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FUNCTIONAL ANALYSIS, ORTHOGONAL POLYNOMIALS AND A THEOREM OF MARKOV

by

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Abstract. The connection between functional analysis and the classical theory of orthogonal polynomials is explored in detail, at least in the bounded case. A functional analytic proof of Markov's theorem, the main link between the two subjects, is given. A special case of Darboux's asymptotic method is presented, and an example showing the power of asymptotic methods to determine orthogonality measures of systems defined by three terms recurrence relations is included.

Resumen. Se estudia en detalle la conexión entre el análisis funcional y la teoría de polinomios ortogonales, al menos en el caso acotado. Se da una demostración del teorema de Markov desde el punto de vista del análisis funcional. Este teorema es el eslabón principal entre las dos teorías. Se examina además un caso especial del método asintótico de Darboux para la determinación de la medida de ortogonalidad de sistemas definidos por relaciones de recurrencia, lo cual se ilustra por medio de un ejemplo.

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§1. Introduction. Recent research (Broad [6]; Reinhardt and Yamani [23], [24]; Bank and Ismail [3]) on linear operators of Quantum Mechanics has called attention to an interesting connection between the spectral theory of linear operators on a Hilbert space and the classical theory of orthogonal polynomials. Roughly speaking the situation is as follows: *some linear operators on a separable Hilbert space, the Jacobi operators, determine systems of orthogonal polynomials whose orthogonality measure, if known, can provide innformation about the spectra and the spectral families of the operators*. This connection is not new and can be traced back to Jacobi in some of his research on the calculus of variations (Guelfand and Fomin [13], Chap.5 §30). Given the variational nature of quantum teory, the interest of Quantum Mechanics in the above mentioned connection is not surprising. The development and refinement in recent years of asymptotics and other techniques of the theory of orthogonal polynomials to determine the orthogonality measures of systems of polynomials given a priori by recurrence relations, on the other hand, make this connection fruitful, since it can provide the means to effectively compute the spectra of quantum mechanical systems.

One of the links between classical orthogonal polynomials and functional analysis is provided by a famous result of Markov, which establishes an important relationship between the asymptotic behavior of the polynomials and the Cauchy-Stieltjes transforma of their orthogonality measure. This allows the Cauchy-Stieltjes inversion formula to recover the measure from the asymptotics. Traditionally Markov's theorem is looked upon as a theorem on continued fractions. The main purpose of this paper is to show that the most widely used form of the theorem is an easy consequence of the underlying functional analysis. We hope this will help to shed further light on the above mentioned link.

In Section 2 we review the relevant facts on orthogonal polynomials and in Section 3 the basic functional analysis.

Section 3 is largely inspired by Ahkiezer's important monograph [1] on the moment problem, but our treatment is more direct and, in view of the most common applications, the results are more complete. In Section 4 we present our point of view on Markov's theorem.

Section 4 was motivated by a talk of M.E.H. Ismail, a few years ago, in the classical analysis seminar at Arizona State University, where he explicitely calculated the orthogonality measure of some classical polynomials by means of what we consider the basic idea of the proof. In Section 5 we take a quick look at Darboux's method, although only to the extent of being able to present a simple but meaningful example of the power of asymptotic methods in orthogonal polynomials. The example we have selected for Section 6 is due to Chihara [10]. It has features which make it specially suited for our purposes.

In view of the above mentioned characteristics, the present paper lies somewhere between a research monograph and an expository article. We hope it can find its place and be of some help to the mathematical community.

§2. Orthogonal polynomials. Let A_n, B_n, C_n, $n \geqslant 0$, be real numbers. The recurrence relation or second order difference equation.

(2.1) $$y_{n+1} = (A_n x + B_n) y_n - C_n y_{n-1}\,, \quad n > 1\,,$$

is called *positive* if

(2.2) $$C_{n+1}/A_nA_{n+1} > 0, \quad n = 0,1,2,\dots .$$

The solutions of (2.1) subject to the initial conditions

(2.3) $$P_0(x) = y_0 = 1, \qquad P_1(x) = y_1 = A_0x + B_0$$

are a system of real polynomials $y_n = P_n(x)$, $n \geq 0$, where $P_n(x)$ has exact degree n. The system $\{P_n(x) \mid n \geq 0\}$ is called a *system of orthogonal polynomials* if (2.2) is satisfied. The set $\{P_n^*(x)\}$ of solutions of (2.1) and initial conditions

(2.4) $$P_0^*(x) = 1, \qquad P_1^*(x) = A_0$$

is called the *associated system of numerator polynomials*. It is also a system of polynomials with real coefficients, $P_n^*(x)$ having exact degree n-1 for $n \geq 1$.

If a_n, b_n, $n \geq 0$, are real numbers and

(2.5) $$b_n > 0, \quad n \geq 0,$$

the recurrence ralation

(2.6) $$xy_n = b_ny_{n+1} + a_ny_n + b_{n-1}y_{n-1}, \quad n \geq 1$$

is a positive second order difference equation. The solutions $y_n = p_n(x)$, $n \geq 0$, of (2.6) subject to the initial conditions

(2.7) $$y_0 = p_0(x) = 1, \quad y_1 = P_1(x) = 1/b_0(x-a_0)$$

are a system of orthogonal polynomials. It is called a *system of orthonormal polynomials.* Their system of *numerator* polynomials satisfies (2.6) and the initial conditions

(2.8) $$p_0^*(x) = 0, \qquad p_1^*(x) = 1/b_0.$$

A positive recurrence relation (2.1) gives rise to a difference equation (2.6) if a_n, b_n are defined through

(2.9) $$a_n = -B_n/A_n, \quad b_n = \sqrt{C_{n+1}/A_nA_{n+1}}, \quad n \geq 0.$$

Then, if $\{P_n(x)\}$ and $\{p_n(x)\}$ are the corresponding systems of orthogonal polynomials for (2.1) and (2.6), $P_n(x)$ and $p_n(x)$ are related through

$$(2.10) \qquad p_n(x) = P_n(x)/\sqrt{\lambda_n}, \quad n \geq 0,$$

where

$$(2.11) \qquad \lambda_0 = 1; \quad \lambda_n = \frac{A_0}{A_n} C_1 \ldots C_n, \quad n \geq 1.$$

This can be easily checked by direct substitution of $P_n(x)/\sqrt{\lambda_n}$ in (2.6). Note that

$$(2.12) \qquad \lambda_n = \prod_{k=0}^{n-1} \frac{A_k C_{k+1}}{A_{k+1}} > 0.$$

The reason for the name orthogonal polynomials given to the solutions $P_n(x)$ of (2.1) and (2.3) is that there is a positive measure μ supported by the real line such that

$$(2.13) \qquad \int_{-\infty}^{+\infty} P_n(x)\; P_m(x) d\mu = \lambda_n \;\; \delta_{mn},$$

where λ_n is given by (2.11).

For the polynomials $p_n(x)$ determined by (2.6) and (2.7) the measure μ can be so chosen that (2.13) takes the form

$$(2.14) \qquad \int_{\infty}^{+\infty} p_n(x) p_m(x)\, d\mu = \delta_{mn}.$$

This explains the name orthonormal given to such polynomials.

The existence of the *orthogonality measure* μ will be discussed in the next section. Observe that once the problem of finding a measure μ satisfying (2.14) has been solved for systems of orthonormal polynomials as defined by (2.6) and (2.7), in which case μ is also called an *orthonormality measure*, the problem for systems determined by (2.1) and (2.3) can be solved by reduction through (2.9). The orthonormality measure μ so obtained will satisfy (2.13) with λ given by (2.11). Hence, it is enough to consider the case of orthonormal polynomials.

REMARK 2.1. If the coefficients of (2.1) and (2.6) are related through (2.9), the corresponding systems of *numerator* polynomials are related through

$$p_n^*(x) = P_n^*(x)/\sqrt{\lambda_n}, \qquad n \geqslant 0. \tag{2.15}$$

REMARK 2.2. Let $\{p_n(x)\}$ be a system of real polynomials such that $p_0(x) = 1$, $p_n(x)$ has exact degree n and positive leading coefficient, and there is a positive measure μ supported by the real line such that (2.14) holds. Since $\{p_n(x)\}$ is an algebraic basis for the $\mathbb{R}$-space $\mathbb{R}[x]$ of real polynomials (and also a basis for the $\mathbb{C}$-space $\mathbb{C}[x]$ of complex polynomials) then, for any $m \geqslant 0$,

$$xp_m(x) = \sum_{k=0}^{m+1} a_{mk}\, p_k(x), \qquad a_{mk} \in \mathbb{R}. \tag{2.16}$$

From (2.16) with $m = n$ it follows that

$$a_{nk} = \int_{-\infty}^{+\infty} xp_n(x)p_k(x)\,d\mu, \qquad k = 0,1,2,\ldots,n+1. \tag{2.17}$$

Since $xp_k(x)$ is a linear combination of $p_0(x),\ldots,p_{k+1}(x)$, from (2.16) with $m = k$ also follows that if $a_{nk} \neq 0$ then $k = n-1,\ n,\ n+1$. Let $a_n = a_{nn}$, $b_n = a_{nn+1}$. From (2.17) we have $a_{nn-1} = a_{n-1}$, so that (2.16) reduces to (2.6) and also $b_0p_1(x) = x-a_0$. If k_n is the leading coefficient of $p_n(x)$ then $k_n = b_nk_{n+1}$, $n \geqslant 0$, follows from (2.16), so that $b_n > 0$. Clearly a_n is real. Hence $\{p_n(x)\}$ is a system of orthonormal polynomials. If $\{P_n(x)\}$ is such that $P_0(x)=1$, $P_n(x)$ is a real polynomial of exact degree n, and (2.13) holds for a positive measure μ with $\lambda_n > 0$, $\lambda_0 = 1$, it follows in the same manner that $\{P_n(x)\}$ will be the system of solutions of a difference equation (2.1) under initial conditions (2.3). In this case

$$(2.18)\qquad A_n = \lambda_{n+1}/A_{n,n+1},\quad B_n = -A_{n,n}A_n/\lambda_n,$$

$$C_n = A_{n,n-1}A_n/\lambda_{n-1},\quad n \geq 1,$$

where

$$(2.19)\qquad A_{j,k} = \int_{-\infty}^{+\infty} x\, P_j(x)\, P_k(x)\, d\mu.$$

Observe that if K_n is the leading coefficient of $P_n(x)$ then

$$(2.20)\qquad A_{n,n+1} = \frac{K_n}{K_{n+1}}\lambda_{n+1} \neq 0,\quad n \geq 0.$$

Also,

$$(2.21)\qquad \frac{C_{n+1}}{A_n A_{n+1}} = \frac{A^2_{n,n+1}}{\lambda_n \lambda_{n+1}} > 0,\quad n \geq 0.$$

Hence $\{P_n(x)\}$ is a system of orthogonal polynomials.

Let $\{p_n(x)\}$ be the system of orthonormal polynomial defined by (2.6) and (2.7), and let $\{p^*_n(x)\}$ be their system of *numerator* polynomials. Induction based on (2.6) and (2.7) readily shows that.

$$(2.22)\qquad b_{n-1}(p_{n-1}(x)p^*_n(x) - p_n(x)p^*_{n-1}(x)) = 1,\quad n \geq 1.$$

This is known as *Abel's formula*. It says that the left hand side is not only different from zero but independent of x. Formula (2.22) is the analogue for difference equations of Abel's formula for second order differential equations. Induction also gives

$$(2.23)\quad b_{n-1}(p_{n-1}(x)p_n(y)-p_n(x)p_{n-1}(y)) = (y-x)\sum_{k=0}^{n-1} p_k(x)p_k(y),\quad n \geq 1.$$

This is the *Christoffel-Darboux identity*. It has important consequences.

From (2.23) and L'Hospital's rule it follows that for all x in $\mathbb{R}$

$$(2.24) \quad b_{n-1}(p_{n-1}(x)p_n'(x)-p_n(x)p_{n-1}'(x)) = \sum_{k=0}^{n-1} p_k^2(x), \quad n \geq 1,$$

and therefore

$$(2.25) \quad p_{n+1}'(x)p_n(x) - p_n'(x)p_{n+1}(x) > 0, \quad n \geq 0, \; x \in \mathbb{R}.$$

Let $\{p_n(x)\}$ be a system of orthonormal polynomials, μ a bounded positive measure such that (2.14) holds. The system $\{p_n(x)\}$ is an algebraic basis of $\mathbb{C}[x]$. If $p(x) \in \mathbb{C}[x]$ has degree n then

$$(2.26) \qquad p(x) = \sum_{k=0}^{n} a_k p_k(x), \quad a_k \in \mathbb{C},$$

so that

$$(2.27) \qquad \int_{-\infty}^{+\infty} p(x)p_k(x)\,d\mu = \begin{cases} a_k, & 0 \leq k \leq n, \\ 0, & k > n. \end{cases}$$

Hence

$$(2.28) \qquad \int_{-\infty}^{+\infty} x^k p_n(x)\,d\mu = 0, \quad 0 \leq k < n,$$

and if $p_n(x) = k_n x^n +$ lower terms, then

$$(2.29) \qquad \int_{-\infty}^{+\infty} x^n p_n(x)\,d\mu = 1/k_n \,.$$

If $p(x) \neq 0$, then

$$(2.30) \qquad \int_{-\infty}^{+\infty} p(x)\,\overline{p(x)}\,d\mu = \sum_{k=0}^{n} |a_k|^2 > 0.$$

Hence, if $p(x) \in \mathbb{R}[x]$ and $p(x) \neq 0$, then

(2.31) $$\int_{-\infty}^{+\infty} p^2(x)\,d\mu > 0.$$

Now assume that $Supp\ \mu \subseteq [\alpha,\beta]$ and let $f(x) \in \mathbb{R}[x]$, $f(x) \neq 0$ and $f(t) \geq 0$ for $t \in [\alpha,\beta]$. If

$$\int_{-\infty}^{+\infty} f(t)\,d\mu = 0$$

and U is the set of $t \in [\alpha,\beta]$ such that $f(t) \neq 0$ then $\mu(U) = 0$. Hence $U \cap Supp\ \mu = \emptyset$, *i.e.*,

$$Supp\ \mu \subseteq \{t \in [\alpha\ \beta] \mid f(t) = 0\} = \{t_1,\dots,t_p\}.$$

Let $q(x) = (x-t_1)(x-t_2)\dots(x-t_p)$. Then $q(t) = 0$ for $t \in Supp\ \mu$ so that

$$\int_{\alpha}^{\beta} q^2(x)\,d\mu = 0.$$

This is absurd, as

$$\int_{\alpha}^{\beta} q^2(x)\,d\mu = \int_{-\infty}^{+\infty} q^2(x)\,d\mu > 0.$$

Hence

THEOREM 2.1. *If μ is an orthogonality measure for the system $\{p_n(x)\}$ of orthogonal polynomials, and if $Supp\ \mu \subseteq [\alpha,\beta]$, then $Supp\ \mu$ is an infinite set and*

$$\int_{\alpha}^{\beta} f(t)\,d\mu > 0$$

for any non-zero polynomial $f(x) \in \mathbb{R}[x]$ such that $f(t) \geq 0$ for $t \in [\alpha,\beta]$.

COROLLARY 2.1. *Let $\{p_n(x)\}$ be a system of orthogonal polynomials, μ an orthogonality measure. Assume that $Supp\ \mu \subseteq [\alpha,\beta]$. Then, for any $n \geq 1$, the roots of $p_n(x)$ are all real and simple, and are contained in (α,β)*

Proof. Let $t_1 < t_2 < \ldots < t_p$ be the odd multimplicity roots of $p_n(x)$ which are contained in (α,β). Since

$$\int_{\alpha}^{\beta} p_n(t)\,d\mu = \int_{-\infty}^{+\infty} p_n(t)\,d\mu = 0 ,$$

then $p \geqslant 1$. Assume $p < n$ and let $q(x) = (x-t_1)\ldots(x-t_p)$. Since $p(x)p_n(x)$ never changes sign in (α,β) then

$$\int_{\infty}^{\infty} q(t)p_n(t)\,d\mu = \int_{\alpha}^{\beta} q(t)p_n(t)\,d\mu \neq 0.$$

This contradicts (2.27). Hence $p = n$, and this completes the proof. ▲

Let $x_{n,1} < x_{n,2} < \ldots < x_{n,n}$ be the n roots of $p_n(x)$. Then

COROLLARY 2.2. *The roots of $p_n(x)$ and $p_{n+1}(x)$, $n \geqslant 1$, are interlaced; that is,*

(2.32) $\quad x_{n+1,1} < x_{n,1} < x_{n+1,2} < \ldots < x_{n+1,n} < x_{n,n} < x_{n+1,n+1}.$

Proof. From (2.25) it follows that

$$p'_{n+1}(x_{n+1,k})p_n(x_{n+1,k}) > 0, \qquad k = 1,2,\ldots,n+1.$$

Since $p'_{n+1}(x) \neq 0$ at $x = x_{n+1,k}$ (because the roots are simple) and has opposite sings at $x_{n+1,k}$ and $x_{n+1,k+1}$, also $p_n(x) \neq 0$ for $x = x_{n+1,k}$ and has opposite signs at $x_{n+1,k}$ and $x_{n+1,k+1}$. Hence, $p_n(x)$ has a roots in each interval $(x_{n+1,k}, x_{n+1,k+1})$, and therefore exactly one. ▲

§3. Jacobi matrices and operators. A matrix

$$(3.1)\qquad J = \begin{pmatrix} a_0 & b_0 & 0 & 0 & 0 & \cdots \\ b_0 & a_1 & b_1 & 0 & 0 & \cdots \\ 0 & b_1 & a_2 & b_2 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix}$$

where a_n, b_n are real numbers and

$$(3.2)\qquad b_n > 0, \qquad n \geqslant 0,$$

is called a *Jacobi matrix*. It is also called a *tridiagonal matrix*. A Jacobi matrix is symmetric. Through (2.6) and (2.7), the Jacobi matrix J defines a system $\{p_n(x)\}$ of orthonormal polynomials, called the J-*polynomials*. Conversely, a system $\{p_n(x)\}$ of orthonormal polynomials determines a Jacobi matrix J such that $\{p_n(x)\}$is the associated system of J-polynomials. the matrix J is called *the matrix of the polynomials*. Direct substitution in (2.6) shows that if J is a Jacobi matrix, and if

$$(3.3)\qquad J_1 = [a_0], \qquad J_n = \begin{pmatrix} a_0 & b_0 & 0 & \cdots & 0 & 0 \\ b_0 & a_1 & b_1 & \cdots & 0 & 0 \\ \vdots & & & & & \\ 0 & 0 & 0 & \cdots & b_{n-2} & a_{n-1} \end{pmatrix}, \quad n \geqslant 2,$$

and I_n is the $n \times n$ identity matrix, then the J-polynomials are given by

$$(3.4)\qquad p_n(x) = \frac{1}{b_0 \ldots b_{n-1}} \operatorname{Det}\,(x I_n - J_n), \qquad n \geqslant 1,$$

so that the eigenvalues of J_n are the roots of $p_n(x)$.

Let H be a separable Hilbert space with hermitian product $(\, ; \,)$ and norm $\| \, \|$. A linear operator L on H is called a *Jacobi operator* if there is a complete orthonormal system $\{e_n \mid n \geqslant 0\}$ of H algebraically spanning the domain $\mathcal{D}(L)$ of L and such that the matrix of L relative to $\{e_n \mid n \geqslant 0\}$ is a Jacobi matrix. The basis $\{e_n \mid n \geqslant 0\}$ is called a *Jacobi basis* for L.

The last statement means that

$$(3.5) \qquad Le_n = b_n e_{n+1} + a_n e_n + b_{n-1} e_{n-1}, \qquad n \geqslant 0,$$

where $b_{-1} = 0$, $e_{-1} = 0$. Note that $\mathcal{D}(L)$ is dense in H, so that L is a densely defined symmetric operator. Hence L is closable and $\bar{L}$ will denote its closure. A Jacobi operator is not necessarily bounded. However, if for some constant $M > 0$.

$$(3.6) \qquad |a_n| \leqslant M/3, \qquad b_n \leqslant M/3, \qquad n \geqslant 0,$$

holds, a simple calculation based on (3.5) shows that $\bar{L}$ is a bounded operator on H such that

$$(3.7) \qquad \|\bar{L}x\| \leqslant M\|x\|, \qquad x \in H.$$

A symmetric operator, and in particular a Jacobi operator, L, has a densely defined adjoint L^*; as a matter of fact, $\mathcal{D}(L) \subseteq \mathcal{D}(\bar{L}) \subseteq \mathcal{D}(L^*)$. More precisely, $L \subseteq \bar{L} \subseteq L^*$, so that $\bar{L}$ is symmetric. Note that $\bar{L} = L^{**}$. Clearly $\bar{L}^* = L^*$ but in general $\bar{L} \neq L^*$, so that $\bar{L}$ may not be self-adjoint. In case $\bar{L}$ is self-adjoint, L is said to be essentially self-adjoint.

A Jacobi operator L defines through its Jacobi matrix J a system $\{p_n(x)\}$ of J-polynomials, called an *associated* *system of orthonormal polynomials for* L. Conversely, given a complete orthonormal system $\{e_n \mid n \geqslant 0\}$ of H, a Jacobi matrix J determines through (3.5) a Jacobi operator L on the algebraic span $\mathcal{D}(L)$ of $\{e_n\}$. Hence, a system $\{p_n(x)\}$ of orthonormal polynomials defines, through its Jacobi matrix and a complete orthonormal system $\{e_n\}$ of H, a Jacobi operator L. The operator L is called the *Jacobi operator of* $\{p_n(x)\}$ *for* $\{e_n \mid n \geqslant 0\}$.

Let L be a Jacobi operator on H with matrix $\mathbf{J}$ for $\{e_n\}$. A vector $y = \sum_{n=0}^{\infty} y_n e_n$ in H is in $\mathcal{D}(L^*)$ if, and only if,

$$(3.8) \qquad y^* = \sum_{n=0}^{\infty} (b_n y_{n+1} + a_n y_n + b_{n-1} y_{n-1}) e_n$$

is in H, *i.e.*, if and only if

$$(3.9) \qquad \sum_{n=0}^{\infty} |b_n y_{n+1} + a_n y_n + b_{n-1} y_{n-1}|^2 < +\infty.$$

In this case, $y^* = L^* y$. All this follows at once from

$$(3.10) \qquad (Le_n; y) = (e_n; y^*).$$

From (3.8) we conclude that $\lambda \in \mathbb{C}$ is an eigenvalue of L^* if, and only if, there is a sequence $\{y_n \mid n \geqslant 0\}$, not all of the y_n's equal to 0, such that

$$(3.11) \qquad \lambda y_n = b_n y_{n+1} + a_n y_n + b_{n-1} y_{n-1}, \quad n \geqslant 0,$$

(here, $b_{-1} = 0$). Thus $\{y_n \mid n \geqslant 0\}$ is a solution of (2.6) with $x = \lambda$, so that

$$(3.12) \qquad y_n = y_0 p_n(\lambda), \quad n \geqslant 0,$$

where $\{p_n(x)\}$ is the associated system of orthonormal polynomials for L. Hence, $y_0 \neq 0$ and

$$(3.13) \qquad \sum_{n=0}^{\infty} |p_n(\lambda)|^2 < +\infty.$$

Conversely, if (3.13) holds, from (3.11) it follows that

$$(3.14) \qquad y_\lambda = \sum_{n=0}^{\infty} p_n(\lambda) e_n$$

is eigenvector of L^* for the eigenvalue λ. We have proved:

THEOREM 3.1. *A complex number λ is an eigenvalue of the adjoint L^* of the Jacobi operators L if and only if*

$$\sum_{n=0}^{\infty} |p_n(\lambda)|^2 < +\infty,$$

where $\{p_n(x)\}$ is an associated system of orthonormal polynomials for L. In such a case, the space of eigenvectors of L^* for λ is spanned by y_λ, where y_λ is given by (3.14).

REMARK 3.1 Observe that λ is an eigenvalue for L^* if and only if also $\bar{\lambda}$ is an eigenvalue. This follows from

$$(3.15) \qquad \sum_{n=0}^{\infty} |p_n(\lambda)|^2 = \sum_{n=0}^{\infty} |p_n(\bar{\lambda})|^2.$$

REMARK 3.2. If $\{p_n(x)\}$ is a system of orthogonal polynomials and $\lambda \in \mathbb{C}$, then $p_n(\lambda) \neq 0$ for an infinite number of values of n. This follows at once from the recurrence relation (2.6). Hence, a Jacobi operator has no eigenvalues. However, $\bar{L}$ may have eigenvalues.

From the general theory of linear operators on a Hilbert space (Gómez [12], Lang [19], Yosida [29]) it follows that for a Jacobi operator L and its closure operator $\bar{L}$ there are but two possibilities:

1) For all $\lambda \in \mathbb{C}-\mathbb{R}$, $Im(\bar{L}-\lambda I)^{\perp} = Ker(L^*-\bar{\lambda}I) = \{0\}$.

2) For all $\lambda \in \mathbb{C}-\mathbb{R}$, $Im(\bar{L}-\lambda I) = Ker(L^*-\bar{\lambda}I)^{\perp}$ is a closed subspace of H of codimension one.

This means that $\bar{L}$ is either self-adjoint or has defficiency indices equal to 1. A well known theorem of J. von Neumann (Gómez [12], Yosida [29]) implies that

THEOREM 3.2. *If L is a Jacobi operator on H, there is a self-adjoint operator $\tilde{L}$ on H which is an extension of L. This extension $\tilde{L}$ is unique if and only if L is essentially self-adjoint, in which case $\tilde{L} = L$.*

If L is a Jacobi operator, inherited from $\tilde{L}$ there is a spectral family $(E_\lambda)_{\lambda\in\mathbb{R}}$ for L, *i.e.*, a family of hermitian projections E_λ of H such that $E_\lambda \leqslant E_\mu$ if $\lambda \leqslant \mu$,

$$(3.16) \qquad \lim_{\lambda\to-\infty} \|E_\lambda x\| = 0, \qquad \lim_{\lambda\to-\infty} \|E_\lambda x-x\|$$

and

$$(3.17) \qquad (Lx\,;\,y) = \int_{-\infty}^{+\infty} \lambda d(E_\lambda x;y)$$

for all $x \in \mathcal{D}(L)$ and all $y \in H$. As a matter of fact (3.17) holds for $\bar{L}$, L^* or $\tilde{L}$, and for x in their respective domains. We recall that

$$(3.18) \qquad \mathcal{D}(\tilde{L}) = \{x \in H \mid \int_{-\infty}^{+\infty} \lambda^2 d\|E_\lambda x\|^2 < +\infty\}$$

and also that, necessarily,

$$(3.19) \qquad E_\lambda E_\mu = E_{min\{\lambda,\mu\}}.$$

It is usual to write

$$(3.20) \qquad Lx = \int_{-\infty}^{+\infty} \lambda dE_\lambda x, \qquad x \in \mathcal{D}(L)$$

or even, at the risk of confussion,

$$(3.21) \qquad L = \int_{-\infty}^{+\infty} \lambda dE_\lambda .$$

If

$$(3.22) \qquad \lim_{\substack{\lambda\to\mu \\ \lambda>\mu}} \|E_\lambda x - E_\mu x\| = 0, \quad x \in H,$$

(E_λ) is said to be *right-continuous*, and (3.22) is usually written

$$(3.23) \qquad E_{\mu+0}x = E_\mu x .$$

Left continuity is defined through

$$(3.24) \qquad \lim_{\substack{\lambda\to\mu \\ \lambda<\mu}} \|E_\lambda x - E_\lambda x| = 0, \quad x \in H,$$

or equivalently,

$$(3.25) \qquad E_{\mu-0}x = E_\mu x.$$

Simultaneous left and right-continuity is equivalent to the absence of eigenvalues for $\tilde{L}$. The uniqueness of a right or left-continuous spectral family for L is equivalent to the essential self-adjointness of L.

Induction based on (3.5) readily shows that

THEOREM 3.3. *If L is a Jacobi operator on H, $\{e_n\}$ is a Jacobi basis for L and $\{p_n(x)\}$ is the corresponding system of orthonormal polynomials of L for $\{e_n\}$, then*

$$(3.26) \qquad p_n(L)e_0 = e_n, \quad n \geqslant 0.$$

Relation (3.26) is the key to the connection between orthogonal polynomials and functional anlysis.

LEMMA 3.1. *Let L be a Jacobi-operator on H, $e \neq 0$ a vector in the domain $\mathcal{D}(L)$ of L, (E_λ) a spectral family for L. Then, for any polynomial $p(x) \in \mathbb{C}[x]$,*

$$(p(L)e\,;\,e) = \int_{-\infty}^{+\infty} p(\lambda)\,d(E_\lambda e\,;\,e). \tag{3.27}$$

Proof. If is enough to prove (3.27) for $p(x) = x^n$, and in such case it is trivial for $n = 0,1$. Notice that $L^n e \in \mathcal{D}(L)$ for all $n \geq 0$. If we assume by induction that (3.27) holds for $n = m$, and if $\tilde{L}$ is the self-adjoint extension of which determines (E_λ), then

$$(L^{m+1}e\,;\,e) = (L^m e; Le) = \int_{-\infty}^{+\infty} \lambda^m \, d(E_\lambda e\,;\,Le).$$

But $(E_\lambda e\,;\,Le) = (\tilde{L}E_\lambda e\,;\,e)$ and

$$(\tilde{L}E_\lambda e\,;\,e) = \int_{-\infty}^{+\infty} t\,d(E_t E_\lambda e\,;\,e) = \int_{-\infty}^{\lambda} t\,d(E_t e\,;\,e) \tag{3.28}$$

(Gómez [12], Yosida [29]). Then

$$d(\tilde{L}E_\lambda e\,;\,e) = \lambda\,d(E_\lambda e\,;\,e),$$

and therefore

$$(L^{m+1}e\,;\,e) = \int_{-\infty}^{+\infty} \lambda^{m+1} d(E_\lambda e\,;\,e).$$

Hence (3.27) also holds for $p(x) = x^n$, $n = m+1$. This proves the lemma. ▲

THEOREM 3.4. *Let L be a Jacobi operator, $\{e_n \mid n \geq 0\}$ a complete orthonormal system for H, $\{p_n(x) \mid n \geq 0\}$ the associated system of orthogonal polynomials of L for $\{e_n\}$ Let (E_λ) be a spectral family for L. then*

$$(3.29)\qquad \int_{-\infty}^{+\infty} p_n(\lambda)p_m(\lambda)d(E_\lambda e_o\,;\,e_o) = \delta_{mn}.$$

Hence, the measure μ *on* $\mathbb{R}$ *determined by the non-decreasing function*

$$(3.30)\qquad \sigma(\lambda) = (E_\lambda e_o\,;\,e_o), \qquad \lambda \in \mathbb{R},$$

is an orthonormality measure for the system $\{p_n(x)\}$.

Proof. In fact, from (3.26) and (3.27),

$$\delta_{mn} = (e_m, e_n) = (p_m(L)e_o\,;\,p_n(L)e_o)$$

$$= (p_n(L)p_m(L)e_o\,;\,e_o) = \int_{-\infty}^{+\infty} p_n(\lambda)p_m(\lambda)d(E_\lambda e_o\,;\,e_o).$$

This proves the theorem. ▲

We recall that if $C_o(\mathbb{R})$ is the space of compactly supported, complex valued, continuous functions on $\mathbb{R}$, then μ is defined by

$$(3.31)\qquad \mu(\phi) = \int_{-\infty}^{+\infty} \phi(\lambda)d\sigma(\lambda)$$

the right hand side being an ordinary Riemann-Stieljes integral. Since

$$(3.32)\qquad \int_{-\infty}^{+\infty} d\sigma(\lambda) = 1\ ,$$

μ is a bounded positive measure on $\mathbb{R}$. Notice that

$$(3.33)\qquad \lim_{\lambda\to+\infty} \sigma(\lambda) = 1,\ \lim_{\lambda\to-\infty} \sigma(\lambda) = 0$$

and, if (E_λ) is right-continuous, then

$$(3.34)\qquad \mu((-\infty,\lambda]) = \sigma(\lambda) = \int_{-\infty}^{\lambda} d\sigma(\lambda),\ \mu((-\infty,\lambda)) = \sigma(\lambda-0)$$

and

(3.35) $$\mu(\{\lambda\}) = \sigma(\lambda)-\sigma(\lambda-0)$$

Right continuity of (E_λ) is convenient in the sense that, from (3.34), we can write

(3.36) $$\int_{-\infty}^{\lambda} d\mu = \int_{-\infty}^{\lambda} d\sigma(t),$$

so that

(3.37) $$\int_{-\infty}^{\lambda} f d\mu = \int_{-\infty}^{\lambda} f d\sigma(t)$$

for any μ-integrable function f. Relation (3.28) would take the form

(3.38) $$\int_{-\infty}^{\lambda} f d\sigma(t) = \int_{(-\infty,\lambda)} f d\mu$$

if left continuity of (E_λ) were assumed.

The following two corollaries of Theorem 3.4 contain properties of the measure μ which are of interest and importance

COROLLARY 3.1. *Under the assumptions of the theorem, the system* $\{p_n(x)\}$ *is a complete orthonormal system of* $L_2(\mathbb{R},\mu)$.

Proof. If $\alpha \leqslant \beta$ then

(3.39) $$(E_\alpha e_o;e_o) = (E_\beta E_\alpha e_o;e_o)$$

$$= (E_\alpha e_o;E_\beta e_o) = \sum_{k=0}^{\infty} (E_\alpha e_o;e_k)\overline{(E_\beta e_o;e_k)}.$$

But

$$(3.40)\qquad (E_\lambda e_o; e_k) = (E_\lambda e_o; p_k(L)e_o) = (p_k(\tilde{L})E_\lambda e_o; e_o)$$

$$= \int_{-\infty}^{+\infty} p_k(t)\,d(E_t E\ e_o; e_o) = \int_{-\infty}^{\lambda} p_k(t)\,d(E_t e_o; e_o)$$

$$= \int_{-\infty}^{\lambda} p_k(t)\,d\mu.$$

If ϕ_λ is the characteristic function of $(-\infty,\lambda]$ then, from (3.39) and (3.40),

$$(3.41)\qquad \int_{-\infty}^{+\infty} \phi_\alpha \phi_\beta d\mu = \sum_{k=0}^{\infty} \Big(\int_{-\infty}^{+\infty} \phi_\alpha p_k d\mu\Big)\Big(\int_{-\infty}^{+\infty} \phi_\beta p_k d\mu\Big).$$

Since any step function is a linear combination of functions ϕ_λ, if f, g are step functions then

$$(3.42)\qquad \int_{-\infty}^{+\infty} f\bar{g}d = \sum_{k=0}^{\infty} \Big(\int_{-\infty}^{+\infty} f p_k d\mu\Big)\overline{\Big(\int_{-\infty}^{+\infty} g p_k d\mu\Big)}.$$

From the density of step functions the above identiy also holds for $f,g \in L_2(\mathbb{R},\mu)$. But relationship (3.42) is the Parseval identity for the orthonormal system $\{p_n(x)\}$ of $L_2(\mathbb{R},\mu)$. Hence, $\{p_n(x)\}$ is a complete orthonormal system of $L_2(\mathbb{R},\mu)$. ▲

COROLLARY 3.2. *Under the assumptions of the theorem, the spectrum of the self-adjoint extension* $\tilde{L}$ *defining* (E_λ) *and the support of the measure* μ *coincide.*

Proof. If $\lambda \in \mathbb{R}$ does not belong to the spectrum $\sigma(\tilde{L})$ of $\tilde{L}$ there are $\lambda_1 < \lambda < \lambda_2$ such that $[\lambda_1,\lambda_2] \cap \sigma(\tilde{L}) = \emptyset$ and therefore (Gómez [11], Yosida [29]) $E_{\lambda_1} = E_{\lambda_2}$. It follows that $E_t = E_{\lambda_1} = E_{\lambda_2}$ for all $t \in (\lambda_1,\lambda_2)$, so that $\sigma(\lambda)$ is constant on (λ_1,λ_2). If $\phi \in C_o(\mathbb{R})$ and vanishes outside a compact subset of (λ_1,λ_2) then

$$\int_{-\infty}^{+\infty} \phi d\mu = \int_{\lambda_1}^{\lambda_2} \phi(t)\, d\sigma(t) = 0 ,$$

so that $\lambda \notin supp\ \mu$. This shows that $supp\ \mu \subseteq \sigma(\tilde{L})$. Conversely, let $\lambda \notin supp\,\mu$ and take $\lambda_1 < \lambda < \lambda_2$ such that $[\lambda_1,\lambda_2] \cap supp\,\mu = \emptyset$. From $E_\mu L = \tilde{L}E_\mu$ (Gómez [12], Yosida [29]) it follows that $p_n(\tilde{L})E_\mu = E_\mu p_n(L)$, so that

$$(3.43)\quad (E_\mu e_n; e_m) = (E_\mu p_n(L)e_o; p_m(L)e_o) = (p_n(\tilde{L})E_\mu e_o, p_m(L)e_o)$$

$$= (p_m(\tilde{L})p_n(\tilde{L})E_\mu e_o; e_o) = \int_{-\infty}^{+\infty} p_m(t)p_n(t)\, d(E_t E_\mu e_o; e_o)$$

$$= \int_{-\infty}^{\mu} p_m(t)p_n(t)\, d\mu .$$

Hence

$$(3.44)\qquad ((E_{\lambda_2} - E_{\lambda_1})e_n ; e_m) = \int_{(\lambda_1,\lambda_2]} p_m(t)p_n(t)\, d\mu = 0$$

so that

$$(3.45)\qquad ((E_{\lambda_2} - E_{\lambda_1})x ; y) = 0$$

for $x \in \mathcal{D}(\tilde{L})$, $y \in H$; *i.e.*, $E_{\lambda_1} = E_{\lambda_2}$. Thus $\lambda \notin \sigma(\tilde{L})$ and the proof is complete. ▲

REMARK 3.3. We remark that in the previous theorem the eigenvalues of $\tilde{L}$ correspond to the points of jump discontinuities of $\sigma(\lambda) = (E_\lambda e_o, e_o)$. This is necessarily the case if λ is an isolated point of $\sigma(\tilde{L})$.

The following consequence of Theorem 3.4 is known as Favard's theorem.

THEOREM 3.5. *Let* $\{p_n(x) \mid n \geqslant 0\}$ *be a system of orthogonal polynomials determined by* (2.6) *and the initial conditions* (2.7). *Then, there is a positive measure* μ *on* $\mathbb{R}$ *such that*

$$\int_{-\infty}^{+\infty} p_n(x)p_m(x)d\mu = \delta_{mn} \tag{3.46}$$

for all $m,n \geqslant 0$. *The measure* μ *can be so chosen that* $\{p_n(x)\}$ *is a complete orthonormal system of* $L_2(\mathbb{R},\mu)$.

Proof. Let J be the Jacobi matrix of the polynomials and take H to be any separable Hilbert space and $\{e_n \mid n \geqslant 0\}$ a complete orthonormal system for H; for example $H = \ell^2$ and $\{e_n\}$ the cannonical basis, $e_n = (\delta_{in})_{i\geqslant 0}$, of this space. Let L be the operator on H with Jacobi matrix J relative to $\{e_n\}$. Then $p_n(L)e_0 = e_n$. Hence, if $\tilde{L}$ is a self adjoint extension of L and (E_λ) is a spectral family for $\tilde{L}$, then

$$\int_{-\infty}^{+\infty} p_n(x)p_m(x)d(E_\lambda e_0;e_0) = (e_m;e_n) = \delta_{mn}.$$

That $\{p_n(x)\}$ is a complete orthonormal system of $L_2(\mathbb{R},\mu)$ follows from Theorem 3.4. ▲

REMARK 3.4. A proof of Favard's theorem from an entirely different and more elementary point of view can be found in Chihara [10].

Given a system $\{p_n(x)\}$ of orthogonal polynomials, there are in general several, and hence infinite, distinct measures μ such that (3.46) holds true. For example, if the associated operator L in some Hilbert space is not essentially self-adjoint, distinct self-adjoint extensions $\tilde{L}$ of L determine distinct spectral families (E_λ). This is so because

$$(3.47) \qquad \tilde{L} = \int_{-\infty}^{+\infty} \lambda dE_\lambda .$$

Since $\{e_n\}$ is a topological basis of H, from

$$(3.48) \qquad E_\lambda e_m = \sum_{n=0}^{\infty} \{ \int_{-\infty}^{\lambda} p_m(t) p_n(t) d\mu \} e_n$$

it follows that distinct spectral families determine distinct measures.

A useful criterion for uniqueness is the following:

THEOREM 3.6. *If the system* $\{p_n(x)\}$ *has a compactly supported orthogonality measure* μ *and* $\mu(\mathbb{R}) = 1$, *then* μ *is the only positive measure for which* (3.46) *holds.*

Proof. Assume (3.46) also holds for the measure ν. Let $\phi \in C_0(\mathbb{R})$ and $M > 0$ be large enough to have

$$Supp\mu \cup Supp\phi \subseteq (-M,M).$$

Let $\{q_n(x)\}$ be a sequence of polynomials $\geqslant 0$ in $[-M,M]$, uniformly convergent to ϕ in $[-M,M]$. The existence of $\{q_n(x)\}$ is granted by the Weierstrass approximation theorem. From

$$\int_{-\infty}^{+\infty} d\mu = \int_{-\infty}^{+\infty} d\nu = 1; \quad \int_{-\infty}^{+\infty} p_n d\mu = \int_{-\infty}^{+\infty} p_n d\nu = 0, \qquad n \geqslant 1,$$

it follows that

$$\int_{-\infty}^{+\infty} q_n d\mu = \int_{-\infty}^{+\infty} q_n d\nu = a_n, \qquad n \geqslant 0,$$

where a_n is the coefficient of $p_0(x)$ in the expansion of $q_n(x)$ with respect to the algebraic basis $\{p_n(x) \mid n \geqslant 0\}$ of $\mathbb{C}[x]$. Hence, assuming $\phi \geqslant 0$.

$$\int_{-\infty}^{+\infty} \phi\, d\mu = \int_{-M}^{M} \phi\, d\mu = \lim_{n\to\infty} \int_{-M}^{M} q_n d\mu = \lim_{n\to\infty} \int_{-\infty}^{+\infty} q_n d\mu = \lim_{n\to\infty} \int_{-\infty}^{+\infty} q_n \; d\nu$$

Since

$$\lim_{n\to\infty} \int_{-M}^{M} q_n d\nu = \int_{-M}^{M} \phi\, d\nu = \int_{-\infty}^{-\infty} \phi\, d\nu \ ,$$

we get

$$\int_{-\infty}^{+\infty} \phi\, d\nu \leqslant \int_{-\infty}^{+\infty} \phi\, d\mu \ ,$$

so that $Supp\, \nu \subseteq Supp\, \mu$. But then

$$\int_{-M}^{M} q_n d\nu = \int_{-\infty}^{+\infty} q_n d\nu \ ,$$

and so

$$\int_{-\infty}^{+\infty} \phi\, d\nu = \int_{-\infty}^{+\infty} \phi\, d\mu \ .$$

This proves the theorem. ▲

REMARK 3.5. If $\{p_n(x)\}$ has a compactly supported orthonormality measure μ and ν is another measure such that

$$\int_{-\infty}^{+\infty} p_n(x) p_m(x)\, d\nu = K_n \delta_{mn} \tag{3.49}$$

where $K_n \geqslant 0$, then $\nu = K_o \mu$. In fact,

$$\int_{-\infty}^{+\infty} d\nu = K_o ; \qquad \int_{-\infty}^{+\infty} p_n(x)\, d\nu = 0, \qquad n \geqslant 1.$$

Hence, for any polynomial $q(x)$,

$$\frac{1}{K_o} \int_{-\infty}^{+\infty} q(x)\, d\nu = a = \int_{-\infty}^{+\infty} q(x)\, d\mu \ ,$$

where a is the coefficient of $p_o(x)$ in the expansion of $q(x)$ with respect to $\{p_n(x) \mid n \geq 0\}$. The argument used in the proof of the theorem shows that $\nu = K_o\mu$. We also deduce that $K_n = K_o$, $n \geq 0$.

The following corollary of Theorem 3.5 is useful.

COROLLARY 3.3. *The Jacobi operator L determined on a Hilbert space H by a system of orthonormal polynomials $\{p_n(x)\}$ is bounded if and only if there is a compactly supported measure μ such that* (3.46) *holds. In such cases μ is the unique orthonormality measure of the system $\{p_n(x)\}$.*

Proof. If L is bounded then $\sigma(\bar{L})$ is a compact set and there is an orthonormality measure of $\{p_n(x)\}$ such that $\sigma(\bar{L}) = Supp\ \mu$. Assume conversely that $\{p_n(x)\}$ has a compactly supported orthonormality measure μ. Then μ is the only orthonormality measure of $\{p_n(x)\}$ and hence is the measure given by Theorem 3.4. But then Corollary 3.2 implies that $\sigma(\bar{L})$ is compact and therefore that L is bounded. ▲

COROLLARY 3.4. *Let $\{p_n(x)\}$ be given by* (2.6) *and the initial conditions* (2.7) *If* (3.6) *holds then there is a unique positive measure μ satisfying* (3.46). *Furthermore $Supp\ \mu \subseteq [-M, M]$.*

Proof. If L on H is a Jacobi operator for the system $\{p_n(x)\}$ then $\sigma(\bar{L}) \subseteq [-\|L\|, \|L\|]$, and $\|\bar{L}\| \leq M$ from (3.7). ▲

If L is self-adjoint but not bounded, uniqueness of μ can not be granted. However, the existence of a unique measure μ such that $\{p_n(x)\}$ is a complete orthonormal system of $L_2(\mathbb{R}, \mu)$ can be asserted. To prove this observe that if $\{e_n \mid n \geq 0\}$ is an orthonormal basis of H, the linear map

$\Gamma: H \to L_2(\mathbb{R},\mu)$ determined on $\{e_n\}$ by

$$(3.50) \qquad \Gamma(e_n) = p_n(x)$$

is isometric and identifies the Jacobi operator L of the $p_n(x)$ *s* in H for $\{e_p\}$ with the operator T of $L_2(\mathbb{R},\mu)$ given by

$$(3.51) \qquad T(f)(x) = xf(x)$$

on the linear span of $\{p_n(x)\}$; *i.e.*, $T = \Gamma L \Gamma^{-1}$. If L is essentially self-adjoint and (E_λ) is the uniquely determined right-continuous spectral family of L then T is essentially self-adjoint and $F_\lambda = \Gamma \mathbb{E}_\lambda \Gamma^{-1}$ is the uniquely determined right-continuous spectral family of T. From (3.48) it is easily shown that

$$(3.52) \qquad (F_\lambda f)(x) = \sum_{n=0}^{\infty} \{\int_{-\infty}^{\lambda} f(t)p_n(t)d\mu(t)\}p_n(x)$$

and if $\sigma(\lambda) = \langle F_\lambda 1 \; ; \; 1\rangle$, where

$$(3.53) \qquad \langle f \; ; \; g\rangle \quad \int_{\infty}^{\infty} f\bar{g}\,d\mu,$$

then

$$(3.54) \qquad d\mu = d\sigma(\lambda)$$

Notice that if (G_λ) is the left continuous spectral family of T then

$$(3.55) \qquad (G_\lambda f)(x) = \sum_{n=0}^{\infty} \{ \int_{(-\infty,\lambda)} f(t)p_n(t)d\mu(t)\}p_n(x).$$

If $\rho(\lambda) = \langle G_\lambda 1 \; ; \; 1\rangle$ then

$$(3.56) \quad \sigma(\lambda)-\sigma(\lambda-0) = \mu(\{\lambda\}) = \rho(\lambda+0) - \rho(\lambda),$$

so that

$$(3.57) \qquad S = \{\lambda \in \mathbb{R} \mid \mu(\{\lambda\}) \neq 0\}$$

is the set of points of discontinuity of both σ and ρ and they have the same jumps at these points. Since

$$(3.58) \qquad \sigma(\lambda) = \rho(\lambda) + \mu(\{\lambda\}),$$

σ and ρ coincide at all points of continuity. Hence, σ and ρ define the same measure, *i.e.*, μ.

REMARK 3.6. Even if $\{p_n(x)\}$ is a complete orthonormal system of $L_2(\mathbb{R},\mu)$, essential self-adjointness of T can not be asserted. If (F_λ) is defined through (3.52), it can be shown to be a spectral family for the self-adjoint extension

$$\tilde{T} = \int_{-\infty}^{+\infty} \lambda dF_\lambda$$

of T. Uniqueness of μ with the above property is equivalent to the essential self-adjointness of T. More generally: under the assumption that $\{p_n(x)\}$ is a complete orthonormal system for $L_2(\mathbb{R},\mu)$, if $\{p_n(x)\}$ is the system of polynomials associated to a general Jacobi operator L for the basis $\{e_n\}$, a spectral family for L can be obtained through (3.48) and linear extension, uniqueness of μ being equivalent to uniqueness of (E_λ) and therefore to essential self adjointness of L. It is in this sense that orthogonal polynomials theory can be of great help in functional analysis.

REMARK 3.7. If $\{p_n(x)\}$ is not a complete orthonormal system of $L_2(\mathbb{R},d\mu)$, the family (F_λ) of non-negative hermitian operators defined through (3.52) is still right-continuous and increasing, but it can not be asserted that the

(F_λ) are projections (*i.e.*, $F^2 = F$). As a matter of fact, $\{p_n(x)\}$ is complete if and only if $F_\alpha F_\beta = F_\alpha$ for $\alpha \leqslant \beta$, as follows from

$$\text{(3.59)} \qquad \langle F_\alpha F_\beta p_m, p_n\rangle \quad \langle F_\alpha p_m, p_n\rangle ,$$

holding if and only if

$$\text{(3.60)} \qquad \sum_{k=0}^{\infty} \left(\int_{-\infty}^{\alpha} p_m p_k d\mu\right)\left(\int_{-\infty}^{\beta} p_n p_k d\mu\right) = \int_{-\infty}^{\alpha} p_m p_n d\mu ,$$

which is equivalent to Parseval's identity for the system $\{p_n(x)\}$. The same observation holds in a general Hilbert space for the family E_λ defined through (3.48).

REMARK 3.8. Uniqueness of the orthonormality measure for a system $\{p_n(x)\}$ of polynomials ensures completeness of $\{p_n(x)\}$ in $L_2(\mathbb{R},\mu)$ and essential self-adjointness of any associated Jacobi operator. That is the case, for example, if μ is compactly supported. The existence and uniqueness problem for orthonormality measures is closely related to the problem of existence and uniqueness of a positive measure supported by the real line and having given moments; *i.e.*, of

$$\int_{-\infty}^{+\infty} t^n d\mu , \qquad n = 0,1,2,\ldots$$

taking prescribed values. This is known as the *moment problem*. A careful study of the moment problem from very diverse points of view is in Ahkiezer [1], where references about the impressive ramifications of this problem throughout all of mathematics can be found.

§4. Markov's theorem. Let $\{p_n(x) \mid n \geq 0\}$ be the system of orthonormal polynomials determined by (2.6) and the initial conditions (2.7). We will require the coefficients a_n, b_n to satisfy

$$(4.1) \qquad |a_n| \leq M/3, \quad b_n \leq M/3, \quad n \geq 0,$$

so that the support of their unique orthonormality measure will be contained in $[-M,M]$.

Now assume that the limit

$$(4.2) \qquad \lim_{n\to\infty} \frac{p_n^*(z)}{p_n(z)} =: \chi(z)$$

can be determined for all $z \in \mathbb{C}-[-M,M]$. The function $\chi(z)$ with domain $\mathbb{C}-[-M,M]$ is then called the *continued fraction* associated to the system $\{p_n(x)\}$ of orthogonal polynomials. The reason for the name is that $p_n^*(z)/p_n(z)$ is the n^{th}-convergent of the continued fraction

$$\frac{1|}{|x_0-a_0} - \frac{b_0^2|}{|x_0-a_1} - \frac{b_1^2|}{|x_0-a_2} - \ldots - \frac{b_{n-1}^2|}{|x_0-a_n} - \ldots$$

as it easily follows from the Wallis formula (Chihara [10]). A very deep study of continued fractions is in Wall [28].

Let $\mathbf{J}$ as in (3.1) be the Jacobi matrix of the system $\{p_n(x)\}$. Let J_n be defined by (3.4), so that the eigenvalues of J_n are the n roots $\lambda_{n1} < \ldots < \lambda_{nn}$ of $p_n(\lambda)$, $n \geq 1$. Then

$$(4.4)\qquad v_k^{(n)} = \begin{pmatrix} p_0(\lambda_{nk}) \\ p_1(\lambda_{nk}) \\ \vdots \\ p_{n-1}(\lambda_{nk}) \end{pmatrix}, \quad k = 1,2,\ldots,\ldots,n$$

is an eigenvector of J_n for λ_{nk}. From the Christoffel-Darboux identity (2.23) it follows that

$$\begin{aligned}(4.5)\qquad v_k^{(n)} \cdot v_j^{(n)} &= \sum_{i=0}^{n-1} p_i(\lambda_{nk}) p_i(\lambda_{nj}) \\ &= b_{n-1}(p_{n-1}(\lambda_{nk}) p_n(\lambda_{nj}) \\ &\quad - p_n(\lambda_{nk}) p_{n-1}(\lambda_{nj}))/(\lambda_{nj} - \lambda_{nk}) \\ &= 0\end{aligned}$$

for $k \neq j$. Hence, if

$$(4.6)\qquad \bar{v}_k^{(n)} = \frac{1}{\sqrt{\sum_{i=0}^{n-1} p_i^2(\lambda_{nk})}}\, v_k^{(n)}, \quad k = 1,2,\ldots,n,$$

then $(\bar{v}_1^{(n)},\ldots,\bar{v}_n^{(n)})$ is an orthonormal system of eigenvectors for J_n. If

$$(4.7)\qquad U_n = (\bar{v}_1^{(n)},\ldots,\bar{v}_n^{(n)})$$

then

$$(4.8\qquad U_n^T J_n U_n = \begin{pmatrix} \lambda_{11} & & 0 \\ & \ddots & \\ 0 & & \lambda_{nn} \end{pmatrix}$$

so that

$$(4.9) \qquad J_n = \sum_{k=1}^{n} \lambda_{nk}\ \mu_{nk}\ F_{nk}$$

where

$$(4.10) \qquad \mu_{nk} = \frac{1}{\sum_{i=0}^{n-1} p_i^2(\lambda_{nk})}$$

and

$$(4.11) \qquad F_{nk} = \nu_k^{(n)} (\nu_k^{(n)})^T = \nu_k^{(n)} \otimes \nu_k^{(n)}, \qquad k = 1,2,\ldots,n.$$

Observe that

$$(4.12) \qquad F_{nk} = (a_{ij})_{n\times n},$$

where

$$(4.13) \qquad a_{ij} = p_i(\lambda_{nk}) p_j(\lambda_{nk}), \qquad 0 \leqslant i,j \leqslant n-1.$$

Let L be the Jacobi operator on H with Jacobi matrix J for the orthonormal basis $\{e_n \mid n \geqslant 0\}$, and let L_n be the operator having matrix

$$(4.14) \qquad J_{(n)} = \begin{pmatrix} J_n & 0 \\ 0 & 0 \end{pmatrix}$$

relative to $\{e_n \mid n \geqslant 0\}$. If

$$(4.15) \qquad U_{(n)} = \begin{pmatrix} U_n & 0 \\ 0 & 0 \end{pmatrix}$$

then

$$(4.16) \qquad U_{(n)}^T J_{(n)} U_{(n)} \left(\begin{array}{ccc|c} \lambda_{n1} & & 0 & \\ & \ddots & & 0 \\ 0 & & \lambda_{nn} & \\ \hline & 0 & & 0 \end{array}\right)$$

Let

(4.17) $$\bar{E}_{nk} = \begin{pmatrix} F_{nk} & 0 \\ 0 & 0 \end{pmatrix}.$$

Then

(4.18) $$L_n = \sum_{k=1}^{n} \lambda_{nk} \mu_{nk} \bar{E}_{nk}.$$

If we define

(4.19) $$E_{n\lambda} = \begin{cases} 0 & \lambda < \lambda_{n1}, \\ \sum_{i=1}^{k} \mu_{ni}\bar{E}_{ni}, & \lambda_{nk} \leqslant \lambda < \lambda_{nk+1}, \quad k \geqslant 1, \\ I, & \lambda \geqslant \lambda_{nn}, \end{cases}$$

Then (4.18) becomes

$$L_n = \int_{-\infty}^{+\infty} \lambda dE_{n\lambda}$$

Since L_n is wholly defined on H, then

(4.20) $$(L_n x ; y) = \int_{-\infty}^{+\infty} \lambda d(E_{n\lambda} x ; y), \qquad x, y \in H.$$

Obviously $(E_{n\lambda})$ is a right-continuous spectral family for L_n. Observe that $L_n x = Lx$ as long as x is in the subspace spanned by $\{e_0,\ldots,e_{n-1}\}$. More generally, $L_n^m x = L^m x$ for x in the span S_{n-m} of $\{e_0,\ldots,e_{n-m}\}$, so that

(4.21) $$p(L_n)x = p(L)x, \qquad x \in S_{n-m}$$

as long as $p(x)$ has degree $\leqslant m$. For $p(x) \in \mathbb{C}[x]$,

(4.22) $$p(L_n) = \int_{-\infty}^{+\infty} p(\lambda) dE_{n\lambda}.$$

Hence, if $k + m \leqslant n$ then

$$(p_m(L)e_o;p_k(L)e_o) = \int_{-\infty}^{+\infty} p_m(\lambda)p_k(\lambda)d(E_{n\lambda}e_o;e_o) = \delta_{mk}$$

If we now define

(4.23) $$\sigma_n(\lambda) = (E_{n\lambda}e_o;e_o), \quad \lambda \in \mathbb{R},$$

then $0 \leq \sigma_n(\lambda) \leq 1$. Also $\sigma_n(\lambda) \leq \sigma_n(\lambda')$ if $\lambda \leq \lambda'$. If for $\lambda \in \mathbb{R}$ we write $I_\lambda = [0,1]$ and $\mathbb{R}_\lambda = \mathbb{R}$, then

(4.24) $$\sigma_n \in \prod_{\lambda\in\mathbb{R}} I_\lambda \subseteq \prod_{\lambda\in\mathbb{R}} \mathbb{R}_\lambda = F(\mathbb{R},\mathbb{R}) ,$$

where $F(\mathbb{R},\mathbb{R})$ is the space of all real valued functions on $\mathbb{R}$. Since $\prod_{\lambda\in\mathbb{R}} I_\lambda$ is a compact subset of $F(\mathbb{R},\mathbb{R})$, as follows from Tichonov's Theorem (Simmons [26], Chap.3, §23), there are $\sigma \in \prod_{\lambda\in\mathbb{R}} I_\lambda$ and a subsequence (σ_{n_k}) of (σ_n) such that

(4.25) $$\sigma_{n_k}(\lambda) \longrightarrow \sigma(\lambda)$$

for all $\lambda \in \mathbb{R}$. Clearly

(4.26) $$\sigma(\lambda) \leq \sigma(\lambda') \qquad \lambda \leq \lambda'.$$

From the so called Helly,s second selection principle (Chihara [10]; Bourbaki [4], Chap.III,IV) we get

(4.27) $$\lim_{k\to\infty} \int_{-\infty}^{+\infty} \phi\, d\sigma_{n_k} = \int_{-\infty}^{+\infty} \phi\, d\sigma , \qquad \phi \in C_o(\mathbb{R}) .$$

By replacing σ by $\hat{\sigma}(\lambda) = \sigma(\lambda+0)$, if neccessary, we may assume that σ is right continuous.

THEOREM 4.1. *The measure μ on $\mathbb{R}$ defined by the bounded non-decreasing function σ is an orthonormality measure for the system $\{p_n(x)\}$. Furthermore, Supp $\mu \in [-M,M]$, and μ is the only orthonormality measure for $\{p_n(x)\}$.*

Proof. In fact, for all $n \geq 1$, $Supp\ d\sigma_n = Z_n = \{\lambda_{n1}, \lambda_{n2}, \ldots, \lambda_{nn}\}$. From Corollary 3.4, there is a unique orthonormality measure ν for $\{p_n(x)\}$ such that $Supp \nu \subseteq [-M,M]$, and from Corollary 2.1, $Z_n \subseteq (-M,M)$. Hence $Supp\ \mu \subseteq [-M,M]$, and from (4.25) and (4.27) we get

$$(4.28) \qquad \int_{-\infty}^{+\infty} p_m(\lambda)p_n(\lambda)\,d\mu = \lim_{k\to\infty} \int_{M}^{M} p_m(\lambda)p_n(\lambda)\,d\sigma_{n_k}(\lambda) = \delta_{mn},$$

so that μ is in fact the orthogonality measure ν for $\{p_n(x)\}$ ▲

REMARK 4.1. If (E_λ) is the right continuous spectral family for L then

$$(4.29) \qquad \sigma(\lambda) = (E_\lambda e_o\,;\,e_o).$$

Now let $\lambda \notin [-M,M]$. From the general theory of operators in Hilbert space (Gómez [12], Lang [19], Yosida[29]),

$$(4.30) \qquad R_\lambda(L_n) = (L_n - \lambda I)^{-1} = \int_{-\infty}^{+\infty} \frac{dE_{n\lambda}}{t-\lambda}$$

is the resolvent operator for L_n. We have

$$(4.31) \qquad (R_\lambda(L_n)e_o;e_o) = \sum_{k=1}^{n} \frac{\mu_{nk}}{\lambda_{nk}-\lambda}(\bar{E}_{nk}e_o;e_o)$$

From (4.13)

$$(4.32) \qquad (\bar{E}_{nk}e_o;e_o) = p_o^2(\lambda_{nk}) = 1,$$

so that

$$(4.33) \qquad (R_\lambda(L_n)e_o;e_o) = \sum_{k=1}^{n} \frac{\mu nk}{\lambda_{nk}-\lambda}.$$

Now let

$$(4.34)\qquad \frac{p_n^*(\lambda)}{p_n(\lambda)} = \sum_{k=1}^{n} \frac{\alpha_{nk}}{\lambda_{nk}-\lambda}$$

be the partial fractions decomposition of $p_n^*(\lambda)/p_n(\lambda)$. Then

$$(4.35)\qquad \alpha_{nk} = \lim_{\lambda\to\lambda_{nk}} (\lambda_{nk}-\lambda)\frac{p_n^*(\lambda)}{p_n(\lambda)} = -\frac{p_n^*(\lambda_{nk})}{p_n'(\lambda_{nk})}$$

From (2.23) and (2.22) it follows that

$$(4.36)\qquad p_n(\lambda_{nk})p_{n-1}(\lambda_{nk}) = \frac{1}{b_{n-1}} \sum_{i=1}^{n-1} p_i^2(\lambda_{nk})$$

and

$$(4.37)\qquad p_{n-1}(\lambda_{nk})p_n^*(\lambda_{nk}) = \frac{1}{b_{n-1}}\,, \qquad n \geqslant 1,$$

so that

$$(4.38)\qquad \alpha_{nk} = -\frac{1}{\sum_{i=0}^{n-1} p_i^2(\lambda_{nk})} = -\mu_{nk}\,.$$

Thus

$$(4.39)\qquad \frac{p_n^*(\lambda)}{p_n(\lambda)} = -(R_\lambda(L_n)e_o;e_o) = -\sum_{k=1}^{n} \frac{\mu_{nk}}{\lambda_{nk}-\lambda} = \int_{-\infty}^{+\infty} \frac{d\sigma_n(t)}{\lambda-t}\,.$$

We can now prove Markov's theorem

THEOREM 4.2. (Markov) *If* (4.1) *holds and if* μ *is the orthonormality measure of the system* $\{p_n(x)\}$, *then*

$$(4.40)\qquad \chi(\lambda) = \int_{-\infty}^{+\infty} \frac{d\mu(t)}{\lambda-t}\,, \qquad \lambda \notin [-M,M].$$

Furthermore, for some subsequence $\{n_k\}$ *of* $\mathbb{N}$,

$$(4.41) \qquad \lim_{k\to\infty} \frac{p^*_{n_k}(\lambda)}{p_{n_k}(\lambda)} = \int_{-\infty}^{+\infty} \frac{d\mu(t)}{\lambda - t}$$

uniformly on compact subsets of $\mathbb{C}-[-M,M]$.

Proof. Let σ_n be given by (4.23) and let (σ_{n_k}) be a subsequence of (σ_n) such that (4.25) and (4.27) fold. Let μ be the positive measure determined by σ and let K be a compact subset of $\mathbb{C}-[-M,M]$ and $a_o < -M < M < b_o$ be such that $K \cap [a_o,b_o] = \emptyset$. If ϕ is continuous, $0 \leqslant \phi \leqslant 1$, and $\phi = 1$ on $[a_o,b_o]$, then

$$\sup_{\lambda\in K} \left| \frac{p^*_{n_k}(\lambda)}{p_{n_k}(\lambda)} - \int_{a_o}^{b_o} \frac{d\sigma(t)}{\lambda - t} \right| \leqslant \sup_{\substack{\lambda\in K\\ t\in[a_o,b_o]}} \left|\frac{1}{\lambda - t}\right| \left| \int_{a_o}^{b_o} \phi\, d\sigma - \int_{a_o}^{b_o} \phi\, d\sigma_{n_k} \right|.$$

Since

$$\int_{a_o}^{b_o} \phi\, d\sigma_{n_k} \longrightarrow \int_{a_o}^{b_o} \phi\, d\sigma$$

and

$$\int_{-\infty}^{+\infty} \frac{d\mu(t)}{\lambda - t} = \int_{a_o}^{b_o} \frac{d\sigma(t)}{\lambda - t}, \qquad \lambda \in K,$$

the proof is complete. ▲

REMARK 4.2. It follows from (4.40) that $\chi(\lambda)$ is analytic in $\mathbb{C}-[-M,M]$. As a mather of fact, $\chi(\lambda)$ is analytic on $\mathbb{C}$-*Supp* μ. Formula (4.40) says that $\chi(\lambda)$ is the *Cauchy-Stieltjes transform* of the measure $-2\pi i\mu$. Hence, *Stieltjes' inversion formula* (Bremermann [5], Gómez [12], Lang [19]) applies to give

$$(4.41) \quad \int_{-\infty}^{+\infty} \phi\, d\mu = \lim_{\varepsilon\to 0^+} \frac{1}{2\pi i} \int_{-\infty}^{+\infty} (\chi(t-i\varepsilon) - \chi(t+i\varepsilon))\phi(t)\, dt, \quad \phi \in C_o(\mathbb{R})$$

or what is the same,

$$(4.42)\quad \sigma(\lambda)-\sigma(\lambda_o-0) = \int_{\lambda_o}^{\lambda} d\mu = \lim_{\varepsilon\to 0^+} \frac{1}{2\pi i}\int_{\lambda_o}^{\lambda} (\chi(t-i\varepsilon)-\chi(t+i\varepsilon))dt\ ,$$

$\lambda,\lambda_o \in \mathbb{R}$.

Formulae (4.41) and (4.42) allow the recovery of μ from the continued fraction $\chi(\lambda)$. They are basic in effectively computing μ.

REMARK 4.3. It can be shown (Ahkiezer [1], Wall [28]) that

$$(4.43)\qquad \lim_{n\to\infty} \frac{p_n^*(\lambda)}{p_n(\lambda)} = \int_{-\infty}^{+\infty} \frac{d\mu(t)}{\lambda-t}$$

uniformly on compact sets of $\mathbb{C}-[-M,M]$, so that the assumption on the existence of $\chi(\lambda)$ is really superfluous. The importance of formula (4.43) explains the name *numerator polynomials* given to the $p_n^*(x)$. In practice the existence of $\chi(\lambda)$ is usually established independently of (4.43) by means of asymptotic methods. We mention that Markov's theorem holds in more general situation than the case of compact supports we have considered. These general situations are intimately related to the question of uniqueness of the orthonormality measure and therefore to the moment problem. Ahkiezer [1] and Shohat and Tamarkin [25] point in this direction.

REMARK 4.4. Let μ_n be the measure determined by σ_n, where σ_n is given by (4.23). Since σ_n is continuous except for $\lambda = \lambda_{nk}$, $k = 1,2,\ldots,n$, and

$$(4.44)\qquad \mu_n(\{\lambda_{nk}\}) = \sigma_n(\lambda_{nk})-\sigma_n(\lambda_{nk}-0) = \mu_{nk} = \frac{1}{\sum_{i=0}^{n-1} p_i^2(\lambda_{nk})}$$

it follows that μ_n has a mass with value μ_{nk} at each $\lambda = \lambda_{nk}$, and no other masses. Recall that the λ_{nk}'s are eigenvalues of L_n.

Also, $\lambda \in \mathbb{R}$ is an eigenvalue of L if and only if $\sum_{n=0}^{\infty} p_n^2(\lambda) < +\infty$. It is a deep result of Tchebichev that in such case the mass μ_λ of the orthonormality measure μ of the $p_n(x)$'s at $x = \lambda$ is given by

$$\text{(4.45)} \qquad \mu_\lambda = \mu(\{\lambda\}) = \frac{1}{\sum_{n=0}^{\infty} p_n^2(\lambda)} .$$

These are the only masses of μ. Ahkiezer [1], Chihara [10], Shohat and Tamarkin [25] and Szego [27] are good references for the results of Tchebichev.

Let λ be an isolated point of $\sigma(\bar{L})$, where $\bar{L}$ is the closure operator of L. Let $\lambda_1 < \lambda < \lambda_2$ be such that $[\lambda_1, \lambda_2] \cap \sigma(\bar{L}) = \{\lambda\}$. Then σ as given by (4.29) is constant on $[\lambda_1, \lambda]$ and on $[\lambda, \lambda_2]$. If $\sigma(\lambda) = \sigma(\lambda-0)$ then σ wold be constant on $[\lambda_1, \lambda_2]$ and $\lambda \notin \sigma(\bar{L})$. Hence, λ is an eigenvalue of $\bar{L}$. Now let R_ε denote the rectangle of vertices $\lambda_2 - i\varepsilon$, $\lambda_2 + i\varepsilon$, $\lambda_1 + i\varepsilon$, $\lambda_1 - i\varepsilon$ and give the boundary ∂R_ε of R_ε the positive orientation. Then

$$\text{(4.46)} \qquad \frac{1}{2\pi i} \int_{\partial R_\varepsilon} \chi(\xi)\, d\xi = Res(\chi, \lambda) .$$

On the other hand

$$\int_{\partial R_\varepsilon} \chi(\xi)\, d\xi = i \int_{-\varepsilon}^{\varepsilon} (\chi(\lambda_2 + it) - \chi(\lambda_1 + it))\, dt$$

$$+ \int_{\lambda_1}^{\lambda_2} (\chi(t - i\varepsilon) - \chi(t + i\varepsilon)\, dt ,$$

and a continity argument shows that

$$\lim_{\varepsilon\to 0} \int_{-\varepsilon}^{\varepsilon} (\chi(\lambda_2+it) - (\lambda_1+it))dt = 0$$

Since $\sigma(\lambda_1-0) = \sigma(\lambda_1)$ it follows from (4.42) that

$$\sigma(\lambda_2)-\sigma(\lambda_1) = \lim_{\varepsilon\to 0} \frac{1}{2\pi 1} \int_{\partial R_\varepsilon} \chi(\xi)\ \xi = Res(\chi,\lambda),$$

i.e., that

$$(4.47)\qquad \mu(\{\lambda\}) = \sigma(\lambda)-\sigma(\lambda-0) = Res(\chi,\lambda).$$

Formula (4.47) provides a way to obtain the masses of μ at isolated points of the spectrum. For non isolated points of discontinuity of σ, (4.45) must be used.

§5. A glimpse at Darboux's asymptotic method. Darboux's asymptotic method is useful in the determination of the continued fraction $\chi(\lambda)$. We give in this section a brief description of that method. Much more detailed analysis can be found in Fields [11], Olver [21], Szego [27]. Multiple applications are in Ismail [14], [15], [16], [17], Ismail and Mulla [18], Bank and Ismail [3] (where applications are given to quantum theory) , in Charris and Ismail [8], [9] and in Bustoz and Ismail [7].

First we recall that

$$(5.1)\qquad \lim_{x\to\pm\infty} \int_a^b e^{ixt}\ g(t)dt = 0,$$

if g is piecewise continuous on $[a, b]$, $-\infty < a < b < +\infty$. This is the Riemann-Lebesgue lemma of Fourier analysis (Olver [21]). Also recall that $a_n = o(b_n)$, $n\to\infty$, means

(5.2) $$\lim_{n\to\infty} \frac{a_n}{b_n} = 0$$

and that $a_n \sim b_n$, $n \to \infty$, stands for

(5.3) $$\lim_{n\to\infty} \frac{a_n}{b_n} = 1$$

Darboux's method is based on the following theorem (Olver [21]).

THEOREM 5.1. (Darboux) *Let*

(5.4) $$f(z) = \sum_{n=0}^{\infty} a_n z^n, \quad g(z) = \sum_{n=0}^{\infty} b_n z^n$$

both have radius of convergence R, $0 < R < +\infty$, *and assume that* $h(z) = f(z) - g(z)$ *is continuous on* $|z| \leq R$. *Then*

(5.5) $$a_n = b_n + o(\frac{1}{R^n}).$$

If we further assume that $\bar{h}(\theta) = h(Re^{i\theta})$, $\theta \in [0,2\pi]$, *is* C^m, $0 \leq m < +\infty$, *then*

(5.6) $$a_n = b_n + o(\frac{1}{n^m R^n}).$$

Proof. Since

(5.7) $$a_n - b_n = \frac{1}{2\pi i} \int_{|z|=R} \frac{h(z)}{z^{n+1}}\, dz = \frac{1}{2\pi R^n} \int_0^{2\pi} \bar{h}(\theta) e^{-in\theta}\, d\theta,$$

(5.5) follows from the Riemann-Lebesgue Lemma. If $\bar{h}$ is C^m, from $\bar{h}(0) = \bar{h}(2\pi)$, $e^{-2in\pi} = 1$, and

$$\int_0^{2\pi} \bar{h}(\theta) e^{-in\theta} d\theta = \frac{1}{n^m i^m} \int_0^{2\pi} \bar{h}^{(m)}(\theta) e^{in\theta} d\theta,$$

(5.6) follows at once.

The following consequence of Theorem 5.1 is specially suited for the applications we have in mind.

COROLLARY 5.1. *If* $\bar{h}$ *is* C^m, $m \geq 0$, *and if*

$$(5.8) \qquad n^{\alpha} R^n b_n \to C \neq 0, \qquad 0 \leq \alpha \leq m,$$

then

$$(5.9) \qquad a_n \sim b_n \qquad n \to \infty$$

Proof. In fact, $a_n - b_n = o(1/n^m R^n)$, so that

$$a_n / b_n = 1 + \frac{1}{n^{m-\alpha}} \frac{o(1/n^m R^n)/(1/n^m R^n)}{n^{\alpha} R^n b_n} \to 1 \ . \quad \blacktriangle$$

The above result usually applies to a subsequense $n_k^{\alpha} R^{n_k} b_{n_k} \to C \neq o$. This is enough for many purposes.

REMARK 5.2. If $a_n \sim b_n$, $a_n^* \sim b_n^*$, $n \to \infty$, and if

$$\lim_{n\to\infty} \frac{b_n^*}{b_n} = c \neq 0,$$

then also

$$\lim_{n\to\infty} \frac{a_n^*}{a_n} = \lim_{n\to\infty} \frac{a_n^*}{b_n^*} \cdot \frac{b_n^*}{b_n} \cdot \frac{b_n}{a_n} = c.$$

In the next section we give an example showing how Darboux's method can be applied to the theory of orthogonal polynomials. We remark that Darboux's method has a much wider scope than the simple ideas of the above description.

§6. An example. In this section we give an example showing how a combination of Darboux's asymptotic method, Markov's theorem and the Stieltjes inversion formula can provide the tools for fully determining the orthogonality measure of systems of polynomials given by recurrence relations.

The system $\{S_n(x) \mid n \geq 0\}$ we have selected has been introduced by Chihara [10]. It is simple enough and has some interesting features. Among them, the orthogonality breaks into disjoint intervals and the discrete spectrum is nonempty. This allows for neat application of some of the basic results of previous sections. The recurrence relation is

$$(6.1) \qquad S_{n+1}(x) = xS_n(x) - \gamma_{n+1}S_{n-1}(x), \quad n \geq 1,$$

where

$$(6.2) \qquad \gamma_{2n} = a, \quad \gamma_{2n+1} = b, \quad a > 0,\ b > 0,\ n \geq 0.$$

The initial conditions are

$$(6.3) \qquad S_0(x) = 1, \quad S_1(x) = x.$$

The *system of numerator* polynomials $\{S_n^*(x) \mid n \geq 0\}$ satisfies (6.1) and the initial conditions

$$(6.4) \qquad S_0^*(x) = 0, \quad S_1^*(x) = 1.$$

To establish the generating function

$$(6.5) \qquad S(z,t) = \sum_{n=0}^{\infty} S_n(z)t^n$$

we write

$$(6.6) \quad S_\rho(z,t) = \sum_{n=0}^{\infty} S_{2n}(z)t^{2n}, \quad S_\sigma(z,t) = \sum_{n=0}^{\infty} S_{2n+1}(z)t^{2n+1},$$

so that $S(z,t) = S_\rho(z,t) + S_\sigma(z,t)$. Multiplying both sides of (6.1) by t^{n+1} and adding we obtain

$$(6.7) \quad \sum_{n=1}^{\infty} S_{n+1}(z)t^{n+1} = zt\sum_{n=1}^{\infty} S_n(z)t^n - t^2\sum_{n=1}^{\infty}\gamma_{n+1}S_{n-1}(z)t^{n-1},$$

so that

$$(6.8) \quad \sum_{n=1}^{\infty} S_{2n+1}(z)t^{2n+1} = zt\sum_{n=1}^{\infty} S_{2n}(z)t^{2n} - bt^2\sum_{n=1}^{\infty} S_{2n-1}(z)t^{2n-1}$$

and also

$$(6.9)\quad \sum_{n=1}^{\infty} S_{2n+2}(z)t^{2n+2} = zt\sum_{n=1}^{\infty} S_{2n+1}(z)t^{2n+1} - at^2\sum_{n=1}^{\infty} S_{2n}(z)t^{2n}.$$

From (6.8) it follows that

$$(6.10)\quad S_\sigma(z,t) - S_1(z)t = zt[S_\rho(z,t) - S_o(z)] - bt^2 S_\sigma(z,t)$$

and from (6.9), that

$$(6.11)\quad S_\rho(z,t) - S_2(z)t^2 - S_o(z) = zt[S_\sigma(z,t) - S_1(z)t] - at^2[S_\rho(z,t) - S_o(z)].$$

Relations (6.10) and (6.11) lead respectively to

$$(6.12)\quad [1+bt^2]S_\sigma(z,t) = ztS_\rho(z,t)$$

and

$$(6.13)\quad [1+at^2]S_\rho(z,t) = zt\,S_\sigma(z,t) + 1.$$

Hence

$$(6.14)\quad S_\rho(z,t) = \frac{1+bt^2}{1+(a+b-z^2)t^2+abt^4}$$

and

$$(6.15)\quad S_\sigma(z,t) = \frac{2t}{1+(a+b-z^2)t^2+abt^4}$$

Thus

$$(6.16)\quad S(z,t) = \frac{1+zt+bt^2}{1+(a+b-z^2)t^2+abt^4}$$

Entirely similar calculations lead to

$$(6.17)\quad S^*(z,t) = \frac{t(1+zt+at^2)}{1+(a+b-z^2)t^2+abt^4} = \sum_{n=o}^{\infty} S_n^*(z)t^n.$$

Now let $\sqrt{z-1}$ be the branch of the square root of z-1 on $\mathbb{C}$ which is analytic for $z \in \mathbb{C}-(-\infty,1]$ and positive for $z \in \mathbb{R}$, $z > 1$. Also let $\sqrt{z+1}$ be analytic for $\mathbb{C}-(-\infty,-1]$ and positive for $z \in \mathbb{R}$, $z > -1$. If is easily verified that

$$(6.18) \qquad \sqrt{z^2-1} = \sqrt{z-1}\,\sqrt{z+1} \quad (*)$$

is continuous on $\mathbb{C}-[-1,1]$. Hence, it is analytic on $\mathbb{C}-[-1,1]$. *A straightforward argument* shows that

$$\sqrt{x^2-1} = \begin{cases} \sqrt{x^2-1}, & x > 1, \\ i\sqrt{1-x^2}, & -1 \leqslant x < 1, \\ -\sqrt{x^2-1}, & x \leqslant -1. \end{cases}$$

Let

$$(6.19) \qquad \alpha(z) = z + \sqrt{z^2-1}, \qquad \beta(z) = z - \sqrt{z^2-1}.$$

Both $\alpha(z)$ and $\beta(z)$ are analytic on $\mathbb{C}-[-1,1]$ and

$$(6.20) \qquad \alpha(z) + \beta(z) = 2z, \qquad \alpha(z)\beta(z) = 1.$$

Hence $\alpha(z)$ and $\beta(z)$ are analytic selections in $\mathbb{C}-[-1,1]$ of the roots of the equation $z^2-2zt+1 = 0$. On the other hand, $[-1,1]$ is a set of branch discontinuities of both $\alpha(z)$ and $\beta(z)$. As a matter of fact, for $-1 \leqslant x \leqslant 1$,

$$(6.21) \qquad \lim_{\varepsilon\to 0} \alpha(x+i\varepsilon) = \begin{cases} \alpha(x), & \varepsilon > 0, \\ \beta(x), & \varepsilon < 0. \end{cases}$$

and also

$$(6.22) \qquad \lim_{\varepsilon\to 0} \beta(x+i\varepsilon) = \begin{cases} \beta(x), & \varepsilon > 0, \\ \alpha(x), & \varepsilon < 0, \end{cases}$$

so that

$$(6.23) \qquad \lim_{\varepsilon\to 0+} \alpha(x+i\varepsilon)-\alpha(x-i\varepsilon) = 2i\sqrt{1-x^2} = \lim_{\varepsilon\to 0+} (\beta(x-i\varepsilon)-\beta(x+i\varepsilon))$$

(*) It is important to observe the difference in the notations $\sqrt{}$ **and** $\sqrt{}$

It is not difficult to prove that $|\alpha(z)| = |\beta(z)|$ if and only if $-1 \leqslant z \leqslant 1$. Since $\alpha(x) > \beta(x)$ for $x > 1$, it follows that $|\beta(z)| < |\alpha(z)|$ for z in $\mathbb{C}-[-1,1]$.

More details about the functions $\alpha(z)$, $\beta(z)$ and $\sqrt{z^2-1}$ can be found in [9]. They seem to play an important role throughout the theory of orthogonal polynomials.

From (6.16) it follows, with

$$\omega = \frac{z^2-a-b}{2\sqrt{ab}}, \tag{6.24}$$

that

$$S(z,t) = \frac{1+zt+bt^2}{ab\left(t^2-\frac{\alpha(\omega)}{\sqrt{ab}}\right)\left(t^2-\frac{\beta(\omega)}{\sqrt{ab}}\right)} \tag{6.25}$$

and

$$S^*(z,t) = \frac{t(1+zt+at^2)}{ab\left(t^2-\frac{\alpha(\omega)}{\sqrt{ab}}\right)\left(t^2-\frac{\beta(\omega)}{\sqrt{ab}}\right)}. \tag{6.26}$$

Now we recall that $|\alpha(\omega)| = |\beta(\omega)|$ if and only if $\omega \in [-1,1]$, which amounts to $z \in I$, where

$$I = [-\sqrt{a}-\sqrt{b}, -|\sqrt{a}-\sqrt{b}|] \cup [|\sqrt{a}-\sqrt{b}|, \sqrt{a}+\sqrt{b}]. \tag{6.27}$$

Note that if $a \neq b$ then I is the union of two disjoint intervals. If $z \notin I$ then $|\beta(\omega)| < |\alpha(\omega)|$, so that $S(z,t)$ and $S^*(z,t)$ are analytic functions of t for $|t| < r(z)$, where

$$r(z) = \sqrt{|\beta(\omega)|/\sqrt{ab}}, \tag{6.28}$$

and both have singularities at $t^2 = \beta(\omega)/\sqrt{ab}$ on $|t| = r(z)$. If

$$\tilde{S}(z,t) \quad \frac{1+zt+bt^2}{\sqrt{ab}(\beta-\alpha)(t^2-\beta/\sqrt{ab})}$$

(6.29)

$$\tilde{S}^*(z,t) = \frac{t(1+zt+at^2)}{\sqrt{ab}(\beta-\alpha)(t^2-\beta/\sqrt{ab})}$$

then

$$S-\tilde{S} = \frac{1+zt+bt^2}{\sqrt{ab}(\alpha-\beta)(t^2-\alpha/\sqrt{ab})}$$

(6.30)

$$S^*-\tilde{S}^* \quad \frac{t(1+zt+at^2)}{\sqrt{ab}(\alpha-\beta)(t^2-\alpha/\sqrt{ab})}$$

are analytic for $|t| < r(z)$ and continuous for $|t| \leq r(z)$; *i.e.*, $\tilde{S}$ and $\tilde{S}^*$ are comparison functions for S and S^*, respectively. Furthermore, $\overline{(S-\tilde{S})}(r(z)e^{i\theta})$ and $(S^*-\tilde{S}^*)(r(z)e^{i\theta})$ are C^∞ functions of θ. Since

(6.31) $$\tilde{S}(z,t) = \frac{1}{\beta(\alpha-\beta)}\{1+zt+\sum_{n=1}^{\infty}\left(\frac{\sqrt{ab}}{\beta}\right)^n\left[\left(1+\beta\sqrt{\frac{b}{a}}\right)t^{2n}+zt^{2n+1}\right]\}$$

and similarly

(6.32) $$S^*(z,t) = \frac{1}{\beta(\alpha-\beta)}\{t+\sum_{n=1}^{\infty}\left(\frac{\sqrt{ab}}{\beta}\right)^n\left[\left(1+\sqrt{\frac{a}{b}}\beta\right)t^{2n+1}+\frac{z}{\sqrt{ab}}\beta t^{2n}\right]\},$$

Darboux's aymptotic method readily gives

(6.33) $$S_{2n}(z) \sim \left(\frac{\sqrt{ab}}{\beta}\right)^n\left[1+\beta\sqrt{\frac{b}{a}}\right]\frac{1}{\beta(\alpha-\beta)},$$

(6.34) $$S_{2n+1}(z) \sim z\left(\frac{\sqrt{ab}}{\beta}\right)^n\frac{1}{\beta(\alpha-\beta)},$$

(6.35) $$S^*_{2n}(z) \sim \left(\frac{\sqrt{ab}}{\beta}\right)^n\frac{z}{\sqrt{ab}}\frac{1}{\alpha-\beta},$$

(6.36) $$S^*_{2n+1}(z) \sim \left(\frac{\sqrt{ab}}{\beta}\right)^n\left[1+\sqrt{\frac{a}{b}}\beta\right]\frac{1}{\beta(\alpha-\beta)}$$

and

(6.37) $$\frac{S^*_{2n}(z)}{S_{2n}(z)} \sim \frac{z\beta(\omega)}{\sqrt{ab}+b\beta(\omega)} \qquad \frac{S^*_{2n+1}(z)}{S_{2n+1}(z)} \sim \frac{a\beta(\omega)+\sqrt{ab}}{\sqrt{ab}\;z}$$

Straightforward calculation shows that

$$(6.38)\quad \chi(z) = \frac{z\beta(\omega)}{\sqrt{ab}+b\beta(\omega)} = \frac{a\beta(\omega)+\sqrt{ab}}{\sqrt{ab}\ z}, \quad z \notin I \cup \{0\}$$

If $x \in \mathbb{R}$ then

$$(6.39)\quad \chi(x \pm i\varepsilon) = \frac{a\beta(\omega(x\pm i\varepsilon))+\sqrt{ab}}{(x\pm i\varepsilon)\sqrt{ab}}$$

Since

$$(6.40)\quad \omega(x\pm i\varepsilon) = \left[\omega(x) - \varepsilon^2/(2\sqrt{ab})\right] \pm i\left[(\varepsilon/\sqrt{ab})x\right],$$

if

$$(6.41)\quad \chi_\varepsilon(x) = \chi(x-i\varepsilon)-\chi(x+i\varepsilon)$$

then, from (6.23),

$$(6.42)\quad \lim_{\varepsilon\to 0^+} \chi_\varepsilon(x) = \begin{cases} \dfrac{2ai}{\sqrt{ab}|x|}\sqrt{1-\left(\dfrac{x^2-a-b}{2\sqrt{ab}}\right)^2}, & x \in I,\ x \neq 0 \\ 0, & x \notin I \cup \{0\} \end{cases}$$

If $a = b$ then

$$(6.43)\quad \lim_{\varepsilon\to 0^+} \chi_\varepsilon(0) = 2i\,\frac{\sqrt{a}}{a}.$$

If $a \neq b$, 0 is an isolated singularity of $\chi(z)$. Since

$$(6.44)\quad \lim_{z\to 0} z\chi(z) = \begin{cases} (b-a)/b, & b > a \\ 0, & a > b, \end{cases}$$

then 0 is a removable isolated singularity with $\chi(0) = 0$ if $a > b$ and a simple pole with residue $(b-a)/b$ if $b > a$. In establishing (6.44) it is important to observe that $\omega(0) = -(a+b)/\ 2\sqrt{ab} < -1$, so that

$$(6.45)\quad \beta(\omega(0)) = \omega(0) - \sqrt{\omega(0)^2-1}$$

$$= \omega(0) + \sqrt{\omega(0)^2-1} = -\frac{a+b-|a-b|}{2\sqrt{ab}}$$

If we define

$$(6.46) \qquad \tilde{\chi}(x) = \lim_{\varepsilon \to 0^+} \chi_\varepsilon(x), \qquad x \quad \mathbb{R}, \in x \neq 0$$

and

$$(6.47) \qquad \tilde{\chi}(0) = \begin{cases} 2i\,\dfrac{\sqrt{a}}{a}\,, & a = b \\ 0\,, & a \neq b\,, \end{cases}$$

Then $\tilde{\chi}(x)$ is continuous on $\mathbb{R}$ if $a \geqslant b$ and on $\mathbb{R}$-{0} if $b > a$. In any case, χ_ε converges to $\tilde{\chi}$ uniformly on I. Since χ vanishes at the end points of I, then

THEOREM 6.1. *The orthogonality measure* μ *for the system* $\{S_n(x) \mid n \geqslant 0\}$ *is given by*

$$(6.48) \qquad \int_{-\infty}^{+\infty} \phi\, d\mu = \frac{1}{2\pi i} \int_{-\infty}^{+\infty} \phi(x)\tilde{\chi}(x)\,dx + \mu_0 \phi(0)$$

where

$$(6.49) \qquad \tilde{\chi}(x) = \begin{cases} \dfrac{2ai}{\sqrt{ab}\,|x|} \sqrt{1 - \left(\dfrac{x^2\ \ a\ \ b}{2\sqrt{ab}}\right)^2}\,, & x \in I \\ 0\,, & x \notin I \end{cases}$$

and

$$(6.50) \qquad \mu_0 = \begin{cases} (b-a)/b\,, & a < b\,, \\ 0\,, & b \leqslant a\,. \end{cases}$$

REMARK 6.1. We recall that another way to determine whether 0 bears a mass of μ is by means of Theorem 3.1 applied to the system of orthonormal polynomials

$$(6.51) \qquad p_n(x) = \frac{S_n(x)}{\sqrt{\lambda_n}}, \qquad n \geqslant 0,$$

where

$$(6.52)\qquad \begin{aligned} \lambda_{2n} &= \gamma_o\gamma_2\cdots\gamma_{2n-1} = (ab)^n , \\ \lambda_{2n+1} &= \gamma_o\cdots\gamma_{2n} = a(ab)^n , \qquad n \geq 0. \end{aligned}$$

Noticing that $S_{2n+1}(0) = 0$ and $S_{2n}(0) = (-1)^n a^n$, we obtain

$$(6.53)\qquad \sum_{n=o}^{\infty} p_n^2(0) = \sum_{n=o}^{\infty} \left(\frac{a^n}{\sqrt{(ab)^n}}\right)^2 \sum_{n=o}^{\infty}\left(\frac{a}{b}\right)^n = \begin{cases} +\infty , & a \geq b \\ b/(b-a), & b > a \end{cases}$$

so that μ only bears a mass if $b > a$. Observe that as predicted by (4.45) we have

$$(6.54)\qquad \mu_o = \frac{1}{\sum_{n=o}^{\infty} p_n^2(0)} = \begin{cases} 0 , & a \geq b \\ (b-a)/b, & b > a \end{cases}$$

REMARK 6.2. The Jacobi matrix J for the system $\{p_n(x)\}$ given by (6.51) is

$$(6.55)\qquad J = \begin{bmatrix} 0 & \sqrt{a} & 0 & 0 & 0 & \cdot & \cdot & \cdot \\ \sqrt{a} & 0 & \sqrt{b} & 0 & 0 & \cdot & \cdot & \cdot \\ 0 & \sqrt{b} & 0 & \sqrt{a} & 0 & \cdot & \cdot & \cdot \\ 0 & 0 & \sqrt{a} & 0 & \sqrt{b} & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{bmatrix}$$

REMARK 6.3. Chihara [10] has given for the system $\{S_n(x)\}$ the representation

$$(6.56)\qquad \begin{cases} S_{2n}(x) = (ab)^{n/2}\left[U_n(\omega) + \sqrt{\frac{b}{a}}\, U_{n-1}(\omega)\right] \\ S_{2n+1}(x) = (ab)^{n/2}\, x\, U_n(\omega). \end{cases}$$

Here ω is given by (6.24) and

$$U_n(\cos\theta) = \frac{\sin(n+1)\theta}{\sin\theta}, \qquad n > 0,$$

is the n^{th} Tchebichev polynomial of the second kind.

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