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A SURJECTIVITY RESULT FOR NONLINEAR MAPPINGS ON UNIFORM SPACES AND APPLICATIONS

by

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Summary. The main purpose of the present paper is to establish a surjectivity result for nonlinear continuous mappings on uniform spaces. Then one can prove that every surjective, continuous and expansive map of an uniform space onto itself has a unique fixed point. Moreover, such a map is a global homeomorphism of the uniform space onto itself. As a consequence of the main theorem a generalization of McCord's theorem to locally convex spaces is proved.

The main purpose of the present paper is to establish a surjectivity result for nonlinear continuous mappings on uniform spaces, using the technique of Kasahara [1]. Then one can prove that every surjective, continuous and expansive map of an uniform space onto itself has a unique fixed point. Moreover, such a map is a global homeomorphism of the uniform space onto itself. As a consequence of the main theorem a generalization of McCord's theorem [2] to locally convex spaces is proved.

§1. **A surjectivity result.** We shall denote by $\mathbf{X}$ a sequentially complete Hausdorff uniform spaces whose uniformity is generated by a saturated family of pseudometrics

$$\{\rho_\lambda(x,y) : \lambda \in \Lambda\},$$

and by $\mathbf{Y}$ a quasicomplete Hausdorff uniform space with a saturated family of pseudometrics $\{\bar{\rho}_\mu(x,y) : \mu \in \mathbf{M}\}$. (cf. [3], [4]). The uniform space Y is said to be *quasicomplete* if every bounded closed subset of Y is complete in the induced topology.

If $\alpha \in Y$ and $S \subset Y$ we shall use the standard notation $\mathcal{D}(\mu)(\alpha,S) = \inf\{\bar{\rho}_\mu(\alpha,x) : x \in S\}$

Let $\phi : M\times Y\times Y \to M$ be a mapping and $\{\bar{y}_k\}_{k=0}^{\infty}$, $\{\bar{\bar{y}}_k\}_{k=0}^{\infty}$ be sequences in Y. We shall use the notation

$$\phi^k(\mu,\bar{y}_k,\bar{\bar{y}}_k) = \phi(\phi^{k-1}(\mu,\bar{y}_{n-1},\bar{\bar{y}}_{k-1}),\bar{y}_k,\bar{\bar{y}}_k),\ (k=1,2,\ldots),\ \phi^0(\mu,\bar{y}_0,\bar{\bar{y}}_0) = \mu.$$

Let $(\Phi) = \{\Phi(\mu)(t) : \mu \in M\}$ be a family of functions $\Phi(\mu)(\cdot) : R^1_+ \to R^1_+$ $(R^1_+ = [0,\infty))$ with the properties:

1) $\Phi(\mu)(t)$ is non-decreasing and continuous from the right with respect to t on R^1_+, for fixed $\mu \in M$.
2) $0 < \Phi(\mu)(t) < t$ for all $t > 0$ and $\mu \in M$.
3) For every $\mu \in M$ there exists $\bar{\Phi}(\mu)(t) \in (\Phi)$ such that

$$\sup\{\Phi(\phi^n(\mu,\bar{y}_n,\bar{\bar{y}}_n))(t) : n = 0,1,2,\ldots\} \leqslant \bar{\Phi}(\mu)(t)$$

where $\bar{\Phi}(\mu)(t)/t$ is non-decreasing in t.

Assume there is a mapping $k:\Lambda \to M$ which is onto, and let $T:X \to Y$ be a continuous mapping and $F \subset Y$ be a bounded and closed subset of Y.

THEOREM 1. *Assume:* (1) *there is a mapping* $\alpha(\lambda):\Lambda \to R^1_+$ *such that for every* $x \in X$ *there exists* $z \in X$ *for which the following inequalities hold*

$$\rho_\lambda(x,z) \leqslant \alpha(\lambda)\mathcal{D}(k(\lambda))(Tx,F)$$

for every $\lambda \in \Lambda$,

$$\mathcal{D}(\mu)(Tz,F) \leqslant \Phi(\mu)(\mathcal{D}(\phi(\mu,Tx,y))(Tx,F))$$

for every $\mu \in M$ *and* $y \in F$. (2) *there exist an element* $x_0 \in X$ *and a constant* $\Delta = \Delta(\mu,F) > 0$ *such that*

$$\mathcal{D}(\phi^n(\mu,Tx_0,y))(Tx_0,F) \leqslant \Delta < \infty$$

for every $y \in F$ *and* $n = 0,1,2,\ldots$. *Then there exists an* $a \in X$ *for which* $Ta \in F$, *and moreover:*

$$\rho_\lambda(x_0,a) \leqslant \alpha(\lambda) \sum_{n=0}^{\infty} \bar{\Phi}^n(k(\lambda))(\Delta) < \infty.$$

(Here $\bar{\Phi}^n(\mu)(t)$ stands for n-th iterate of $\bar{\Phi}(\mu)(t)$, as function of t, and $\bar{\Phi}^0(\mu)(t) = t$).

Proof. By the assumption 1 for every $x \in X$ the set

$$S(x) = \{z \in X : \rho_\lambda(x,z) \leqslant \alpha(\lambda)\mathcal{D}(k(\lambda))(Tx,F)$$
$$\text{and } \mathcal{D}(k(\lambda))(Tz,F) \leqslant \Phi(k(\lambda))(\mathcal{D}(\phi(k(\lambda),Tx,y))(Tx,F))$$
$$\text{for every } \lambda \in \Lambda \text{ and } y \in F\}$$

is nonempty. Begining with $x_0 \in X$ from assumption 2, we define the sequence $\{x_n\}_{n=0}^{\infty}$ by chosing $x_1 \in S(x_0), x_2 \in S(x_1),\ldots,x_{n+1} \in S(x_n),\ldots$. Now the sequence is independent on λ. Consequently, we have for every $\lambda \in \Lambda$ and $y \in F$

$$\rho_\lambda(x_n,x_{n+1}) \leqslant \alpha(\lambda)\mathcal{D}(k(\lambda))(Tx_n,F),$$

and

$$\mathcal{D}(k(\lambda))(Tx_{n+1},F) \leqslant \Phi(k(\lambda))(\mathcal{D}(\phi(k(\lambda),Tx_n,y))(Tx_n,F)).$$

Then

$$\rho_\lambda(x_n,x_{n+1}) \leqslant \alpha(\lambda)\Phi(k(\lambda))(\mathcal{D}(\phi(k(\lambda),Tx_{n-1},y))(Tx_{n-1},F))$$
$$\leqslant \alpha(\lambda)\Phi(k(\lambda))(\Phi(\phi(k(\lambda),Tx_{n-1},y))(\mathcal{D}(\phi^2(k(\lambda),Tx_{n-2},y))(Tx_{n-2},F)))$$
$$\leqslant \ldots \leqslant \alpha(\lambda)\Phi(k(\lambda))(\Phi(\phi(k(\lambda),Tx_{n-1},y))$$

$$(\ldots\Phi(\phi^{n-1}(k(\lambda),Tx_1,y))(\mathcal{D}(\phi^{n}(k(\lambda),Tx_o,y))(Tx_o,F))\ldots)$$
$$\leqslant \alpha(\lambda)\bar{\Phi}^{n}(k(\lambda))(\Delta),$$

and

$$\mathcal{D}(\mu)(Tx_n,F) \leqslant \Phi(\mu)(\mathcal{D}(\phi(\mu,Tx_{n-1},y))(Tx_{n-1},F))$$
$$\leqslant \Phi(\mu)(\Phi(\phi(\mu,Tx_{n-1},y))(\mathcal{D}(\phi^2(\mu,Tx_{n-2},y))(Tx_{n-2},F))) \leqslant \ldots$$
$$\leqslant \Phi(\mu)(\Phi(\phi(\mu,Tx_{n-1},y))\ldots\Phi(\phi^{n-1}(\mu,Tx_o,y))(Tx_o,F))\ldots))$$
$$\leqslant \bar{\Phi}^{n}(\mu)(\Delta).$$

Since

$$\bar{\Phi}^{n+1}(\mu)(\Delta)/\bar{\Phi}^{n}(\mu)(\Delta) \leqslant \bar{\Phi}(\mu)(\Delta)/\Delta < 1$$

then the serie $\sum_{n=o}^{\infty} \bar{\Phi}^{n}(\mu)(\Delta)$ is convergent and hence

$$\lim_{n\to o} \bar{\Phi}^{n}(\mu)(\Delta) = 0.$$

Hence

$$\rho_\lambda(x_n,x_{n+m}) \leqslant \sum_{k=1}^{m}\rho_\lambda(x_{n+k-1},x_{n+k}) \leqslant \alpha(\lambda) \sum_{k=1}^{m} \bar{\Phi}^{n+k-1}(\mu)(\Delta)$$

and therefore $\{x_n\}_{n=o}^{\infty}$ is a Cauchy sequence that in view of the sequential completeness of X tends to the limit $a \in X$. The continuity of T implies $\lim_{n\to\infty} Tx_n = Ta$ in Y. But Y is quasicomplete and F is bounded and closed. Therefore $Ta \in F$. Finally, we have

$$\rho_\lambda(x_o,a) \leqslant \rho_\lambda(x_o,x_1) + \rho_\lambda(x_1,x_2)+\ldots+\rho_\lambda(x_{n-1},x_n)+ \rho_\lambda(x_n,a)$$
$$\leqslant \alpha(\lambda) \sum_{k=o}^{n-1} \bar{\Phi}^{k}(\mu)(\Delta) + \rho_\lambda(x_n,a)$$

and then, when $n \to \infty$,

$$\rho_\lambda(x_o,a) \leqslant \alpha(\lambda) \sum_{k=o}^{\infty} \bar{\Phi}^{k}(\mu)(\Delta) < \infty$$

Theorem 1 is thus proved. ▲

Theorem 2 is a consequence of Theorem 1. Here the assumption for a quasicompleteness of Y is not necessary.

THEOREM 2. *$T:X \to Y$ be a continuous map. Suppose there are a map $\alpha(\lambda):\Lambda \to R^1_+$ and one element $x_o \in X$ such that for every $x \in X$ there is $\bar{x} \in X$ for which*

$$\rho_\lambda(x,\bar{x}) \leqslant \alpha(\lambda)\bar{\rho}_{k(\lambda)}(Tx,y)$$

and

$$\bar{\rho}_{k(\lambda)}(T\bar{x},y) \leqslant \Phi(k(\lambda))(\bar{\rho}_{\phi(k(\lambda),Tx,y)}(Tx,y))$$

for every $\lambda \in \Lambda$ and $y \in Y$, and for $n = 0,1,2,\ldots$

$$\bar{\rho}_{\phi^n(\mu,Tx_o,y)}(Tx_o,y) \leqslant \Delta < \infty, \text{ where } \Delta = \Delta(\mu,y) > 0,$$

Then T is surjective. Moreover, if $Tx = \ell$ then

$$\rho_\lambda(x_o,a) \leqslant \alpha(\lambda)\sum_{n=o}^{\infty}\bar{\Phi}^n(k(\lambda))(\Delta(k(\lambda),\ell)) < \infty \,.$$

§2. Application 1 - Fixed points. Now we are going to obtain fixed point theorem for surjective continuos and expansive mappings.

We shall consider a uniform space X with a saturated family of pseudometrics $\{\rho_\mu(x,y) : \mu \in M\}$. Here the mapping ϕ is defined only on M, that is $\phi:M \to M$.

The mapping $T:X \to X$ is said to be a Φ-*expansion* on X if

$$\rho_\mu(x,y) \leqslant \Phi(\mu)(\rho_{\phi(\mu)}(Tx,Ty))$$

for every $x,y \in X$ and $\mu \in M$.

If, for instance, we choose $\phi(\mu) = \mu$ and $\Phi(\mu)(t) = k_\mu t$ with $0 < k_\mu < 1$, then

$$\rho_\mu(Tx,Ty) \geqslant (1/k_\mu)\rho_\mu(x,y).$$

THEOREM 3. *Let $T:X \to X$ be a continuous, surjective and Φ-expansive mapping. If there is $x_o \in X$ whose inverse image $T^{-1}x_o$ contains an element x_o such that*

$$0 < \rho_{\phi^n(\mu)}(Tx_1,x_1) \leqslant \Delta(\mu) < \infty \qquad (n = 0,1,2,\ldots)$$

then T has at least one fixed point.

Proof. Since T is surjective we may define the sequence $\{x_n\}_{n=0}$ by the equality $x_n = Tx_{n+1}$, begining with x_o and with x_1 satisfying the assumption of the theorem. Setting $c_n^\mu = \rho_\mu(x_n, x_{n+1})$ we obtain

$$c_n^\mu = \rho_\mu(x_n,x_{n+1}) \leqslant \Phi(\mu)(\rho_{\phi(\mu)}(Tx_n,Tx_{n+1})) = \Phi(\mu)(\rho_{\phi(\mu)}(x_{n-1},x_n))$$
$$\leqslant \Phi(\mu)(\Phi(\phi(\mu))(\rho_{\phi^2(\mu)}(x_{n-2},x_{n-1}))) \leqslant \dots$$
$$\leqslant \Phi(\mu)(\Phi(\phi(\mu))(\dots\Phi(\phi^{n-1}(\mu))(\rho_{\phi^n(\mu)}(x_o,x_1))\dots)) \leqslant \bar{\Phi}^n(\mu)(\Delta).$$

The inequalities

$$\rho_\mu(x_n,x_{n+m}) \leqslant \sum_{k=1}^{m} c_{n+k-1}^{\mu} \leqslant \sum_{k=1}^{m} \bar{\Phi}^{n+k-1}(\mu)(\Delta)$$

show that $\{x_n\}_{n=0}^\infty$ is a Cauchy sequence because

$$\bar{\Phi}^{n+1}(\mu)(\Delta)/\bar{\Phi}^{n}(\mu)(\Delta) \leqslant \bar{\Phi}(\mu)(\Delta)/\Delta < 1, \quad (\Delta = \Delta(\mu)).$$

Then the completeness of X implies $\lim_{n\to\infty} x_n = x \subset X$. The element x is the desired fixed point of T. In fact, since T is continuous, $x_n = Tx_{n+1}$ and $\lim_{n\to\infty} x_n = x$, then $x = Tx$.

THEOREM 4. *Let the condititons of Theorem 3 hold true. If, in addition, we suppose that for every fixed points x and y of T the sequence $\{\rho_{\phi^n(\mu)}(x,y)\}_{n=0}^\infty$ is bounded, that is, $\rho_{\phi^n(\mu)}(x,y) \leqslant Q(\mu,x,y)$ $(Q > 0)$, then T has a unique fixed point.*

Proof. The desired conclusion follows immediately from the inequalities

$$\rho_\mu(x,y) \leqslant \Phi(\mu)(\rho_{\phi(\mu)}(Tx,Ty)) = \Phi(\mu)(\rho_{\phi(\mu)}(x,y)) \leqslant$$
$$\dots\leqslant \Phi(\mu)(\Phi(\phi(\mu))(\dots\Phi(\phi^{n-1}(\mu)(\rho_{\phi^n(\mu)}(x,y))\dots)) \leqslant$$
$$\leqslant \bar{\Phi}^n(\mu)(\ (\mu,x,y)).$$

COROLLARY 1. *Let the conditions of theorems 3 and 4 hold good for T^s, which is a Φ-expansion for some positive* *integer s. Then T has a unique fixed point.*

Let us redefine the family (Φ), using the conditions (1) and (2) and replacing the condition (3) in §1 by the following one:

3') for every $\mu \in M$,

$$\lim_{n\to\infty} \Phi(\mu)(\Phi(\phi(\mu))(\ldots\Phi(\phi^n(\mu))(t)\ldots)) = 0, \qquad t > 0.$$

In Theorem 5 the notation of Φ-expansion is taken with respect to the redefined family (Φ).

THEOREM 5. *Let $T:X \to X$ be continuous, surjective and Φ-expansive mapping. If (1) the mapping $\phi:M \to M$ is surjective and $\rho_\mu(x_n, x_{n+m}) \geqslant \rho_{\phi(\mu)}(x_n, x_{n+m})$ for some $x_o \in X$ and every $\mu \in M$, where $x_n = Tx_{n+1}$, and (2) for every two fixed points of T, x and y,*

$$\rho_{\phi^n(\mu)}(x,y) \leqslant Q(\mu,x,y) < \infty \qquad (n = 0,1,2,\ldots) ;$$

then T has a unique fixed point.

Proof. Let us put $c_n^\mu = \rho_\mu(x_n, x_{n+1})$, $p(\mu) = \rho_\mu(x_o, x_1)$. Then

$$c_n^\mu = \rho_\mu(x_n, x_{n+1}) \leqslant \Phi(\mu)(\rho_{\phi(\mu)}(Tx_n, Tx_{n+1}))$$

$$= \Phi(\mu)(\rho_{\phi(\mu)}(x_{n-1}, x_n)) \leqslant \Phi(\mu)(\Phi(\phi(\mu))(\rho_{\phi^2(\mu)}(x_{n-2}, x_{n-1}))) \leqslant$$

$$\ldots \leqslant \Phi(\mu)(\Phi(\phi(\mu))(\ldots\Phi(\phi^{n-1}(\mu))(\rho_{\phi^n(\mu)}(x_o, x_1))\ldots))$$

$$\leqslant \Phi(\mu)(\Phi(\phi(\mu))(\ldots\Phi(\phi^{n-1}(\mu))(p(\mu))\ldots)).$$

Therefore $\lim_{n\to\infty} c_n^\mu = 0$ for every $\mu \in M$.

We suppose, by contradiction, that $\{x_n\}_{n=o}^\infty$ is not a Cauchy sequence. Then there exists $\varepsilon_o > 0$ and finite number of pseudometrics $\{\rho_{\bar\mu}\}$ such that for every ν there is $m = m(\nu) > \nu$ and $p = p(\nu) > 0$ for which $\rho_{\bar\mu}(x_m, x_{m+p}) \geqslant \varepsilon_o$. But ϕ is surjective and we can find $\mu \in M$ such that $\bar\mu = \phi(\mu)$. Moreover, $\rho_\mu(x_m, x_{m+p}) \geqslant \rho_{\phi(\mu)}(x_m, x_{m+p}) \geqslant \varepsilon_o$. Let p be the smallest positive integer for which $\rho_{\phi(\mu)}(x_{m+\bar{p}}, x_m) \geqslant \varepsilon_o$. It follows that $\rho_{\phi(\mu)}(x_{m+\bar{p}-1}, x_m) < \varepsilon_o$. Setting $a_\nu^{\phi(\mu)} = \rho_{\phi(\mu)}(x_{m+\bar{p}}, x_m)$, we have

$$\varepsilon_o \leqslant a_\nu^{\phi(\mu)} \leqslant \rho_{\phi(\mu)}(x_{m+\bar{p}}, x_{m+\bar{p}-1}) + \rho_{\phi(\mu)}(x_{m+\bar{p}-1}, x_m) \leqslant c_{m+\bar{p}-1}^{\phi(\mu)} + \varepsilon_o$$

Passing to the limit in the last inequality when $\nu \to \infty$ we obtain $\lim\limits_{\nu\to\infty} a_\nu^{\phi(\mu)} = \varepsilon_o$. On the other hand we have:

$$\varepsilon_o \leqslant \rho_\mu(x_{m+\bar{p}}, x_m) \leqslant \rho_\mu(x_{m+\bar{p}}, x_{m+\bar{p}+1})$$

$$+ \rho_\mu(x_{m+\bar{p}+1}, x_{m+1}) + \rho_\mu(x_{m+1}, x_m)$$

$$\leqslant c_{m+\bar{p}} + \Phi(\mu)(\rho_{\mu(\mu)}(x_{m+\bar{p}}, x_m)) + c_m^\mu$$

which yields for $\nu \to \infty$, $\varepsilon_o \leqslant \Phi(\mu)(\varepsilon_o)$. The obtained contradiction shows that $\{x_n\}_{n=o}^{\infty}$ is a Cauchy sequence. From there on, the proof can be accomplished as those of Theorem 3 and 3. ▲

REMARK. The mapping T is continuous, surjective and inversible. The inverse mapping T^{-1} is continuous, because it is a contraction. Therefore T is a global homeomorphism of X onto itself. Moreover, if we state $x_o = T^{-1}x_o$, one can obtain a fixed point theorem for Φ-contractive mappings, as in [5]. Then by choosing particular values of $\Phi(\mu)$, we obtain the fixed point theorem of Tarafdar [6], Theorem 11.

§3. Application 2 - McCord's Theorem. Here we establish an extension of McCord's Theorem [2]. Let E and G be Hausdorff locally convex topological vector spaces with saturated families of seminorms, $\{|\cdot\|_\lambda : \lambda \in \Lambda\}$ and $\{\|\cdot\|_\mu : \mu \in M\}$, respectively. Assume there exists an onto $k:\Lambda \to M$, and let $T:E \to G$ be a continuous linear mapping.

THEOREM 6. *Suppose that for each* $y \in G$ *with* $|y|_\mu = 1$

for some $\mu \in M$, *there exists an element* $x \in E$ *and mapppings* $\alpha(\lambda): \Lambda \to R^1_+$, $\beta(\mu): M \to [0,\beta]$, $0 < \beta < 1$, *such that* $\|x\|_\lambda \leqslant \alpha(\lambda)$ *and* $\|y-Tx\|_{k(\lambda)} \leqslant \beta(k(\lambda))$ *for every* $\lambda \in \Lambda$. *Then* T *is surjective. If* $C = Ta$, *then* $\|a\|_\lambda \leqslant \alpha(\lambda) \sum_{n=0}^{\infty} \beta^n$.

Proof. Let $x \in E$ and $y \in G$. One can suppose that $y \neq Tx$, then there exists $\mu_o \in M$ such that $\|y-Tx\|_{\mu_o} \neq 0$, because is a Hausdorff space. Then we can find an element $z \in E$ for which $\|z\|_\lambda \leqslant \alpha(\lambda)$ and

$$\left\| \frac{y - Tx}{\|y-Tx\|_{\mu_o}} - Tz \right\|_{k(\lambda)} \leqslant \beta(k(\lambda)).$$

Let us put $\bar{x} = x + \|y-Tx\|_{k(\lambda)} z$. We have

$$\|\bar{x}-x\|_\lambda \leqslant \|z\|_\lambda \|y-Tx\|_{k(\lambda)} \leqslant \alpha(\lambda)\|y-Tx\|_{k(\lambda)}$$

and then

$$T\bar{x} - y = Tx - y + \|y-Tx\|_{k(\lambda)} Tz,$$

for every $\lambda \in \Lambda$, and so for every $\mu \in M$, the following inequality is satisfied:

$$\|T\bar{x}-y\|_{k(\lambda)} = \|y-Tx - \|y-Tx\|_{\mu_o} Tz\|_{k(\lambda)} \leqslant \beta(\mu)\|y-Tx\|_{\mu_o}.$$

In order to obtain the assumptions of Theorem 2 we define the mapping $\phi: G\times G \to M$ in the following way: for every $\bar{y}, \bar{\bar{y}} \in G$, $\phi(\bar{y},\bar{\bar{y}})$ is the index $\mu \in M$ of the norm for which $\|\bar{y}-\bar{\bar{y}}\|_\mu \neq 0$. Then we put $\phi(Tx,y) = \mu_o$. Since ϕ does not depend on μ, we have $\Delta(\mu,y) = \rho_\mu(T\bar{0},y)$ where $\bar{0}$ is the null element of E. Applying Theorem 2 we obtain the assertion of Theorem 6. ▲

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