral multiplicity theory is generalized for projections in an arbitrary abelian von Neumann algebra. The well known type I_n direct sum decomposition of a type I von Neumann algebra follows as a consequence.

Resumen. La teoría de multiplicidad espectral se generaliza para las proyecciones en un álgebra conmutativa de von Neumann. La bien conocida descomposición de un álgebra de tipo I de von Neumann en una suma directa de álgebras de tipo I_n se deduce como una consecuencia.

§1. Introduction. Suppose (X,S) is a measure space with S a σ-algebra. If $E(\cdot)$ is a spectral measure on S with values in projections on a Hilbert space H, let W be the commutant of the range of $E(\cdot)$ in $B(H)$. Then W is a von Neumann algebra on H with W', the commutant of W, being abelian. In [2] Halmos develops a (spectral) multiplicity theory for projections in W' (§54-64), making use of the countable additivity

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of $E(\cdot)$. In Section 6 we conclude the present article with a discussion of the results of Halmos [2], in the set up of the von Neumann algebra W.

By the results of Stone [5] the maximal ideal space M of an abelian von Neumann algebra A on a Hilbert space H is extremally disconnected. Thus, it is possible to construct a spectral measure $E(\cdot)$ on the Borel subsets of M such that the range of $E(\cdot)$ coincides with the Boolean algbera of all projections in A (vide pp.158-160,[3]). Consequently, by means of $E(\cdot)$ thus constructed, the spectral multiplicity theory developed in Halmos [2] can also be interpreted as a multiplicity theory for projections in A. The object of the present work is to give a direct alternate method of study for the multiplicity theory of projections in an arbitrary abelian von Neumann algebra and thus deduce the results of Halmos in §54-64 of [2] as a particular case. Besides, the present study offers a new proof of the well known type I_n direct sum decomposition of a type I von Neumann algebra.

§2. **Preliminaries.** This section is devoted to fix the terminology and notation to be followed in the sequel, and to list a few definitions and results from the theory of von Neumann algebras, which we repeatedly-use in the body of this article.

H will denote a Hilbert space, W a von Neumann algebra on H with abelian commutant W', R an arbitrary von Neumann algebra on H with centre Z, and I the identity operator on H. If M is a nonempty subset in H, $[M]$ denotes the closed linear subspace spanned by M. If E is a projection, E is also denoted by its range and vice-versa. Thus for a vector $x \in H$, $[Rx]$ is the projection with range given by $[Tx : T \in R]$, that is, the cyclic projection generated by x under R. Clearly, $[Rx] \in R'$ and $[R'x] \in R$.

Let $A \in R$. If

$$F = \{Q \in Z : Q \text{ a projection}, \quad QA = A\},$$

then $C_A = \bigwedge\{Q : Q \in F\}$ belongs to Z and it is a proyection. C_A is called the *central support* of A. For C_A it is true that $C_A A = A$.

PROPOSITION 2.1. *If A is an operator in R and Q a central projection of R, then $C_{QA} = QC_A$.*

DEFINITION 2.2. For two projections E and F in R we say $E \sim F$ if there exists a partial isometry V in R such that $V^*V = E$ and $VV^* = F$. We say $E \preccurlyeq F$ if there exists a subprojection E_1 of F in R such that $E \sim E_1$, and $E \prec F$ if $E \preccurlyeq F$ and $E \not\sim F$.

DEFINITION 2.3. A projection E in R is said to be *finite* if there exists no subprojection E_1 in R of E such that $E \sim E_1 < E$. Otherwise E is called *infinite*.

DEFINITION 2.4. A projection E in R is said to be *abelian* if ERE is an abelian algebra.

PROPOSITION 2.5. *If E and F are projections in R, then the following assertions hold:*

(i) *If E is finite and $E \sim F$, then F is also finite.*

(ii) *E is abelian in R if and only if every subprojection F of E in R is of the form $F = C_F E$.*

(iii) *If E and F are abelian in R and $C_E = C_F$, then $E \sim F$.*

(iv) *If $E \preccurlyeq F$, then $C_E \leqslant C_F$; if $E \sim F$, then $C_E = C_F$.*

(v) *If E is abelian in R, then E is finite.*

Proof. (i) Trivial.

(ii) Use the observation that E is abelian in R if and only if $ERE = ZE$.

(iii) An immediate consequence of Lemma 1, §3, Chapter III of Dixmier [1].

(iv) Vide Corollary 1 to Proposition 7, §1, Chapter I and Proposition 1, §1, Chapter III of Dixmier [1].

(v) If $E \sim E_1 < E$, with E_1 a projective in R, then by (iv), $C_E = C_{E_1}$. But, on the other hand, by (ii), $E_1 = C_{E_1} E = E$.

This is a contradiction; hence, E is finite.

PROPOSITION 2.6. (i) *If* $\{E_\alpha\}$ *is an orthogonal family of projections (respectively,* (ii) *abelian projections) in* R, *with the property that* $\{C_{E_\alpha}\}$ *is an orthogonal family, and if* $E = \sum E_\alpha$, *then* $C_E = \sum C_{E_\alpha}$ *(respectively, and* E *is abelian in* R*).*

Proof. If $Q = C_{E_\alpha}$, then Q is a central projection of R and $Q \geqslant C_{E_\alpha} \geqslant E_\alpha$ for all α. Thus $Q \geqslant E$ and hence, $Q \geqslant C_E$. Since $C_E \geqslant E \geqslant E_\alpha$, $C_E \geqslant C_{E_\alpha}$ for all α and hence $C_E \geqslant Q$. Thus $Q = C_E$. In consequence,

$$E\,R\,E = \sum \oplus C_{E_\alpha} E\,R\,E\; C_{E_\alpha} = \sum \oplus E_\alpha\, R\, E_\alpha.$$

If each E_α is abelian, then $E\,R\,E$ is abelian and hence E is abelian. (Vide 2, §2, Chapter I of Dixmier [1], where Π is used instead of $\sum\oplus$).

THEOREM 2.7. *Suppose* E, F *are non-zero projections in* R, F *being finite. If* $(F_\alpha)_{\alpha\in A}$ *and* $(G_\beta)_{\beta\in B}$ *are two maximal orthogonal families of projections in* R *such that* $F \sim F_\alpha \leqslant E$ *and* $F \sim G_\beta \leqslant E$ *for all* $\alpha \in A$ *and* $\beta \in A$, *then* Card.A = Card.B.

Proof. Vide Ringrose [4] where this theorem is known as theorem of generalized invariance of dimension.

§3. Some lemmas.

As mentioned in §2, W will denote a von Neumann algebra on H with abelian commutant W'. In this section we shall give two lemmas which generalize, respectively, §60 Theorem 2, §61 Theorem 3, of Halmos [2] for projections in W.

LEMMA 3.1. *For each vector* x *in* H *the cyclic projection* $E = [W'x]$ *is abelian in* W.

Proof. Suppose x is a non-zero vector in H. If $E = [W'x]$, then $W'E$ is an abelian von Neumann algebra on $E(H)$ and has a generating vector x. Then from Corollary 2 to Proposition 4, §6, Chapter I, of Dixmier [1], it follows that

$(W'E)' = W'E$ and hence $(W'E)'$ is abelian. But by Proposition 1(i), §2, Chapter I of [1], $(W'E)' = EWE$, so that EWE is abelian and hence E is abelian in W.

LEMMA 3.2. *Given a projection E in W, there exists an abelian projection $F \leqslant E$ in W such that $C_F = C_E$.*

Proof. Let E be a non-zero projection in W. By lemma 3.1 and Zorn's lemma there exists a maximal orthogonal family $\{E_\alpha\}$ of non-zero abelian subprojections of E in W such that $\{C_{E_\alpha}\}$ is an orthogonal family. Then $E \leqslant \sum C_{E_\alpha}$. On the contrary, there would exist a non-zero vector x in the range of E such that $C_{E_\alpha}x = 0$ for all α, then for $T' \in W'$ and for all α, $C_{E_\alpha}T'x = T'C_{E_\alpha}x = 0$ and hence $C_{E_\alpha}[W'x] = 0$; by Proposition 2.1, $C_{[W'x]}C_{E_\alpha} = C_{[W'x]C_{E_\alpha}} = 0$ for all α. As $[W'x]$ is abelian in W by Lemma 3.1, this would contradict the maximality of $\{E_\alpha\}$. As $E_\alpha \leqslant E$, then $C_{E_\alpha} \leqslant C_E$ for all α and thus $C_E = \sum C_{E_\alpha}$. If $F = \sum E_\alpha$, then by Proposition 2.6(ii), F is abelian in W, and $C_F = \sum C_{E_\alpha} = C_E$.

§4. Multiplicity of projections.

Making use of the lemmas in §3 and Theorem 2.7, we shall associate with each projection E' in W' a well defined cardinal number $u(E')$ called *the multiplicity of* E', and prove that $u(\vee E'_j) = \min\{u(E'_j)\}$, whenever $\{E'_j\}$ is an orthogonal family of non-zero projections in W'.

LEMMA 4.1. *Let E' be a projection in W'. Then there exists a maximal orthogonal family $\{E_\alpha\}$ of abelian projections in W such that $C_{E_\alpha} = E'$ for all α.*

Proof. Suppose E' is a non-zero projection in W'. By Lemma 3.2 there exists an abelian projection F in W such that $C_F = E'$. Then the collection of all orthogonal families of abelian projections in W, such that each of their members has central support E', is a non-empty partially ordered set under set inclusion, and every linearly ordered set in this collection has an upper bound which corresponds to the union of all the members of the family. Appealing to Zorn's lemma we obtain the result.

PROPOSITION 4.2. *Let E' be a projection in W'. Then for two maximal orthogonal families $(E_\alpha)_{\alpha \in J}$ and $(F_\beta)_{\beta \in K}$ of abelian projections in W such that $C_{E_\alpha} = C_{F_\beta} = E'$ for all $\alpha \in J$ and $\beta \in K$,* Card.J = Card.K.

Proof. By Proposition 2.5(iii), $E_\alpha \sim F_\beta$ for all $\alpha \in J$ and $\beta \in K$. Besides, by Proposition 2.5(v), they are all finite projections. The assertion follows from Theorem 2.7.

By virtue of the above proposition, we are justified in giving the following definition.

DEFINITION 4.3. A non-zero projection E' in W' is said to have *multiplicity* n if there exists a maximal orthogonal family of n abelian projections $\{E_i\}$ in W, with $C_{E_i} = E'$ for each i. The multiplicity of the zero projection is defined to be zero.

Let B denote the collection of all projections in W'. Since W' is abelian and strongly closed, B is a complete Boolean algebra of projections on H.

THEOREM 4.4. *Let u be the function associating each projection in W' with its multiplicity and $N = \{n : n$ a cardinal number $\leqslant$ dim$H\}$. Then:*

(i) *u is well defined, the domain of u is B and the range of u is a subset of N.*

(ii) *If E', F' are members of B and $0 < E' \leqslant F'$, then $u(E') \geqslant u(F')$.*

(iii) *If $\{F_j'\}$ is an orthogonal family of non-zero members in B and if $F' = \vee F_j'$, then $u(F') = \min\{u(F_j')\}$.*

Proof. (i) Immediate from Lema 4.1, Proposition 4.2, and Definition 4.3.

(ii) If $\{F_j\}$ is a maximal orthogonal family of abelian projections in W such that $C_{F_j} = F'$, then by Proposition 2.1,

$C_{F_jE'} = E'$, and F_jE' is abelian for all j. Hence the conclusion (ii).

(iii) Since $\mathcal{B}$ is a complete Boolean algebra of projections, F' belongs to $\mathcal{B}$. Let $\min\{u(F'_j)\} = n$. By (ii) it is obvious that $u(F') \leqslant n$. To prove the reverse inequality, let I_0 be a set with Card.$I_0 = n$. As $u(F'_j) \geqslant n$, for each j there exists an ortogonal family $\{F_{jk}\}_{k\in I_0}$ of abelian projections in W, with $C_{F_{jk}} = F'_j$ for all $k \in I_0$. Since $F'_jF'_{j'} = 0$ for $j \neq j'$, it follows from Proposition 2.6 that $G_k = \sum_j F_{jk}$ is abelian in W with $C_{G_k} = F'$, for all $k \in I_0$. In other words, $u(F') \geqslant n$, and hence the conclusion (iii).

§5. Uniform multiplicity of projections. In this section the concept of uniform multiplicity for projections in W' is introduced. We give a theorem characterizing those projections in W' which have uniform multiplicity, and deduce the well known type I_n decomposition of a type I von Neumann algebra as a corollary to another theorem on uniform multiplicity (vide Theorems 5.4, 5.6 and Corollary 5.7).

DEFINITION 5.1. A non-zero projection E' in W' is said to have *uniform multiplicity* n if every non-zero projection $F' \leqslant E'$ in W' has multiplicity n.

DEFINITION 5.2. Let R be a von Neumann algebra on H. An orthogonal family $\{E_\alpha\}$ of projections in R is said to be a *complete orthogonal system*, for a central projection Q of R if $\sum E_\alpha = Q$.

LEMMA 5.3. *Let E' be a non-zero projection in W' and $\{E_\alpha\}$ a maximal orthogonal family of abelian projections in W such that $C_{E_\alpha} = E'$ for all α. Then there exists a non-zero vector x in H such that $\{C_{[W'x]}E_\alpha\}$ is a complete orthogonal system of abelian projections in W, for $C_{[W'x]}$, and the central support of $C_{[W'x]}E_\alpha$ is $C_{[W'x]}$ for each α.*

Proof. If $E_0 = \sum E_\alpha$, and $F = E'-E_0$, then by the maximality of $\{E_\alpha\}$ and by Lemma 3.2, $G' = E'-C_F \neq 0$. Then $G'C_F = 0$, and hence by Proposition 2.1, $G'F = 0$. Consequently, $G' = G'E' = G'E_0 = \sum G'E_\alpha$, $\{G'E_\alpha\}$ is an orthogonal family of abelian projections in W, and $C_{G'E_\alpha} = G'C_{E_\alpha} = G'E' = G'$ by Proposition 2.1. Now let x be a non-zero vector in the range of G'. Then as W' is abelian, and $G' \in W'$, it follows that $[W'x] \leqslant G'$. Evidently $\{C_{[W'x]}E_\alpha\}$ is a complete system of abelian projections in W for $C_{[W'x]}$. Besides, $C_{E_\alpha}C_{[W'x]} = C_{[W'x]}C_{E_\alpha} = C_{[W'x]}E' = C_{[W'x]}$, since $C_{[W'x]} \leqslant G' \leqslant E'$.

The following theorem gives a necessary and sufficient condition for a non-zero projection E' in W' to have uniform multiplicity.

THEOREM 5.4. *A non-zero projection E' in W' has uniform multiplicity n if and only if there exists a complete orthogonal system $\{E_i\}$ of n abelian projections in W for E' with $C_{E_i} = E'$ for all i. Thus E' has uniform multiplicity n if and only if $E'W'$ is a von Neumann algebra of type I_n.*

Proof. Suppose there exists an orthogonal family $\{E_\alpha\}$ of abelian projections in W such that $C_{E_\alpha} = E'$, for all α, and such that $\sum E_\alpha = E'$. Let F' be a non-zero subprojection of E' in W'. As $C_{F'E_\alpha} = F$, then $F'E_\alpha \neq 0$ for each α. Evidently $\{F'E_\alpha\}$ is a complete orthogonal system of abelian proyections in W for F'. Since a complete orthogonal system of projections for F' is also a maximal orthogonal system of projections for F', and since $C_{F'E_\alpha} = F'C_{E_\alpha} = F'E' = F'$ for all α, by Proposition 2.1, it follows that $u(F') = n$. Thus the condition is sufficient.

Conversely, suppose E' has uniform multiplicity n. By Lemma 5.3 and Zorn's lemma there exists a maximal orthogonal family $\{F'_j\}_{j\in J}$ of non-zero subprojections of E' in W such that each F'_j has a complete orthogonal system $\{F_{jk}\}_{k\in I_0}$ of abelian projections in W, with $C_{F_{jk}} = F'_j$ for all $k \in I_0$, where Card.$I_0 = n$. In consequence, $\sum_{k\in I_0} F_{jk} = F'_j$, $j \in J$. Then by the maximimality of $\{F'_j\}_{j\in J}$ and by Lemma 5.3 we conclude

$\sum_{j\in J} F'_j = E'$. Let $G_k = \sum_{j\in J} F_{jk}$. Then by Proposition 2.6 G_k is abelian in W, and $C_{G_k} = \sum_{j\in J} F'_j = E'$ for all $k \in I_0$. Besides,

$$\sum_{k\in I_0} G_k = \sum_{k\in I_0} \sum_{j\in J} F_{jk} = \sum_{j\in J} \sum_{k\in I_0} F_{jk} = \sum_{j\in J} F'_j = E'.$$

Hence, the necessity of the condition in the first part of the theorem follows.

The second part of the theorem is an immediate consequence of the first if one appeals to Proposition 2.5(iv) and the definition of type I_n von Neumann algebras in [1]. This completes the proof of the theorem.

PROPOSITION 5.5. *Suppose* $\{E'_j\}$ *is an orthogonal family in* B *such that each* E'_j *has uniform multiplicity* n. *If* $E' = \sum E'_j$, *then* $E' \in B$ *and* E' *has uniform multiplicity* n.

Proof. As B is a complete Boolean algebra of projections and $E' = VE'_j$, it follows that $E' \in B$. Let E'_0 be a non-zero subprojection of E' in B. Then $E'_0 = VE'_0E'_j$, where we consider only those indices j for wich $E'_0E'_j \neq 0$. By hypothesis, for each of such j, $u(E'_0E'_j) = n$. The proposition is an immediate consequence of Theorem 4.4.(iii).

THEOREM 5.6. *Suppose, for each cardinal number* n *not exceeding the dimension of* H, P'_n *is the supremum of all those projections in* W' *which have uniform multiplicity* n. *Then* $\{P'_n\}$ *is an orthogonal family of projections in* W', $\sum P'_n = I$, *and for each* n, *either* $P'_n = 0$ *or* P'_n *has uniform multiplicity* n.

Proof. For a fixed cardinal number $n \leqslant \dim H$, let $\{E'_\alpha\}$ be a maximal orthogonal family of projections in W' such that each E'_α has uniform multiplicity n. If no such family exists we take the supremum P'_n to be the zero projection. If $P'_n - VE'_\alpha > 0$, then there exists some non-zero projection F' in W' such that $0 < F' \leqslant P'_n - VE'_\alpha$, and such that F' has uniform multiplicity n. Since this contradicts the maximality of $\{E'_\alpha\}$, we conclude $P'_n = VE'_\alpha$. Thus, if $P'_n \neq 0$, then by Proposition 5.5, P'_n has uniform multiplicity n. In consequence for $n \neq m$ with $P'_n P'_m \neq 0$, we have n = the multiplicity of P'_n = the multiplicity of $P'_n P'_m$ = the multiplicity of $P'_m = m$. This contradiction establishes that $\{P'_n\}$ is an orthogonal family. Finally, $\sum P'_n = I$, by Lemmas 4.1 and 5.3. This completes the proof of the theorem.

COROLLARY 5.7. *Let R be a type I von Neumann algebra on H. Then $R = \sum_{i \in I_0} \oplus R_i$, where the R_i's are von Neumann algebra of type I_{n_i} with $n_i \neq n_{i'}$, for $i \neq i'$ in I_0.*

Proof. By the definition of a type I von Neumann algebra, there exists a von Neumann algebra W, with W' abelian, on a Hilbert space H', such that R and W are *-isomorphic. Let ϕ be a *-isomorphism from R onto W. Let $I_0 = \{n_i : n_i \leq \dim H'$. with $P'_{n_i} \neq 0\}$ in Theorem 5.6 with respect to W. Then $W = \sum_{i \in I_0} \oplus WP'_{n_i}$ and by Theorem 5.4, WP'_{n_i} is of type I_{n_i} for each $i \in I_0$. As a consequence,

$$R = \phi^{-1} W = \sum_{i \in I_0} \oplus \phi^{-1}(WP'_{n_i}) = \sum_{i \in I_0} \oplus RQ_{n_i},$$

where $Q_{n_i} = \phi^{-1}(P'_{n_i})$ is a central projection of R and each RQ_{n_i} is a von Neumann algebra of type I_{n_i}.

§6. Concluding remarks. Suppose (X,S) is a measure space with S a σ-algebra, and $E(\cdot)$ a spectral measure on S with values in projections on H. In [2] Halmos uses the following terminology and notation:

$\mathbf{E} = \{E(M) : M \in S\}$

$\mathbf{P} = \{F : FE(M) = E(M)F$ for all $M \in S$, F a projection$\}$

$\mathbf{F} = \{G : GP = PG$ for all $P \in \mathbf{P}$, G a projection$\}$

For $P \in \mathbf{P}$, the *column generated by* P, in symbols $C(P)$, is defined as $C(P) = \wedge\{F : P \leq F \in \mathbf{F}\}$. A *row projection* R *in* $\mathbf{P}$ is one such that if $R \geq P \in \mathbf{P}$, then $P = C(P)R$. For a vector x in H the *cycle generated by* x, in symbols $Z(x)$, is the projection $[E(M)x : M \in S]$, and it is seen that $Z(x) \in \mathbf{P}$.

The above definitions of rows, cycles and generated columns can be related to abelian projections, cyclic projections, and central supports, respectively, if we define the von Neumann algebra W suitably. In fact, let W be the von Neumann algebra of all operators T in $B(H)$ which commute with the members of **P**. Then the commutant W' of W is the von Neumann algebra generated by **F**, and **F** is the Boolean algebra of all projections in W'. Thus for a projections $P \in \mathbf{P}$ the column $C(P)$ generated by P coincides with the central support C_P, since the centre of W is W'. By Proposition 2.5(ii), a row $P \in \mathbf{P}$ is nothing else but an abelian projection in W. Finally, the cycle $Z(x)$ is the cyclic projection $[W'x]$, since W' is closed in the strong topology of operators. Consequently, $Z(x) \in W'' = W$ and hence $Z(x)$ belongs to **P**.

As a consequence of the above remarks, it is evident, as commented in Section 1, that our results subsume those of Halmos in §54-64 of [2].

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