GENERALIZED URYSOHN SPACES

by

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Abstract. In this paper two new generalizations of Urysohn spaces are investigated and used to obtain new characterizations of Urysohn spaces.

§1. Introduction. In 1961 A. Davis was interested in obtaining properties of topological spaces, which together with $T_{i-1}$ would be equivalent to $T_i$, $i = 1, 2$, respectively. Davis' 1961 investigation led to the following definitions and discoveries. A space $(X, T)$ is $R_0$ iff for each $0 \subseteq T$ and each $x \subseteq 0$, $\overline{\{x\}} = 0$ and $(X, T)$ is $R_1$ iff for each pair $x, y \subseteq X$ such that $\overline{\{x\}} \neq \overline{\{y\}}$, there exist disjoint open sets $U$ and $V$ such that $\overline{\{x\}} \subseteq U$ and $\overline{\{y\}} \subseteq V$ [3]. A space $(X, T)$ is $T_i$ iff it is $R_i-1$ and $T_{i-1}$, $i = 1, 2$, respectively [3].

The results above led to the introduction of weakly Urysohn spaces in this paper. A space $(X, T)$ is weakly Urysohn iff for each pair $x, y \subseteq X$ such that $\overline{\{x\}} \neq \overline{\{y\}}$, there exist disjoint open sets $U$ and $V$ such that $\overline{\{x\}} \subseteq U$, $\overline{\{y\}} \subseteq V$, and $\overline{U} \cap \overline{V} = \phi$.

In 1963 N. Levine introduced semi open sets. Let $(X, T)$ be a space and let $A \subseteq X$. Then $A$ is semi open, denoted by
A ∈ SO(X,T), iff there exists 0 ∈ T such that 0 ⊆ A ⊆ △, [17]. In 1970 semi open sets were used to define semi closed sets, which were used to define the semi closure of a set. Let (X,T) be a space and let A,B ⊆ X. Then A is semi closed iff X-A is semi open and the semi closure of B, denoted by scl B, is the intersection of all semi closed sets containing B [1]. In 1975 [18] T_≤ spaces were generalized to semi-T_≤ spaces by replacing the word open in the definition of T_≤ by semi open, i = 0,1,2, respectively. These new separation axioms raised questions about properties of topological spaces, which together with semi-T_≤, would be equivalent to T_≤, i = 0, 1,2, respectively. The investigation of these questions led to the following definitions and discoveries, which give additional answers to Davis' 1961 questions. Let (X,T) be a space and let R be the equivalence relation on X defined by xRy iff scl{x} = scl{y}. Then the semi-T_0-identification space of (X,T) is (X_S,Q_S(X,T)), where X_S is the set of equivalence classes of R and Q_S(X,T) is the decomposition topology on X_S [5]. The space (X,T) is s-essentially T_≤ iff (X_S,Q_S(X,T)) is T_≤ and (X,T) is T_≤ iff (X,T) is s-essentially T_≤ and semi-T_≤, i = 0 [6], i = 1 [7], and i = 2 [8]. These results led to the introduction of s-essentially Urysohn spaces in this paper. The space (X,T) is s-essentially Urysohn iff (X_S,Q_S(X,T)) is Urysohn.

In this paper weakly Urysohn and s-essentially Urysohn spaces are investigated and used to give new characterizations of Urysohn spaces. Throughout the remainder of this paper, for each space (X,T) the T_0-identification space of (X,T) will be denoted by (X_0,Q(X,T)), the natural map from (X,T) onto (X_0,Q(X,T)) will be denoted by P, the natural map from (X,T) onto (X_S,Q_S(X,T)) will be denoted by P_S, and for each x ∈ X, the element of X_0 containing x will be denoted by C_x and the element of X_S containing x will be denoted by K_x.
§2. Weakly Urysohn Spaces.

THEOREM 2.1. Let \((X,T)\) be a space. Then the following are equivalent: (a) for each pair \(x,y \in X\) such that \(\{x\} \neq \{y\}\), there exist disjoint open sets \(U\) and \(V\) such that \(x \in U\), \(y \in V\), and \(U \cap V = \emptyset\). (b) \((X,T)\) is weakly Urysohn, (c) \((X_0, Q(X,T))\) is Urysohn, (d) \((X_S, Q_S(X,T))\) is weakly Urysohn, and (e) every homeomorphic image of \((X,T)\) is weakly Urysohn, and (f) for each \(Y \subseteq X\), \((Y,T_Y)\) is weakly Urysohn.

Proof. (a) implies (b). Let \(0 \in T\) and let \(a \in 0\). Let \(b \in X - 0\). Then \(\{a\} \neq \{b\}\) and there exist disjoint open sets \(A\) and \(B\) such that \(a \in A\) and \(b \in B\), which implies \(b \notin \{a\}\). Thus \(\{a\} = 0\) and \((X,T)\) is \(R_0\). Let \(x,y \in X\) such that \(\{x\} \neq \{y\}\). Then there exist disjoint open sets \(U\) and \(V\) such that \(x \in U\), \(y \in V\), and \(U \cap V = \emptyset\). Since \((X,T)\) is \(R_0\), then \(\{x\} \subseteq U\) and \(\{y\} \subseteq V\).

(b) implies (c). Let \(C_x, C_y \in X_0\) such that \(C_x \neq C_y\). Then \(\{x\} \neq \{y\}\) and there exist disjoint open sets \(U\) and \(V\) such that \(\{x\} \subseteq U\), \(\{y\} \subseteq V\), and \(U \cap V = \emptyset\). Since \(P\) is continuous, closed, and open [9], then \(C_x \subseteq P(U) \subseteq Q(X,T)\), \(C_y \subseteq P(V) \subseteq Q(X,T)\), \(P(U) = \overline{P(U)}\), and \(P(V) = \overline{P(V)}\). Since \(P^{-1}(P(W)) = W\) for each \(W \subseteq T\) [9], then for each \(W \subseteq T\), \(P^{-1}(P(W)) = P^{-1}(\overline{P(W)}) = P^{-1}(\overline{P(W)}) = W\), which implies \(P(U) \cap P(V) = \emptyset\).

(c) implies (d). Let \(x,y \in X\) such that \(\{x\} \neq \{y\}\). Then \(C_x \neq C_y\) and there exist disjoint open sets \(U\) and \(V\) such that \(\{x\} \subseteq U\), \(\{y\} \subseteq V\), and \(U \cap V = \emptyset\). Since \(P_s\) is continuous and closed [5], then \(\{x\} \neq \{y\}\) and there exist disjoint open sets \(U\) and \(V\) such that \(\{u\} \subseteq U\), \(\{v\} \subseteq V\), and \(U \cap V = \emptyset\). Since \(P_s\) is continuous, closed, and open, and \(P_s^{-1}(P_s(W)) = W\) for all \(W \subseteq SO(X,T)\) [5], then \(\{K_{x}^T\} \subseteq P_s(U) \subseteq Q_s(X,T), \{K_{y}^T\} \subseteq P_s(V) \subseteq Q_s(X,T), \) and \(P_s(U) \cap P_s(V) = \emptyset\).

(d) implies (e). Let \(x,y \in X\) such that \(\{x\} \neq \{y\}\). Then \(\{K_x^T\} \neq \{K_y^T\}\) and there exist disjoint open sets \(U\) and \(V\)
such that \( \{K_x\} \subseteq U, \{K'_y\} \subseteq V \) and \( U \cap V = \emptyset \). Then \( x \in P_{S^{-1}(U)} \subseteq T, y \in P_{S^{-1}(V)} \subseteq T \), and \( P_{S^{-1}(U)} \cap P_{S^{-1}(V)} = P_{S^{-1}(U)} \cap P_{S^{-1}(V)} = \emptyset \). Thus \((X,T)\) is weakly Urysohn. The remainder of the proof is straightforward and omitted.

The straightforward proof that (e) implies (f) and (f) implies (a) is also omitted.

**Theorem 2.2.** For each \( \alpha \in A \) let \((X_\alpha,T_\alpha)\) be a nonempty space and let \( S \) be the product topology on \( \prod_{\alpha \in A} X_\alpha \). Then \((\prod_{\alpha \in A} X_\alpha,S)\) is weakly Urysohn iff \((X_\alpha,T_\alpha)\) is weakly Urysohn for each \( \alpha \in A \).

**Proof.** Let \( W \) be the product topology on the product of \( \{(X_\alpha) \circ Q(X_\alpha,T_\alpha) \mid \alpha \in A\} \). Then \((\prod_{\alpha \in A} X_\alpha) \circ Q(\prod_{\alpha \in A} X_\alpha,S)\) and \((\prod_{\alpha \in A} X_\alpha) \circ W\) are homeomorphic [10]. Suppose \((\prod_{\alpha \in A} X_\alpha,S)\) is weakly Urysohn. Then \((\prod_{\alpha \in A} X_\alpha) \circ Q(\prod_{\alpha \in A} X_\alpha,S)\) is Urysohn and since Urysohn is a topological property [2], then \((\prod_{\alpha \in A} X_\alpha) \circ W\) is Urysohn, which implies \((X_\alpha) \circ Q(X_\alpha,T_\alpha)\) is Urysohn for each \( \alpha \in A \) and \((X_\alpha,T_\alpha)\) is weakly Urysohn.

Conversely, suppose \((X_\alpha,T_\alpha)\) is weakly Urysohn for each \( \alpha \in A \). Then \((X_\alpha) \circ Q(X_\alpha,T_\alpha)\) is Urysohn for each \( \alpha \in A \), which implies \((\prod_{\alpha \in A} X_\alpha) \circ W\) is Urysohn. Thus \((\prod_{\alpha \in A} X_\alpha) \circ Q(\prod_{\alpha \in A} X_\alpha,S)\) is Urysohn, which implies \((\prod_{\alpha \in A} X_\alpha,S)\) is weakly Urysohn.

**Theorem 2.3.** Every regular space is weakly Urysohn.

**Proof.** If \((X,T)\) is regular, then \((X_0,Q(X,T))\) is \( T_2 \) [9], which implies \((X_0,Q(X,T))\) is Urysohn [21] and \((X,T)\) is weakly Urysohn.

**Theorem 2.4.** Every Urysohn space is weakly Urysohn.

**Proof.** If \((X,T)\) is Urysohn, then \((X,T)\) is \( T_2 \) [21] and \( P:(X,T) \rightharpoonup (X_0,Q(X,T))\) is one-to-one, which implies \( P \) is a homeomorphism, \((X_0,Q(X,T))\) is Urysohn, and \((X,T)\) is weakly Urysohn.

**Theorem 2.5.** Every weakly Urysohn space is \( R_1 \).
The straightforward proof is omitted. ▲

In 1937 M. Stone introduced regular open sets. If \((X, T)\) is a space and \(A \subseteq X\), then \(A\) is regular open, denoted by \(A \subseteq RO(X, T)\), iff \(A = \text{Int}(A)\) [20]. In the 1937 investigation, Stone showed that for a space \((X, T)\), \(RO(X, T)\) is a base for a topology \(T_S\) on \(X\) coarser than \(T\), and called \((X, T_S)\) the semiregularization of \((X, T)\). A space \((X, T)\) is semiregular iff \(T = T_S\). Below the semiregularization of a weakly Urysohn space is investigated.

**THEOREM 2.6.** Let \((X, T)\) be weakly Urysohn, let \((X^*_0, Q(X, T_S))\) denote the \(T_0\)-identification space of \((X, T_S)\), and let \((X^*_S, Q_S(X, T_S))\) denote the semi-\(T_0\)-identification space of \((X, T_S)\). Then \((X^*_0, Q(X, T_S)) = (X^*_0, Q(X, T_S))\), which is Urysohn, and \((X^*_S, Q_S(X, T_S)) = (X^*_S, Q_S(X, T_S))\), which is weakly Urysohn semi-\(T_1\); which implies \((X, T_S)\) is weakly Hausdorff.

**Proof.** Since \((X, T)\) is weakly Urysohn, then \((X, T)\) is \(R_1\) and \((X^*_0, Q(X, T_S)) = (X^*_0, Q(X, T_S))\) [11] and \((X^*_S, Q_S(X, T_S)) = (X^*_S, Q_S(X, T_S))\), which is semi-\(T_1\) [12]. Since \((X^*_0, Q(X, T))\) is Urysohn, then \((X^*_0, Q(X, T_S))\) is Urysohn [13]. Thus \((X^*_0, Q(X, T_S))\) is Urysohn, which implies \((X, T_S)\) is weakly Urysohn. Since \((X^*_S, Q_S(X, T_S))\) is weakly Urysohn, then \((X^*_S, Q_S(X, T_S))\) is weakly Urysohn. ▲

Examples can be given of a non weakly Urysohn spaces whose semiregularization is weakly Urysohn.

§3. **S-Essentially Urysohn Spaces.** In 1978 the semi closure operator was used to define feebly open sets, which were used to define feebly closed sets and the feebly closure of a set. Let \((X, T)\) be a space and let \(A, B, C \subseteq X\). Then \(A\) is feebly open, denoted by \(A \subseteq FO(X, T)\), iff there exists \(\sigma \subseteq T\) such that \(\sigma \subseteq A \subseteq \text{scI} \sigma\), \(B\) is feebly closed iff \(X - B\) is feebly open, and the feebly closure of \(C\) is the intersection
of all feebly closed sets containing $C[19]$. Further investigation of feebly open sets showed that for a space $(X, T)$, $F_0(X, T)$ is a topology on $X$ and $T = F_0(X, T) = F_0(X, F_0(X, T))$ [14]. Below feebly induced spaces are used to obtain additional characterizations of $\delta$-essentially Urysohn spaces.

**Theorem 3.1.** Let $(X, T)$ be a space. Then the following are equivalent: (a) $(X, T)$ is $\delta$-essentially Urysohn, (b) $(X, T)$ is weakly Urysohn and $\delta$-essentially $T_2$, (c) for each $Y \subseteq X$, $(Y, T_Y)$ is $\delta$-essentially Urysohn, (d) $(X, T)$ is weakly Urysohn and $\delta$-essentially $T_0$, (e) $(X, T)$ is weakly Urysohn and $X_0 = X_S$, (f) $(X, F_0(X, T))$ is $\delta$-essentially Urysohn, and (g) $(X, F_0(X, T))$ is weakly Urysohn.

**Proof.** Let $(X_6S, Q_8(X, F_0(X, T)))$ denote the semi-$T_0$-identification space of $(X, F_0(X, T))$.

(a) implies (b). Since $(X, T)$ is $\delta$-essentially Urysohn, then $(X_S, Q_8(X, T))$ is Urysohn. Thus $(X_S, Q_8(X, T))$ is weakly Urysohn and $\delta$-essentially $T_2$, which implies $(X, T)$ is weakly Urysohn and $\delta$-essentially $T_2$.

(b) implies (c). Since $(X, T)$ is $\delta$-essentially $T_2$, then $(X, T)$ is $\delta$-essentially $T_0$ and $X_0 = X_S$ [6] and since $(X, T)$ is weakly Urysohn, then $(X_0, Q(X, T))$ is Urysohn. Thus $(X_S, Q_8(X, T)) = (X_0, Q(X, T))$ is Urysohn, which implies $(X, T)$ is $\delta$-essentially Urysohn. Let $Y \subseteq X$. Then $(Y, T_Y)$ is weakly Urysohn and $\delta$-essentially $T_2$ [8], which implies $(Y, T_Y)$ is $\delta$-essentially Urysohn.

(c) implies (d). Since $X \subseteq X$, then $(X, T)$ is $\delta$-essentially Urysohn. Then by the argument above, $(X, T)$ is weakly Urysohn and $\delta$-essentially $T_2$, which implies $(X, T)$ is weakly Urysohn and $\delta$-essentially $T_0$.

Clearly (d) implies (e).

(e) implies (f). Since $X_0 = X_S$ and $(X, T)$ is weakly Urysohn, then $(X_S, Q_8(X, T)) = (X_0, Q(X, T))$ is Urysohn and $(X_S, F_0(X_S, Q_8(X, T)))$ is Urysohn [15]. Then $(X_6S, Q_8(X, F_0(X, T))) = (X_S, F_0(X_S, Q_8(X, T)))$ [16] is Urysohn, which implies $(X, F_0(X, T))$ is $\delta$-essentially Urysohn.

Clearly by the arguments above, (f) implies (g).
(g) implies (a). Since \((X, F_0(X, T))\) is weakly Urysohn, then \((X, F_0(X, T))\) is \(R_1\), which implies \((X, F_0(X, T))\) is \(\delta\)-essentially \(T_2\) [16]. Then by the argument above \((X, F_0(X, T))\) is \(\delta\)-essentially Urysohn and \((X_S, F_0(X_S, Q_S(X, T))) = (X_S, Q_S(X, F_0(X, T)))\) is Urysohn, which implies \((X_S, Q_S(X, T))\) is Urysohn [15] and \((X, T)\) is \(\delta\)-essentially Urysohn. ▲

Combining the results above gives the following corollary.

**COROLLARY 3.1.** Every regular \(\delta\)-essentially \(T_0\) space is \(\delta\)-essentially Urysohn.

**THEOREM 3.2.** Let \((X, T)\) be a space. Then the following are equivalent: (a) \((X, T)\) is Urysohn, (b) \((X, T)\) is weakly Urysohn and \(T_0\), and (c) \((X, T)\) is \(\delta\)-essentially Urysohn and semi-\(T_0\).

**Proof.** Clearly (a) implies (b).

(b) implies (c). Since \((X, T)\) is \(T_0\), then \((X, T)\) is \(\delta\)-essentially \(T_0\) and semi-\(T_0\). Thus \((X, T)\) is weakly Urysohn and \(\delta\)-essentially \(T_0\), which implies \((X, T)\) is \(\delta\)-essentially Urysohn.

(c) implies (a). Since \((X, T)\) is \(\delta\)-essentially Urysohn, then \((X_S, Q_S(X, T))\) is Urysohn and since \((X, T)\) is semi-\(T_0\), then \(P_S\) is one-to-one. Thus \(P_S\) is a homeomorphism and \((X, T)\) is Urysohn. ▲

Combining definitions and results above with the fact that for a space \((X, T)\), \((X_0, Q(X, T))\) is \(T_0\) [21] and \((X_S, Q_S(X, T))\) is semi-\(T_0\) [5] gives the next result.

**COROLLARY 3.2.** Let \((X, T)\) be a space. Then \((X, T)\) is weakly Urysohn iff \((X_0, Q(X, T))\) is weakly Urysohn and \((X, T)\) is \(\delta\)-essentially Urysohn iff \((X_S, Q_S(X, T))\) is \(\delta\)-essentially Urysohn.

**THEOREM 3.3.** For each \(\alpha \in A\), let \((X_\alpha, T_\alpha)\) be a nonempty space, let \(S\) be the product topology on \(\prod_{\alpha \in A} X_\alpha\), and let
B = \{a \in A \mid T_a \text{ is not the indiscrete topology on } X_a \}. Then
\left( \prod_{a \in A} X_a, S \right) \text{ is } \delta\text{-essentially Urysohn iff (1) } (X_a, T_a) \text{ is Urysohn for all } a \in A, \text{ or (2) } B \text{ is finite and for each } a \in A \text{ and each } x_a \in X_a, \{x_a\} \in T_a, \text{ or (3) } (X_a, T_a) \text{ is } \delta\text{-essentially Urysohn for all } a \in A, B \text{ is finite, } X_a \text{ is a singleton set for all } a \in A-B, \text{ and except for one element of } A, T_a \text{ is the discrete topology on } X_a.\]

**Proof.** Suppose \( \left( \prod_{a \in A} X_a, S \right) \) is \( \delta\text{-essentially Urysohn.} \) Then
\( \left( \prod_{a \in A} X_a, S \right) \) is weakly Urysohn and \( \delta\text{-essentially } T_2. \) Then \( (X_a, T_a) \) is weakly Urysohn for all \( a \in A \) and since \( \left( \prod_{a \in A} X_a, S \right) \) is \( \delta\text{-essentially } T_2, \) then (a) \( (X_a, T_a) \) is \( T_2 \) for all \( a \in A, \) or (b) \( B \) is finite and for each \( a \in A \) and each \( x_a \in X_a, \{x_a\} \in T_a, \) or (c) \( (X_a, T_a) \) is \( \delta\text{-essentially } T_2 \) for all \( a \in A, B \) is finite, \( X_a \) is a singleton set for all \( a \in A-B, \) and, except for one element of \( A, T_a \) is the discrete topology on \( X_a \) [8]. Thus (1), or (2), or (3) is true.

Conversely, suppose (1), or (2), or (3) is satisfied. Then \( (X_a, T_a) \) is weakly Urysohn for each \( a \in A \) and \( \left( \prod_{a \in A} X_a, S \right) \) is weakly Urysohn and condition (a), or (b), or (c) above is satisfied, which implies \( \left( \prod_{a \in A} X_a, S \right) \) is \( \delta\text{-essentially } T_2 \) [8]. Thus \( \left( \prod_{a \in A} X_a, S \right) \) is \( \delta\text{-essentially Urysohn.} \)

The results above can be combined to obtain the next result.

**COROLLARY 3.3.** Let \( (X, T) \) be \( \delta\text{-essentially Urysohn.} \) Then
\[ (X^*, Q(X, T_S)) = (X_o, Q(X, T_S)) = (X_S, Q_S(X, T_S)) = (X^*, Q_S(X, T_S)), \]
which is Urysohn, and \( (X, T_S) \) is \( \delta\text{-essentially Urysohn.} \)

Examples can be given of non \( \delta\text{-essentially Urysohn} \) spaces whose semiregularization is \( \delta\text{-essentially Urysohn.} \)

In 1972 [2] homeomorphisms were generalized to semi homeomorphisms by replacing the word open in the definition of homeomorphisms by semi open and properties preserved by semi homeomorphisms were called semi topological properties. In investigations of feebly open sets, it has been shown that certain properties are simultaneously shared by both a
space and its feebly induced space. In [15] a topological property simultaneously shared by both a space and the feebly-induced space was called a feebly property and it was shown that a property is a feebly property iff it is a semi topological property. Thus Theorem 3.1 above shows that \( \delta \)-essentially Urysohn is a semi topological property.

Also, the investigation of feebly open sets has led to several new characterizations of regular open sets and in [11] it was shown that for a space \((X, T)\), \(RO(X, T) = \{ \text{scl } 0 \mid 0 \in T \}\). This new characterization will be used below to further investigate semiregular, weakly Urysohn, and \( \delta \)-essentially Urysohn spaces.

**THEOREM 3.4.** Let \((X, T)\) be a space and let \(0 \in T\). Then
\[
\text{P}(\text{scl } 0) = \text{scl } P(0) \text{ and } \text{P}_S(\text{scl } 0) = \text{scl } P_S(0).
\]

**Proof.** Since \(P^{-1}(\text{scl } U) = \text{scl } P^{-1}(U)\) for each \(U \in Q(X, T)\) [11], then \(P(\text{scl } 0) = P(\text{scl } P^{-1}(P(0))) = P(P^{-1}(\text{scl } P(0))) = \text{scl } P(0)\) and since \(P_S^{-1}(\text{scl } U) = \text{scl } P_S^{-1}(U)\) for each \(U \in Q_S(X, T)\) [12], then, similarly, \(P_S(\text{scl } 0) = \text{scl } P_S(0)\). △

**THEOREM 3.5.** Let \((X, T)\) be a space. Then the following are equivalent: (a) \((X, T)\) is semiregular, (b) \((X^o, Q(X, T))\) is semiregular, and (c) \((X_S, Q_S(X, T))\) is semiregular.

**Proof.** (a) implies (b). Let \(0 \in Q(X, T)\). Let \(C_x \in 0\). Then \(P^{-1}(0) \in T = T_S\), \(x \in P^{-1}(0)\), and there exists \(0 \in T\) such that \(x \in \text{scl } 0 \in P^{-1}(Q)\). Then \(C_x \in P(\text{scl } 0) = \text{scl } P(0) \subseteq 0\) and \(\text{scl } P(0) \subseteq RO(X_0, Q(X, T))\), which implies \(0 \in Q(X, T)_S\). Thus \(Q(X, T) \subseteq Q(X,T)_S\), which implies \(Q(X,T) = Q(X,T)_S\).

(b) implies (c). Let \(0 \in Q_S(X, T)\). Let \(K_x \subseteq 0\). Then \(x \in P_S^{-1}(0) \subseteq T\) and \(C_x \in P(P_S^{-1}(0)) \subseteq Q(X, T)_S\) and there exists \(U \in Q(X, T)\) such that \(C_x \subseteq \text{scl } U \subseteq P(P_S^{-1}(0))\). Then \(x \in P^{-1}(\text{scl } U) = \text{scl } P^{-1}(U) = P_S^{-1}(0)\) and \(K_x \subseteq P_S(\text{scl } P^{-1}(U)) = P_S(\text{scl } P_S^{-1}(P^{-1}(U))) = \text{scl } P_S(P^{-1}(U)) \subseteq 0\), where \(\text{scl } P_S(P^{-1}(U)) \subseteq RO(X_S, Q_S(X, T))\), which implies \(0 \in Q_S(X, T)_S\). Thus \(Q_S(X, T) \subseteq Q_S(X, T)_S\), which implies \(Q_S(X, T) = Q_S(X, T)_S\).

(c) implies (a). Let \(0 \subseteq T\). Let \(x \subseteq 0\). Then \(K_x \subseteq P_S(0) \subseteq Q_S(X, T)_S\) and there exists \(\forall \subseteq Q_S(X, T)\) such that
$K_x \subseteq \text{scl } \mathcal{V} \subseteq P_S(0)$. Then $x \in P_S^{-1}(\text{scl } \mathcal{V}) = \text{scl } P_S^{-1}(\mathcal{V}) \subseteq 0$, where $\text{scl } P_S^{-1}(\mathcal{V}) \subseteq T_S$, which implies $0 \in T_S$. Thus $T \subseteq T_S$ and $T = T_S$. △

**THEOREM 3.6.** The following are equivalent: (a) every Urysohn space is semiregular, (b) every weakly Urysohn space is semiregular, and (c) every $\delta$-essentially Urysohn space is semiregular.

**Proof.** (a) implies (b). Let $(X,T)$ be weakly Urysohn. Then $(X,\mathcal{Q}(X,T))$ is Urysohn and $(X,\mathcal{Q}(X,T))$ is semiregular, which implies $(X,T)$ is semiregular. Clearly, from the results above, (b) implies (c) and (c) implies (a). △

Since not every Urysohn space is semiregular [21], then not every weakly Urysohn or $\delta$-essentially Urysohn space is semiregular, which implies not every weakly Urysohn or $\delta$-essentially Urysohn space is regular.

**THEOREM 3.7.** The following are equivalent: (a) every semiregular $T_2$ space is Urysohn, (b) every semiregular $R_1$ space is weakly Urysohn, and (c) every semiregular $\delta$-essentially $T_2$ space is $\delta$-essentially Urysohn.

**Proof.** (a) implies (b). Let $(X,T)$ be semiregular $R_1$. Then $(X,\mathcal{Q}(X,T))$ is semiregular and $T_2$ [4], which implies $(X,\mathcal{Q}(X,T))$ is Urysohn and $(X,T)$ is weakly Urysohn.

(b) implies (c). Since every $\delta$-essentially $T_2$ space is $R_1$ [8], then every semiregular $\delta$-essentially $T_2$ space is weakly Urysohn and $\delta$-essentially $T_2$, which implies every semiregular $\delta$-essentially $T_2$ space is $\delta$-essentially Urysohn. Similarly, (c) implies (a). △

Since not every semiregular $T_2$ space is Urysohn [21], then not every semiregular $R_1$ space is weakly Urysohn and not every semiregular $\delta$-essentially $T_2$ space is $\delta$-essentially Urysohn. In [10] it was shown that for rim-compact spaces, regular and $R_1$ are equivalent. Combining this result with those above give the following corollaries.
COROLLARY 3.4. If \((X, T)\) is rim-compact, then the following are equivalent: (a) \((X, T)\) is \(T_3\), (b) \((X, T)\) is Urysohn (c) \((X, T)\) is \(T_2\), and (d) \((X, T)\) is semiregular \(T_2\).

COROLLARY 3.5. If \((X, T)\) is rim-compact, then the following are equivalent: (a) \((X, T)\) is regular, (b) \((X, T)\) is weakly Urysohn, (c) \((X, T)\) is \(R_1\), and (d) \((X, T)\) is semiregular \(R_1\).

COROLLARY 3.6. If \((X, T)\) is rim-compact, then the following are equivalent: (a) \((X, T)\) is regular \(s\)-essentially \(T_0\), (b) \((X, T)\) is \(s\)-essentially Urysohn, (c) \((X, T)\) is \(s\)-essentially \(T_2\), and (d) \((X, T)\) is \(s\)-essentially \(T_2\) and semiregular.

There are examples of compact, regular, non \(s\)-essentially \(T_0\) spaces.

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