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# THE LIMITED EXISTENCE OF $\alpha$-DERIVATIVES WITH $\alpha \neq 1$

by

David F. FINDLEY

Let $f:[0,1] \to \mathbf{R}$ have the property that

$$f^{\alpha}(x_o) = \lim_{x \downarrow x_o} \frac{f(x)-f(x_o)}{(x-x_o)^{\alpha}} \tag{1}$$

exists at $x_o \in [0,1)$, where $\alpha > 0$ and $\alpha \neq 1$. Moore and Nashed [4] have found interesting applications for a generalization of this *derivative of degree* $\alpha$, as it might be called. We wish to show that such a derivative cannot be non-zero on a set of positive measure. More precisely, let $E \in [0,1)$ be given, and define $f_E^{\alpha}(x_o)$ as in (1), with the additional requirement that $x, x_o \in E$. Our result is:

**THEOREM.** *Suppose* $f_E^{\alpha}(x_o)$ *exists at all points* $x_o \in E$.

(a) *If* $0 < \alpha < 1$, *then* $f_E^{\alpha}(x) = 0$, *a.e. in* $E$.

(b) *Suppose* $\alpha > 1$ *and* $E = O \cap S$, *where* $O$ *is an open set on which* $f$ *is continuous, and* $S$ *has countable complement in* $[0,1)$. *Then* $f_E^{\alpha}(x) = 0$ *for all* $x \in E$.

**Proof.** Observe that

$$\frac{f(x)-f(x_o)}{x-x_o} = (x-x_o)^{\alpha-1}\,\frac{f(x)-f(x_o)}{(x-x_o)^{\alpha}} \qquad (x > x_o). \tag{2}$$

consequently, when $0 < \alpha < 1$, the right derivative $f'_+(x_o)$ is defined and infinite at all points of the set $F = \{x_o : f'_E(x_o) \neq 0\}$. It follows from the generalized Denjoy-Young-Saks Theorem (cf.[2]), that $F$ has Lebesgue measure zero.

If $\alpha > 1$ and $E$ is a subinterval, open from the right, on which $f$ is continuous, we see from (2) that $f'_+(x_o) = 0$ at all $x_o \in E$. A simple argument (cf. (17.23) of [3] , for example) shows that $f$ must be constant on $E$, so that $f^{\alpha}_E \equiv 0$. Since an open set is a countable disjoint union of intervals, our assertion in (b) follows from 7.2.2.1 of [1](p.222).

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*Census Bureau*

*Statistical Research Division*

*Room 3000, FOB #4*

*Washington, D.C. 20233*

*U.S.A.*



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