Revista Colombiana de Matemáticas
Vol. XXII (1988), págs. 61-66

A CHARACTERIZATION OF PRIMAL NOETHERIAN RINGS

by

P. L. MANLEY

Abstract. A Noetherian ring with identity is shown to be a primal ring if and only if the prime ideals are totally ordered by inclusion.

In this paper we prove that a Noetherian ring with identity is a primal ring if and only if the prime ideals are totally ordered by inclusion. This partially extends the following theorem proved in [2].

THEOREM. *If R is a commutative ring with identity and no zero divisor, then R is primal if and only if the set of all prime ideals of R is simply ordered by inclusion.*

Barnes [1] formulated a definition of a primal ideal for non-commutative rings which reduces to the definition of Fuchs [4] for commutative rings and to that of Curtis [3] for Noetherian rings with identity.

Let R be a ring with identity, and A and B be proper two-sided ideals in R, including the zero ideal but not R itself. The subset

$$A \setminus B = \{x \in R : xB \subseteq A\}$$

is a two-sided ideal in R, and $A \subseteq A \setminus B$.

An ideal B is said to be *(right) prime with respect to* A if and only if $A \setminus B = A$. The ring R itself is regarded as a prime ideal.

Let $\{B_i : i \in J\}$, J a non empty index set, be the set of all two-sided ideals not (right) prime with respect to A. Then

$$\bar{A} = \sum(B_i : i \in J) \qquad (1)$$

is a two-sided ideal of R.

An ideal A is called a *right primal ideal,* if whenever B_i $(i \in J)$ are any two-sided ideals which are not right prime with respect to A, then $\sum(B_i : i \in J)$ is not right prime with respect to A, equivalently, if $\bar{A}$ is not right prime with respect to A. If all the proper two-sided ideals of R are right primal ideals, then R is said to be a *right primal ring.*

In the sequel, *primal ideal* (ring) and *prime with respect to* A will be understood to mean right primal ideal (ring) and right prime with respect to A.

LEMMA 1. *If* A *is a primal ideal, then* $\bar{A}$ *is either equal to* R *or a prime ideal.*

Proof. Let $BC \subseteq \bar{A}$. Since $A \setminus \bar{A} \supset A$ (proper inclusion), then $A \setminus (BC) \supset A$. Since $A \setminus (BC) = (A \setminus C) \setminus B$, then if $A \setminus C = A$, it must be that $A \setminus B \supset A$. Therefore $B \subseteq \bar{A}$. On the other hand, if $A \setminus C \supset A$, it follows that $C \subseteq \bar{A}$.

An ideal A of R is called irreducible if $A = R$, or if A is different from R and different from the intersection of the ideals of R which are proper factors of A. We state an elementary lemma without proof.

LEMMA 2. *An irreducible ideal is a primal ideal.*

Now we state the main theorem.

THEOREM 1. *A ring R with identity, satisfying the ascending chain condition on two-sided ideals, is a primal ring if and only if the prime ideals of R are totally ordered by set inclusion.*

Proof. To prove the necessity of the condition, we proceed by contradiction. Let P and Q be prime ideals in the primal ring R, and suppose that $P \not\subseteq Q$ and $Q \not\subseteq P$.

Now, if $A = P \cap Q$, then $P \not\subseteq A$ and $Q \not\subseteq A$. Since $P \subseteq A \setminus Q$ and $Q \subseteq A \setminus P$, then $A \setminus P \neq A$ and $A \setminus Q \neq A$. Hence neither P nor Q is prime with respect to A, and, because R is a primal ring, then $P + Q$ is not prime with respect to A. Hence $A \setminus (P + Q) \supset A$. Therefore $A \setminus (P + Q) \not\subseteq P$ or $A \setminus (P + Q) \not\subseteq Q$.

Without loss of generality, suppose that $A \setminus (P + Q) \not\subseteq P$ Then we consider the fact that $[A \setminus (P + Q)](P + Q) \subseteq A \subseteq P$. Because P is a prime ideal, it follows that $(P + Q) \subseteq P$, so that $Q \subseteq P$, contrary to the hypothesis. Whence this contradiction proves that the prime ideals of R are totally ordered by set inclusion.

For the converse, let A be a proper ideal of R. From the ascending chain condition, for any $x \notin A$, there exists a maximal ideal A_x such that $x \notin A_x$ and $A \subseteq A_x$. Since every proper factor of A_x includes x, then A_x is an irreducible ideal, and by the Lemma 2 is a primal ideal.

Now, since $A = \cap (A_x : x \in R - A)$, then from the ascending chain condition of R, it suffices to consider only a finite number of the A_x, say $A = A_{x_1} \cap \ldots \cap A_{x_n}$. We assume that there is no redundancy in this expression and that each A_{x_k} $(k = 1, \ldots, n)$ in it cannot be exchanged into a larger ideal.

Let $\bar{A}_{x_k}$, $k = 1, \ldots, n$, be defined as in (1). Then, from the condition in the theorem and Lema 1 $\{\bar{A}_{x_k} : k = 1, \ldots, n\}$ must be ordered by inclusion. Suppose $\bar{A}_{x_1} \supseteq \ldots \supseteq \bar{A}_{x_n}$. Now, if $A \subset A \setminus B_i$, then

$$A \subset A \setminus B_i = (A_{x_1} \cap \ldots \cap A_{x_n}) \setminus B_i$$

$$= A_{x_1} \setminus B_i \cap \ldots \cap A_{x_n} \setminus B_i.$$

Thus, there exists a k $(1 \leqslant k \leqslant n)$ such that $A_{x_k} \subset A_{x_k} \setminus B_i$ or $\bar{A}_{x_1} \supseteq \bar{A}_{x_k} \supseteq B_i$. This shows that $\bar{A} \subseteq \bar{A}_{x_1}$. The reverse inclusion also holds since

$$A \setminus \bar{A}_{x_1} = A_{x_1} \setminus \bar{A}_{x_1} \cap \ldots \cap A_{x_n} \setminus \bar{A}_{x_1} \supset A_{x_1} \cap \ldots \cap A_{x_n}.$$

This shows that $\bar{A}_{x_1} \subseteq \bar{A}$. Whence $\bar{A} = \bar{A}_{x_1}$ and $\bar{A}$ is not prime with respect to A because A_{x_1} is primal, that is, A is primal ideal. This completes the proof of the theorem.

Let R_n denote the matrix ring of order n over R, and A_n be a two-sided ideal of R_n.

The set A_n^{ij} of the ij^{th} elements of all the matrices of A_n forms a two-sided ideal of R. For any i $(1 \leqslant i \leqslant n)$ and j $(1 \leqslant j \leqslant n)$, the ideal A_n^{ij} is always identical, and A_n^{ij} is independent of i $(i = 1, \ldots, n)$ and j $(j = 1, \ldots, n)$.

If we denote A_n^{ij} by $\mathbf{A}$ and arbitrarily take $n \times n$ elements x_{ij} $(i, j = 1, \ldots, n)$ in $\mathbf{A}$, then $[x_{ij}]$ is a matrix in A_n, and A_n is a ring formed by all the $n \times n$ matrices over $\mathbf{A}$. In this way, we have established the one-to-one correspondence between all the two-sided ideals of R_n and all of the two-sided ideals of R. Then it is easy to show that $[\sum(B_i : i \in J)]_n = \sum([B_i]_n : i \in J)$.

THEOREM 2. *A ring R with identity is a primal ring if and only if the matrix ring R_n over R is primal.*

Proof. Suppose R is a primal ring. Now, if $\mathbf{A} \setminus B \supset \mathbf{A}$, then $A_n \setminus B_n \supset A_n$. In fact, $x \notin \mathbf{A}$ and $xB \subseteq \mathbf{A}$, so that $xI_n B_n \subseteq A_n$ and $xI_n \notin A_n$, where I_n is the unit matrix.

Conversely, suppose R_n is a primal ring. Now, if $A_n \setminus B_n \supset A_n$, then there is a square matrix $P = [x_{ij}] \notin A_n$ such that $PB_n \subseteq A_n$. By the hypothesis there exists an ij^{th} $(i, j \leqslant n)$ element x in P such that $x \notin \mathbf{A}$. Since $A_n \setminus B_n$ is an ideal, then $P \in A_n \setminus B_n$, so that $xI_n \in A_n \setminus B_n$ and $xB \subseteq \mathbf{A}$, that is, $x \in A \setminus B$. Therefore, whenever $A \setminus \bar{A} \supset A$, we have that $A_n \setminus \bar{A}_n \supset A_n$. This completes the proof of the theorem.

*

REFERENCES

[1] Barnes, W.E., *Primal ideals and isolated components in noncommutative rings*, Trans. Amer. Math. Soc. 82 (1956), 1-16.

[2] Cheo, P.H., *On primal rings*, Acta Math. Sinica 6 (1956), 542-547 (MR (1961)#8010).

[3] Curtis, C.W., *On additive ideal theory in general rings*, Amer. J. Math. 74 (1952), 687-700.

[4] Fuchs, L., *On primal ideals*, Proc. Amer. Math. Soc. 1 (1950), 1-6.

*

Department of Mathematics & Statistics
University of Windsor
Windsor, Ontario N9B 3P4
CANADA.

(Recibido en mayo de 1984, la versión revisada en octubre de 1986).