A SURJECTIVITY RESULT FOR NONLINEAR MAPPINGS ON UNIFORM SPACES AND APPLICATIONS

by

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Summary. The main purpose of the present paper is to establish a surjectivity result for nonlinear continuous mappings on uniform spaces. Then one can prove that every surjective, continuous and expansive map of an uniform space onto itself has a unique fixed point. Moreover, such a map is a global homeomorphism of the uniform space onto itself. As a consequence of the main theorem a generalization of McCord's theorem to locally convex spaces is proved.

The main purpose of the present paper is to establish a surjectivity result for nonlinear continuous mappings on uniform spaces, using the technique of Kasahara [1]. Then one can prove that every surjective, continuous and expansive map of an uniform space onto itself has a unique fixed point. Moreover, such a map is a global homeomorphism of the uniform space onto itself. As a consequence of the main theorem a generalization of McCord's theorem [2] to locally convex spaces is proved.
§1. A surjectivity result. We shall denote by $X$ a sequentially complete Hausdorff uniform spaces whose uniformity is generated by a saturated family of pseudometrics

$$\{\rho_\lambda(x,y) : \lambda \in \Lambda\},$$

and by $Y$ a quasicomplete Hausdorff uniform space with a saturated family of pseudometrics $\{\rho_\mu(x,y) : \mu \in M\}$. (cf. [3], [4]). The uniform space $Y$ is said to be quasicomplete if every bounded closed subset of $Y$ is complete in the induced topology.

If $\alpha \in Y$ and $S \subset Y$ we shall use the standard notation

$$\mathcal{D}(\alpha)(\alpha,S) = \inf\{\rho_\mu(\alpha,x) : x \in S\}$$

Let $\phi: M \times Y \times Y \to M$ be a mapping and $\{\tilde{y}_k\}_{k=0}^\infty$, $\{\tilde{y}_k\}_{k=0}^\infty$ be sequences in $Y$. We shall use the notation

$$\phi_k(\mu,\tilde{y}_k,\tilde{y}_k) = \phi(\phi_{k-1}(\mu,\tilde{y}_{k-1},\tilde{y}_{k-1}),\tilde{y}_k,\tilde{y}_k), \ (k=1,2,\ldots), \ \phi^0(\mu,\tilde{y}_0,\tilde{y}_0) = \mu.$$

Let $(\phi) = \{\phi(\mu)(\cdot) : \mu \in M\}$ be a family of functions $\phi(\mu)(\cdot): \mathbb{R}_+^1 \to \mathbb{R}_+^1$ ($\mathbb{R}_+^1 = [0,\infty)$) with the properties:

1) $\phi(\mu)(t)$ is non-decreasing and continuous from the right with respect to $t$ on $\mathbb{R}_+^1$, for fixed $\mu \in M$.
2) $0 < \phi(\mu)(t) < t$ for all $t > 0$ and $\mu \in M$.
3) For every $\mu \in M$ there exists $\hat{\phi}(\mu)(t) \in (\phi)$ such that

$$\sup \{ \phi(\phi^n(\mu,\tilde{y}_n,\tilde{y}_n))(t) : n = 0,1,2,\ldots \} \leq \hat{\phi}(\mu)(t)$$

where $\hat{\phi}(\mu)(t)/t$ is non-decreasing in $t$.

Assume there is a mapping $k: \Lambda \to M$ which is onto, and let $T: X \to Y$ be a continuous mapping and $F \subset Y$ be a bounded and closed subset of $Y$.

**Theorem 1.** Assume: (1) there is a mapping $\alpha(\lambda): \Lambda \to \mathbb{R}_+^1$ such that for every $x \in X$ there exists $z \in X$ for which the following inequalities hold
\[ \rho_\lambda(x,z) \leq \alpha(\lambda)D(k(\lambda))(Tx,F) \]
for every \( \lambda \in \Lambda, \)

\[ D(\mu)(Tz,F) \leq \Phi(\mu)(D(\phi(\mu,Tx,y))(Tx,F))) \]
for every \( \mu \in M \) and \( y \in F. \)

(2) there exist an element \( x_0 \in X \) and a constant \( \Delta = \Delta(\mu,F) > 0 \) such that

\[ D(\phi^n(\mu,Tx_0,y))(Tx_0,F) \leq \Delta < \infty \]
for every \( \mu \in F \) and \( n = 0,1,2,... \). Then there exists an \( a \in X \) for which \( Ta \in F \), and moreover:

\[ \rho_\lambda(x_0,a) \leq \alpha(\lambda) \sum_{n=0}^{\infty} \phi^n(k(\lambda))(\Delta) < \infty. \]

(Here \( \tilde{\phi}^n(\mu)(t) \) stands for \( n \)-th iterate of \( \tilde{\phi}(\mu)(t) \), as function of \( t \), and \( \tilde{\phi}^0(\mu)(t) = t \).)

**Proof.** By the assumption 1 for every \( x \in X \) the set

\[ S(x) = \{ z \in X : \rho_\lambda(x,z) \leq \alpha(\lambda)D(k(\lambda))(Tx,F) \]
and \( D(k(\lambda))(Tz,F) \leq \Phi(k(\lambda))(D(\phi(k(\lambda),Tx,y))(Tx,F)) \)

for every \( \lambda \in \Lambda \) and \( y \in F \)

is nonempty. Beginning with \( x_0 \in X \) from assumption 2, we define the sequence \( \{x_n\}_{n=0}^{\infty} \) by chosing \( x_1 \in S(x_0), x_2 \in S(x_1),...,x_{n+1} \in S(x_n),... \). Now the sequence is independent on \( \lambda \). Consequently, we have for every \( \lambda \in \Lambda \) and \( y \in F \)

\[ \rho_\lambda(x_n,x_{n+1}) \leq \alpha(\lambda)D(k(\lambda))(Tx_n,F), \]

and

\[ D(k(\lambda))(Tx_{n+1},F) \leq \Phi(k(\lambda))(D(\phi(k(\lambda),Tx_n,y))(Tx_n,F)). \]

Then

\[ \rho_\lambda(x_n,x_{n+1}) \leq \alpha(\lambda)\Phi(k(\lambda))(D(\phi(k(\lambda),Tx_{n-1},y))(Tx_{n-1},F)) \]

\[ \leq \alpha(\lambda)\Phi(k(\lambda))(\Phi(k(\lambda),Tx_{n-1},y))(D(\phi^2(k(\lambda),Tx_{n-2},y))(Tx_{n-2},F))) \]

\[ \leq \ldots \leq \alpha(\lambda)\Phi(k(\lambda))(\Phi(k(\lambda),Tx_{n-1},y)) \]

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\[ \ldots \phi^n(k(\lambda), Tx_1, y) \) (D(\phi^n(k(\lambda), Tx_0, y)) (Tx_0, F)) \ldots \] 
\[ \leq \alpha(\lambda) \Phi^n(k(\lambda))(\delta), \]
and
\[ D(\mu)(Tx_n, F) \leq \Phi(\mu)(D(\phi(\mu, Tx_{n-1}, y))(Tx_{n-1}, F)) \]
\[ \leq \Phi(\mu)(\Phi(\phi(\mu, Tx_{n-1}, y)))(D(\phi^2(\mu, Tx_{n-2}, y))(Tx_{n-2}, F))) \leq \ldots \]
\[ \leq \Phi(\mu)(\Phi(\phi(\mu, Tx_{n-1}, y))\ldots \Phi(kn-1(\mu, Tx_0, y))(Tx_0, F))) \]
\[ \leq \Phi^n(\mu)(\delta). \]

Since
\[ \Phi^n+1(\mu)(\delta)/\Phi^n(\mu)(\delta) \leq \Phi(\mu)(\delta)/\delta < 1 \]
then the serie \[ \sum_{n=0}^{\infty} \Phi^n(\mu)(\delta) \] is convergent and hence
\[ \lim_{n \to \infty} \Phi^n(\mu)(\delta) = \delta. \]

Hence
\[ \rho^1_{\lambda}(x_n, x_{n+m}) \leq \sum_{k=1}^{m} \rho_{\lambda}(x_{n+k-1}, x_{n+k}) \leq \alpha(\lambda) \sum_{k=1}^{m} \Phi^{n+k-1}(\mu)(\delta) \]
and therefore \( \{x_n\}_{n=0}^{\infty} \) is a Cauchy sequence that in view of the sequential completeness of \( X \) tends to the limit \( a \in X \).

The continuity of \( T \) implies \( \lim_{n \to \infty} Tx_n = Ta \) in \( Y \). But \( V \) is quasicomplete and \( F \) is bounded and closed. Therefore \( Ta \in F \).

Finally, we have
\[ \rho^1_{\lambda}(x_0, a) = \rho_{\lambda}(x_0, x_1) + \rho_{\lambda}(x_1, x_2) + \ldots + \rho_{\lambda}(x_{n-1}, x_n) + \rho_{\lambda}(x_n, a) \]
\[ \leq \alpha(\lambda) \sum_{k=0}^{n-1} \Phi^k(\mu)(\delta) + \rho_{\lambda}(x_n, a) \]
and then, when \( n \to \infty \),
\[ \rho_{\lambda}(x_0, a) \leq \alpha(\lambda) \sum_{k=0}^{\infty} \Phi^k(\mu)(\delta) < \infty \]

Theorem 1 is thus proved. ▲

Theorem 2 is a consequence of Theorem 1. Here the assumption for a quasicompleteness of \( V \) is not necessary.
THEOREM 2. \( T: X \to Y \) be a continuous map. Suppose there are a map \( \alpha(\lambda): \Lambda \to \mathbb{R}_+^1 \) and one element \( x_0 \in X \) such that for every \( x \in X \) there is \( \bar{x} \in X \) for which
\[
\rho(\lambda, x, \bar{x}) \leq \alpha(\lambda) \rho_{k(\lambda)}(T x, y)
\]
and
\[
\tilde{\rho}_{k(\lambda)}(T \bar{x}, y) \leq \Phi(k(\lambda))(\rho_{\Phi}(k(\lambda), T x, y)(T x, y))
\]
for every \( \lambda \in \Lambda \) and \( y \in Y \), and for \( n = 0, 1, 2, \ldots \)
\[
\tilde{\rho}_{\Phi}(n, T x_0, y)(T x_0, y) \leq \Delta < \infty, \text{ where } \Delta = \Delta(\mu, y) > 0,
\]
Then \( T \) is surjective. Moreover, if \( T x = \ell \) then
\[
\rho(\lambda, x_0, \ell) \leq \alpha(\lambda) \sum_{n=0}^{\infty} \Phi(k(\lambda)) \Delta(k(\lambda), \ell) < \infty.
\]

§2. Application 1 - Fixed points. Now we are going to obtain fixed point theorem for surjective continuous and expansive mappings.

We shall consider a uniform space \( X \) with a saturated family of pseudometrics \( \{\rho_\mu(x, y): \mu \in M\} \). Here the mapping \( \phi \) is defined only on \( M \), that is \( \phi:M \to M \).

The mapping \( T:X \to X \) is said to be a \( \Phi \)-expansion on \( X \) if
\[
\rho_\mu(x, y) \leq \Phi(\mu)(\rho_{\Phi}(\mu)(T x, T y))
\]
for every \( x, y \in X \) and \( \mu \in M \).

If, for instance, we choose \( \phi(\mu) = \mu \) and \( \phi(\mu)(t) = k_\mu t \) with \( 0 < k_\mu < 1 \), then
\[
\rho_\mu(T x, T y) \geq (1/k_\mu)\rho_\mu(x, y).
\]

THEOREM 3. Let \( T:X \to X \) be a continuous, surjective and \( \Phi \)-expansive mapping. If there is \( x_0 \in X \) whose inverse image \( T^{-1}x_0 \) contains an element \( x_0 \) such that
\[
0 < \rho_{\Phi}(n, (T x, x_1)) \leq \Delta(\mu) < \infty \quad (n = 0, 1, 2, \ldots)
\]
Then $T$ has at least one fixed point.

**Proof.** Since $T$ is surjective, we may define the sequence $\{x\}$ by the equality $x = x \cdot T + 1$, beginning with $x_0$ satisfying the assumption of the theorem. Setting $p(\cdot) = x_0, x_1, \ldots$, we obtain a sequence and with $c. = p(\cdot)$.

The inequalities show that $\{x\}$ is a Cauchy sequence because $\lim_{n \to \infty} x_n = x \in X$. Then the completeness of $X$ implies $\lim_{n \to \infty} x_n = x$. Thus $x = \lim_{n \to \infty} x_n$, and $x$ is the desired fixed point of $T$.

**Theorem 4.** Let the conditions of Theorem 3 hold true. If, in addition, we suppose that for every fixed points $x$ and $y$ of $T$ the sequence $\{p(\cdot)\} = \{x, y\}$ is bounded, that is, $p(\cdot) = \{x, y\}$ is bounded, then $T$ has a unique fixed point.

**Proof.** The desired conclusion follows immediately from the inequalities

$$
\forall n \geq 0, \quad (x, y) = x + y
$$

show that $\{x\}$ is a Cauchy sequence because

$$
\forall (\forall) \quad (x, y) = x + y
$$

The inequalities

$$
(\forall) \quad (x, y) = x + y
$$

Proof. Since $T$ is surjective, we may define the sequence $\{x\}$ by the assumption of the theorem. Since $T$ is a positive a fixed point $x$ with $x = x \cdot T + 1$, then $x = x \cdot T + 1$, and $x$ is the desired fixed point of $T$. In fact, since $T$ is continuous, the element $x \in X$ is a unique fixed point of $T$.
integer $s$. Then $T$ has a unique fixed point.

Let us redefine the family ($\phi$), using the conditions (1) and (2) and replacing the condition (3) in §1 by the following one:

3') for every $\mu \in M$,

$$\lim_{n \to \infty} \phi(\mu)(\cdots \phi^n(\mu)(t)) = 0, \quad t > 0.$$  

In Theorem 5 the notation of $\phi$-expansion is taken with respect to the redefined family ($\phi$).

**Theorem 5.** Let $T:X \to X$ be continuous, surjective and $\phi$-expansive mapping. If (1) the mapping $\phi:M \to M$ is surjective and $\rho_{\mu}(x_n, x_{n+m}) > \rho_{\phi}(x_n, x_{n+m})$ for some $x_0 \in X$ and every $\mu \in M$, where $x_n = T^m x_0$, and (2) for every two fixed points of $T$, $x$ and $y$,

$$\rho_{\phi}(x, y) \leq Q(\mu, x, y) < \infty \quad (n = 0, 1, 2, \ldots);$$

then $T$ has a unique fixed point.

**Proof.** Let us put $c_\mu^n = \rho_{\mu}(x_n, x_{n+1})$, $p(\mu) = \rho_{\mu}(x_0, x_1)$.

Then

$$c_\mu^n = \rho_{\mu}(x_n, x_{n+1}) \leq \phi(\mu)(\rho_{\phi}(T^n x_n, T x_{n+1}))$$

$$= \phi(\mu)(\rho_{\phi}(T^n x_n, x_{n+1})) \leq \phi(\mu)(\phi(\mu))(\rho_{\phi^2}(x_{n-2}, x_{n-1})) \leq \cdots \leq \phi(\mu)(\phi(\mu))(\cdots \phi(\mu^n(\mu))(\rho_{\phi^n}(x_0, x_1)) \cdots)$$

$$\leq \phi(\mu)(\phi(\mu))(\cdots \phi(\mu^n(\mu))(p(\mu)) \cdots).$$

Therefore $\lim_{n \to \infty} c_\mu^n = 0$ for every $\mu \in M$.

We suppose, by contradiction, that $\{x_n\}_{n=0}^{\infty}$ is not a Cauchy sequence. Then there exists $\varepsilon_0 > 0$ and finite number of pseudometrics $\{\rho_{\mu}\}$ such that for every $\nu$ there is $m = m(\nu) > \nu$ and $p = p(\nu) > 0$ for which $\rho_{\mu}(x_m, x_{m+p}) > \varepsilon_0$.

But $\phi$ is surjective and we can find $\mu \in M$ such that $\mu = \phi(\mu)$.

Moreover, $\rho_{\mu}(x_m, x_{m+p}) > \rho_{\phi}(x_m, x_{m+p}) > \varepsilon_0$. Let $p$ be the
smallest positive integer for which \( \rho_\Phi(\mu)(x_{m+p_+}, x_m) \leq \varepsilon_0 \), it follows that \( \rho_\Phi(\mu)(x_{m+p_+}, x_m) < \varepsilon_0 \). Setting \( a_\Phi(\mu) = \rho_\Phi(\mu)(x_{m+p_+}, x_m) \), we have

\[
\varepsilon_0 \leq a_\Phi(\mu) = \rho_\Phi(\mu)(x_{m+p_+}, x_m) < \varepsilon_0
\]

Passing to the limit in the last inequality when \( \nu \to \infty \) we obtain \( \lim_{\nu \to \infty} a_\Phi(\mu) = \varepsilon_0 \). On the other hand we have:

\[
\varepsilon_0 \leq \rho_\mu(x_{m+p_+}, x_m) \leq \rho_\mu(x_{m+p+1}, x_{m+1}) + \rho_\mu(x_{m+1}, x_m)
\]

which yields for \( \nu \to \infty \), \( \varepsilon_0 \leq \Phi(\varepsilon_0) \). The obtained contradiction shows that \( \{x_n\}_{n=0}^\infty \) is a Cauchy sequence. From there on, the proof can be accomplished as those of Theorem 3 and 3. ▲

**REMARK.** The mapping \( T \) is continuous, surjective and invertible. The inverse mapping \( T^{-1} \) is continuous, because it is a contraction. Therefore \( T \) is a global homeomorphism of \( X \) onto itself. Moreover, if we state \( x_0 = T^{-1} x_0 \), one can obtain a fixed point theorem for \( \Phi \)-contractive mappings, as in [5]. Then by choosing particular values of \( \Phi(\mu) \), we obtain the fixed point theorem of Tarafdar [6], Theorem 11.

§3. Application 2 - McCord's Theorem. Here we establish an extension of McCord's Theorem [2]. Let \( E \) and \( G \) be Hausdorff locally convex topological vector spaces with saturated families of seminorms, \( \{ \cdot \}_{\lambda} : \lambda \in \Lambda \} \) and \( \{ \cdot \}_{\mu} : \mu \in \mathcal{M} \}, \) respectively. Assume there exists an onto \( k : \Lambda \to \mathcal{M} \), and let \( T : E + G \) be a continuous linear mapping.

**THEOREM 6.** Suppose that for each \( y \in G \) with \( \|y\|_\mu = 1 \)
for some $\mu \in \mathcal{M}$, there exists an element $x \in E$ and mappings $\alpha(\lambda) : \Lambda \to \mathbb{R}^1$, $\beta(\mu) : \mathcal{M} \to [0, \beta]$, $0 < \beta < 1$, such that $|x|_\lambda \leq \alpha(\lambda)$ and $|y-Tx|_{k(\lambda)} \leq \beta(k(\lambda))$ for every $\lambda \in \Lambda$. Then $T$ is surjective. If $C = Ta$, then $|a|_\lambda \leq \alpha(\lambda) \sum_{n=0}^{\infty} \beta^n$.

**Proof.** Let $x \in E$ and $y \in G$. One can suppose that $y \neq Tx$, then there exists $\mu_0 \in \mathcal{M}$ such that $|y-Tx|_{\mu_0} \neq 0$, because is a Hausdorff space. Then we can find an element $z \in E$ for which $|z|_\lambda \leq \alpha(\lambda)$ and

$$\left| \frac{y-Tx}{|y-Tx|_{\mu_0}} - Tz \right|_{k(\lambda)} \leq \beta(k(\lambda)).$$

Let us put $\tilde{x} = x + |y-Tx|_{k(\lambda)} z$. We have

$$|\tilde{x}-x|_\lambda \leq |z|_\lambda |y-Tx|_{k(\lambda)} \leq \alpha(\lambda) |y-Tx|_{k(\lambda)}$$

and then

$$T\tilde{x} - y = Tx - y + |y-Tx|_{k(\lambda)} Tz,$$

for every $\lambda \in \Lambda$, and so for every $\mu \in \mathcal{M}$, the following inequality is satisfied:

$$|T\tilde{x}-y|_{k(\lambda)} = |y-Tx|_{\mu_0} - |y-Tx|_{\mu_0} Tz|_{k(\lambda)} \leq \beta(\mu) |y-Tx|_{\mu_0}.$$

In order to obtain the assumptions of Theorem 2 we define the mapping $\phi : G \times G \to \mathcal{M}$ in the following way: for every $\tilde{y}, \bar{y} \in G$, $\phi(\tilde{y}, \bar{y})$ is the index $\mu \in \mathcal{M}$ of the norm for which $|\tilde{y}-\bar{y}|_{\mu} \neq 0$. Then we put $\phi(Tx, y) = \mu_0$. Since $\phi$ does not depend on $\mu$, we have $\Delta(\mu, y) = \rho_{\mu}(T\tilde{0}, y)$ where $\tilde{0}$ is the null element of $E$.

Applying Theorem 2 we obtain the assertion of Theorem 6. △

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