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GENERALIZED URYSOHN SPACES

by

Charles DORSETT

Abstract. In this paper two new generalizations of Urysohn spaces are investigated and used to obtain new characterizations of Urysohn spaces.

§1. Introduction. In 1961 A. Davis was interested in obtaining properties of topological spaces, which together with T_{i-1} would be equivalent to T_i, $i = 1,2$, respectively. Davis' 1961 investigation led to the following definitions and discoveries. A space (X,T) is R_0 iff for each $0 \in T$ and each $x \in 0$, $\overline{\{x\}} \subset 0$ and (X,T) is R_1 iff for each pair $x,y \in X$ such that $\overline{\{x\}} \neq \overline{\{y\}}$, there exist disjoint open sets U and V such that $\overline{\{x\}} \subset U$ and $\overline{\{y\}} \subset V$ [3]. A space (X,T) is T_i iff it is R_{i-1} and T_{i-1}, $i = 1,2$, respectively [3].

The results above led to the introduction of weakly Urysohn spaces in this paper. A space (X,T) is *weakly Urysohn* iff for each pair $x,y \in X$ such that $\overline{\{x\}} \neq \overline{\{y\}}$, there exist disjoint open sets U and V such that $\overline{\{x\}} \subset U$, $\overline{\{y\}} \subset V$, and $\bar{U} \cap \bar{V} = \phi$.

In 1963 N. Levine introduced semi open sets. Let (X,T) be a space and let $A \subset X$. Then A is *semi open*, denoted by

$A \in SO(X,T)$, iff there exists $O \in T$ such that $O \subset A \subset \bar{O}$, [17]. In 1970 semi open sets were used to define semi closed sets, which were used to define the semi closure of a set. Let (X,T) be a space and let $A,B \subset X$. Then A is *semi closed* iff $X-A$ is semi open and the *semi closure of* B, denoted by scl B, is the intersection of all semi closed sets containing B [1]. In 1975 [18] T_i spaces were generalized to semi-T_i spaces by replacing the word open in the definition of T_i by semi open, $i = 0,1,2$, respectively. These new separation axioms raised questions about properties of topological spaces, which together with semi-T_i, would be equivalent to T_i, $i = 0, 1,2$, respectively. The investigation of these questions led to the following definitions and discoveries, which give additional answers to Davis' 1961 questions. Let (X,T) be a space and let R be the equivalence relation on X defined by xRy iff $\text{scl}\{x\} = \text{scl}\{y\}$. Then the *semi-$T_0$-identification space of* (X,T) is $(X_S, \mathcal{Q}_S(X,T))$, where X_S is the set of equivalence classes of R and $\mathcal{Q}_S(X,T)$ is the decomposition topology on X_S [5]. The space (X,T) is *s-essentially* T_i iff $(X_S, \mathcal{Q}_S(X,T))$ is T_i and (X,T) is T_i iff (X,T) is s-essentially T_i and semi-T_i, $i = 0$ [6], $i = 1$ [7], and $i = 2$ [8]. These results led to the introduction of s-essentially Urysohn spaces in this paper. The space (X,T) is *s-essentially Urysohn* iff $(X_S, \mathcal{Q}_S(X,T))$ is Urysohn.

In this paper weakly Urysohn and s-essentially Urysohn spaces are investigated and used to give new characterizations of Urysohn spaces. Throughout the remainder of this paper, for each space (X,T) the T_0-identification space of (X,T) will be denoted by $(X_0, \mathcal{Q}(X,T))$, the natural map from (X,T) onto $(X_0, \mathcal{Q}(X,T))$ will be denoted by P, the natural map from (X,T) onto $(X_S, \mathcal{Q}_S(X,T))$ will be denoted by P_S, and for each $x \in X$, the element of X_0 containing x will be denoted by C_x and the element of X_S containing x will be denoted by K_x.

§2. Weakly Urysohn Spaces.

THEOREM 2.1. *Let* (X,T) *be a space. Then the following are equivalent:* **(a)** *for each pair* $x,y \in X$ *such that* $\overline{\{x\}} \pm \overline{\{y\}}$, *there exist disjoint open sets* U *and* V *such that* $x \in U$, $y \in V$, *and* $\bar{U} \cap \bar{V} = \phi$, **(b)** (X,T) *is weakly Urysohn,* **(c)** $(X_0,\mathcal{Q}(X,T))$ *is Urysohn,* **(d)** $(X_S,\mathcal{Q}_S(X,T))$ *is weakly Urysohn,* **(e)** *every homeomorphic image of* (X,T) *is weakly Urysohn, and* **(f)** *for each* $Y \subset X$, (Y,T_Y) *is weakly Urysohn.*

Proof. **(a)** *implies* **(b)**. Let $O \in T$ and let $a \in O$. Let $b \in X-O$. Then $\overline{\{a\}} \neq \overline{\{b\}}$ and there exist disjoint open sets A and B such that $a \in A$ and $b \in B$, which implies $b \notin \overline{\{a\}}$. Thus $\overline{\{a\}} \subset O$ and (X,T) is R_0. Let $x,y \in X$ such that $\overline{\{x\}} \neq \overline{\{y\}}$. Then there exist disjoint open sets U and V such that $x \in U$, $y \in V$, and $\bar{U} \cap \bar{V} = \phi$. Since (X,T) is R_0, then $\overline{\{x\}} \subset U$ and $\overline{\{y\}} \subset V$.

(b) *implies* **(c)**. Let $C_x, C_y \in X_0$ such that $C_x \neq C_y$. Then $\overline{\{x\}} \neq \overline{\{y\}}$ and there exist disjoint open sets U and V such that $\overline{\{x\}} \subset U$, $\overline{\{y\}} \subset V$, and $\bar{U} \cap \bar{V} = \phi$. Since P is continuous, closed, and open [9], then $C_x \in P(U) \in \mathcal{Q}(X,T)$, $C_y \in P(V) \in \mathcal{Q}(X,T)$, $P(\bar{U}) = \overline{P(U)}$, and $P(\bar{V}) = \overline{P(V)}$. Since $P^{-1}(P(W)) = W$ for each $W \in T$ [9], then for each $W \in T$, $P^{-1}(P(\bar{W})) = P^{-1}(\overline{P(W)}) = \overline{P^{-1}(P(W))} = \bar{W}$, which implies $\overline{P(U)} \cap \overline{P(V)} = \phi$.

(c) *implies* **(d)**. Let $x,y \in X$ such that $\overline{\{x\}} \neq \overline{\{y\}}$. Then $C_x \neq C_y$ and there exist disjoint open sets $\mathfrak{U}$ and $\mathfrak{V}$ such that $C_x \in \mathfrak{U}$, $C_y \in \mathfrak{V}$, and $\bar{\mathfrak{U}} \cap \bar{\mathfrak{V}} = \phi$. Then $x \in P^{-1}(\mathfrak{U}) \in T$, $y \in P^{-1}(\mathfrak{V}) \in T$, and $\overline{P^{-1}(\mathfrak{U})} \cap \overline{P^{-1}(\mathfrak{V})} = P^{-1}(\bar{\mathfrak{U}}) \cap P^{-1}(\bar{\mathfrak{V}}) = \phi$. Thus (X,T) is weakly Urysohn. Let $K_u, K_v \in X_S$ such that $\overline{\{K_u\}} \neq \overline{\{K_v\}}$. Since P_S is continuous and closed [5], then $\overline{\{u\}} \neq \overline{\{v\}}$ and there exist disjoint open sets U and V such that $\overline{\{u\}} \subset U$, $\overline{\{v\}} \subset V$, and $\bar{U} \cap \bar{V} = \phi$. Since P_S is continuous, closed, and open, and $P_S^{-1}(P_S(W)) = W$ for all $W \subset SO(X,T)$ [5], then $\overline{\{K_u\}} \subset P_S(U) \in \mathcal{Q}_S(X,T)$, $\overline{\{K_v\}} \subset P_S(V) \in \mathcal{Q}_S(X,T)$, and $\overline{P_S(U)} \cap \overline{P_S(V)} = \phi$

(d) *implies* **(e)**. Let $x,y \in X$ such that $\overline{\{x\}} \neq \overline{\{y\}}$. Then $\overline{\{K_x\}} \neq \overline{\{K_y\}}$ and there exist disjoint open sets $\mathfrak{U}$ and $\mathfrak{V}$

such that $\overline{\{K_x\}} \subset \mathcal{U}$, $\overline{\{K_y\}} \subset \mathcal{V}$, and $\bar{\mathcal{U}} \cap \bar{\mathcal{V}} = \phi$. Then $x \in P_S^{-1}(\mathcal{U}) \in T$, $y \in P_S^{-1}(\mathcal{V}) \in T$, and $\overline{P_S^{-1}(\mathcal{U})} \cap \overline{P_S^{-1}(\mathcal{V})} = P_S^{-1}(\bar{\mathcal{U}}) \cap P_S^{-1}(\bar{\mathcal{V}}) = \phi$. Thus (X,T) is weakly Urysohn. The remainder of the proof is straightforward and omitted.

The straightforward proof that (e) implies (f) and (f) implies (a) is also omitted. ▲

THEOREM 2.2. *For each* $\alpha \in A$ *let* (X_α, T_α) *be a nonempty space and let* S *be the product topology on* $\prod_{\alpha\in A} X_\alpha$. *Then* $(\prod_{\alpha\in A} X_\alpha, S)$ *is weakly Urysohn iff* (X_α, T_α) *is weakly Urysohn for each* $\alpha \in A$.

Proof. Let W be the product topology on the product of $\{((X_\alpha)_0, \mathcal{Q}(X_\alpha, T_\alpha)) \mid \alpha \in A\}$. Then $((\prod_{\alpha\in A} X_\alpha)_0, \mathcal{Q}(\prod_{\alpha\in A} X_\alpha, S))$ and $(\prod_{\alpha\in A} (X_\alpha)_0, W)$ are homeomorphic [10]. Suppose $(\prod_{\alpha\in A} X_\alpha, S)$ is weakly Urysohn. Then $((\prod_{\alpha\in A} X_\alpha)_0, \mathcal{Q}(\prod_{\alpha\in A} X_\alpha, S))$ is Urysohn and since Urysohn is a topological property [2], then $(\prod_{\alpha\in A} (X_\alpha)_0, W)$ is Urysohn, which implies $((X_\alpha)_0, \mathcal{Q}(X_\alpha, T_\alpha))$ is Urysohn for each $\alpha \in A$ and (X_α, T_α) is weakly Urysohn.

Conversely, suppose (X_α, T_α) is weakly Urysohn for each $\alpha \in A$. Then $((X_\alpha)_0, \mathcal{Q}(X_\alpha, T_\alpha))$ is Urysohn for each $\alpha \in A$, which implies $(\prod_{\alpha\in A} (X_\alpha)_0, W)$ is Urysohn. Thus $((\prod_{\alpha\in A} X_\alpha)_0, \mathcal{Q}(\prod_{\alpha\in A} X_\alpha, S))$ is Urysohn, which implies $(\prod_{\alpha\in A} X_\alpha, S)$ is weakly Urysohn. ▲

THEOREM 2.3. *Every regular space is weakly Urysohn.*

Proof. If (X,T) is regular, then $(X_0, \mathcal{Q}(X,T))$ is T_3 [9], which implies $(X_0, \mathcal{Q}(X,T))$ is Urysohn [21] and (X,T) is weakly Urysohn. ▲

THEOREM 2.4. *Every Urysohn space is weakly Urysohn.*

Proof. If (X,T) is Urysohn, then (X,T) is T_2 [21] and $P:(X,T) \to (X_0, \mathcal{Q}(X,T))$ is one-to-one, which implies P is a homeomorphism, $(X_0, \mathcal{Q}(X,T))$ is Urysohn, and (X,T) is weakly Urysohn. ▲

THEOREM 2.5. *Every weakly Urysohn space is* R_1.

The straightforward proof is omitted. ▲

In 1937 M. Stone introduced regular open sets. If (X,T) is a space and $A \subset X$, then A is *regular open*, denoted by $A \in RO(X,T)$, iff $A = \text{Int}(\bar{A})$ [20]. In the 1937 investigation, Stone showed that for a space (X,T), $RO(X,T)$ is a base for a topology T_S on X coarser than T, and called (X,T_S) the *semiregularization of* (X,T). A space (X,T) is *semiregular* iff $T = T_S$. Below the semiregularization of a weakly Urysohn space is investigated.

THEOREM 2.6. *Let* (X,T) *be weakly Urysohn, let* $(X_0^*,Q(X,T_S))$ *denote the* T_0*-identification space of* (X,T_S)*, and let* $(X_S^*,Q_S(X,T_S))$ *denote the semi-*T_0*-identification space of* (X,T_S)*. Then* $(X_0,Q(X,T)_S)) = (X_0^*,Q(X,T_S))$*, which is Urysohn, and* $(X_S,Q_S(X,T)_S) = (X_S^*,Q_S(X,T_S))$*, which is weakly Urysohn semi-*T_1*; which implies* (X,T_S) *is weakly Hausdorff.*

Proof. Since (X,T) is weakly Usysohn, then (X,T) is R_1 and $(X_0,Q(X,T)_S) = (X_0^*,Q(X,T_S))$ [11] and $(X_S,Q_S(X,T)_S) = (X_S^*,Q_S(X,T_S))$, which is semi-$T_1$ [12]. Since $(X_0,Q(X,T))$ is Urysohn, then $(X_0,Q(X,T)_S)$ is Urysohn [13]. Thus $(X_0^*,Q(X,T_S))$ is Urysohn, which implies (X,T_S) is weakly Urysohn. Since $(X_S,Q_S(X,T))$ is weakly Urysohn, then $(X_S^*,Q_S(X,T_S))$ is weakly Urysohn. ▲

Examples can be given of a non weakly Urysohn spaces whose semiregularization is weakly Urysohn.

§3. S-Essentially Urysohn Spaces. In 1978 the semi closure operator was used to define feebly open sets, which were used to define feebly closed sets and the feebly closure of a set. Let (X,T) be a space and let $A,B,C \subset X$. Then A is *feebly open*, denoted by $A \in FO(X,T)$, iff there exists $\mathcal{O} \in T$ such that $\mathcal{O} \subset A \subset \text{scl}\ \mathcal{O}$, B is *feebly closed* iff $X-B$ is feebly open, and the *feebly closure* of C is the intersection of all feebly closed sets containing C [19]. Further investigation of feebly open sets showed that for a space (X,T), $FO(X,T)$ is a topology on X and $T \subset FO(X,T) = FO(X,FO(X,T))$ [14]. Below feebly induced spaces are used to obtain additional characterizations of δ-essentially Urysohn spaces.

THEOREM 3.1. *Let* (X,T) *be a space. Then the following are equivalent:* (a) (X,T) *is* δ*-essentially Urysohn,* (b) (X,T) *is weakly Urysohn and* δ*-essentially* T_2, (c) *for each* $Y \subset X$, (Y,T_Y) *is* δ*-essentially Urysohn,* (d) (X,T) *is weakly Urysohn and* δ*-essentially* T_0, (e) (X,T) *is weakly Urysohn and* $X_0 = X_S$, (f) $(X,FO(X,T))$ *is* δ*-essentially Urysohn, and* (g) $(X,FO(X,T))$ *is weakly Urysohn.*

Proof. Let $(X_{\delta S},\mathcal{Q}_S(X,FO(X,T)))$ denote the semi-T_0-identification space of $(X,FO(X,T))$.

(a) *implies* **(b).** Since (X,T) is δ-ssentially Urysohn, then $(X_S,\mathcal{Q}_S(X,T))$ is Urysohn. Thus $(X_S,\mathcal{Q}_S(X,T))$ is weakly Urysohn and T_2, which implies (X,T) is weakly Urysohn and δ-essentially T_2.

(b) *implies* **(c).** Since (X,T) is δ-essentially T_2, then (X,T) is δ-essentially T_0 and $X_0 = X_S$ [6] and since (X,T) is weakly Urysohn, then $(X_0,\mathcal{Q}(X,T))$ is Urysohn. Thus $(X_S,\mathcal{Q}_S(X,T)) = (X_0,\mathcal{Q}(X,T))$ is Urysohn, which implies (X,T) is δ-essentially Urysohn. Let $Y \subset X$. Then (Y,T_Y) is weakly Urysohn and δ-essentially T_2 [8], which implies (Y,T_Y) is δ-essentially Urysohn.

(c) *implies* **(d).** Since $X \subset X$, then (X,T) is δ-essentially Urysohn. Then by the argument above, (X,T) is weakly Urysohn and δ-essentally T_2, which implies (X,T) is weakly Urysohn and δ-essentially T_0.

Clearly (d) implies (e).

(e) *implies* **(f).** Since $X_0 = X_S$ and (X,T) is weakly Urysohn, then $(X_S,\mathcal{Q}_S(X,T)) = (X_0,\mathcal{Q}(X,T))$ is Urysohn and $(X_S,FO(X_S,\mathcal{Q}_S(X,T)))$ is Urysohn [15]. Then $(X_{\delta S},\mathcal{Q}_S(X,FO(X,T)))$ $= (X_S,FO(X_S,\mathcal{Q}_S(X,T)))$ [16] is Urysohn, which implies $(X,FO(X,T))$ is δ-essentially Urysohn.

Clearly by the arguments above, (f) implies (g).

(g) *implies* **(a)**. Since $(X,FO(X,T))$ is weakly Urysohn, then $(X,FO(X,T))$ is R_1, which implies $(X,FO(X,T))$ is s-essentially T_2 [16]. Then by the argument above $(X,FO(X,T))$ is s-essentially Urysohn and $(X_S,FO(X_S,Q_S(X,T))) = (X_S,Q_S(X,FO(X,T)))$ is Urysohn, which implies $(X_S,Q_S(X,T))$ is Urysohn [15] and (X,T) is s-essentially Urysohn. ▲

Combining the results above gives the following corollary.

COROLLARY 3.1. *Every regular s-essentially T_0 space is s-essentially Urysohn.*

THEOREM 3.2. *Let (X,T) be a space. Then the following are equivalent:* (a) (X,T) *is Urysohn,* (b) (X,T) *is weakly Urysohn and* T_0, *and* (c) (X,T) *is s-essentially Urysohn and semi-T_0.*

Proof. Clearly (a) implies (b).

(b) *implies* **(c)**. Since (X,T) is T_0, then (X,T) is s-essentially T_0 and semi-T_0. Thus (X,T) is weakly Urysohn and s-essentially T_0, which implies (X,T) is s-essentially Urysohn.

(c) *implies* **(a)**. Since (X,T) is s-essentially Urysohn, then $(X_S,Q_S(X,T))$ is Urysohn and since (X,T) is semi-T_0, then P_S is one-to-one. Thus P_S is a homeomorphism and (X,T) is Urysohn. ▲

Combining definitions and results above with the fact that for a space (X,T), $(X_0,Q(X,T))$ is T_0 [21] and $(X_S,Q_S(X,T))$ is semi-T_0 [5] gives the next result.

COROLLARY 3.2. *Let (X,T) be a space. Then (X,T) is weakly Urysohn iff $(X_0,Q(X,T))$ is weakly Urysohn and (X,T) is s-essentially Urysohn iff $(X_S,Q_S(X,T))$ is s-essentially Urysohn.*

THEOREM 3.3. *For each $\alpha \in A$, let (X_α,T_α) be a nonempty space, let S be the product topology on $\prod_{\alpha\in A} X_\alpha$, and let*

$B = \{\alpha \in A \mid T_\alpha$ *is not the indiscrete topology on* $X_\alpha\}$. *Then* $(\prod_{\alpha\in A} X_\alpha, S)$ *is s-essentially Urysohn iff* (1) (X_α, T_α) *is Urysohn for all* $\alpha \in A$, *or* (2) B *is finite and for each* $\alpha \in A$ *and each* $x_\alpha \in X_\alpha$, $\overline{\{x_\alpha\}} \in T_\alpha$, *or* (3) (X_α, T_α) *is s-essentially Urysohn for all* $\alpha \in A$, B *is finite*, X_α *is a singleton set for all* $\alpha \in A-B$, *and except for one element of* A, T_α *is the discrete topology on* X_α.

Proof. Suppose $(\prod_{\alpha\in A} X_\alpha, S)$ is s-essentially Urysohn. Then $(\prod_{\alpha\in A} X_\alpha, S)$ is weakly Urysohn and s-essentially T_2. Then (X_α, T_α) is weakly Urysohn for all $\alpha \in A$ and since $(\prod_{\alpha\in A} X_\alpha, S)$ is s-essentially T_2, then (a) (X_α, T_α) is T_2 for all $\alpha \in A$, or (b) B is finite and for each $\alpha \in A$ and each $x_\alpha \in X_\alpha$, $\overline{\{x_\alpha\}} \in T_\alpha$, or (c) (X_α, T_α) is s-essentially T_2 for all $\alpha \in A$, B is finite, X_α is a singleton set for all $\alpha \in A-B$, and, except for one element of A, T_α is the discrete topology on X_α [8]. Thus (1), or (2), or (3) is true.

Conversely, suppose (1), or (2), or (3) is satisfied. Then (X_α, T_α) is weakly Urysohn for each $\alpha \in A$ and $(\prod_{\alpha\in A} X_\alpha, S)$ is weakly Urysohn and condition (a), or (b), or (c) above is satisfied, which implies $(\prod_{\alpha\in A} X_\alpha, S)$ is s-essentially T_2 [8]. Thus $(\prod_{\alpha\in A} X_\alpha, S)$ is s-essentially Urysohn. ▲

The results above can be combined to obtain the next result.

COROLLARY 3.3. *Let* (X,T) *be s-essentially Urysohn. Then* $(X_o^*, Q(X,T_S)) = (X_o, Q(X,T)_S) = (X_S, Q_S(X,T)_S) = (X_S^*, Q_S(X,T_S))$, *which is Urysohn, and* (X,T_S) *is s-essentially Urysohn.*

Examples can be given of non s-essentially Urysohn spaces whose semiregularization is s-essentially Urysohn.

In 1972 [2] homeomorphisms were generalized to semi homeomorphisms by reaplacing the word open in the definition of homeomorphisms by semi open and properties preserved by semi homeomorphisms were called semi topological properties. In investigations of feebly open sets, it has been shown that certain properties are simultaneously shared by both a space and its feebly induced space. In [15] a topological property simultaneously shared by both a space and the feebly induced space was called a feebly property and it was shown that a property is a feeble property iff it is a semi topological property. Thus Theorem 3.1 above shows that δ-essentially Urysohn is a semi topological property.

Also, the investigation of feebly open sets has led to several new characterizacion of regular open sets and in [11] it was shown that for a space (X,T), $RO(X,T) = \{\text{scl } O \mid O \in T\}$. This new characterizacion will be used below to further investigate semiregular, weakly Urysohn, and δ-essentially Urysohn spaces.

THEOREM 3.4. *Let* (X,T) *be a space and let* $O \in T$. *Then* $P(\text{scl } O) = \text{scl } P(O)$ *and* $P_S(\text{scl } O) = \text{scl } P_S(O)$.

Proof. Since $P^{-1}(\text{scl } \mathcal{U}) = \text{scl } P^{-1}(\mathcal{U})$ for each $\mathcal{U} \in Q(X,T)$ [11], then $P(\text{scl } O) = P(\text{scl } P^{-1}(P(O))) = P(P^{-1}(\text{scl } P(O))) = \text{scl } P(O)$ and since $P_S^{-1}(\text{scl } \mathcal{U}) = \text{scl } P_S^{-1}(\mathcal{U})$ for each $\mathcal{U} \in Q_S(X,T)$ [12], then, similarly, $P_S(\text{scl } O) = \text{scl } P_S(O)$. ▲

THEOREM 3.5. *Let* (X,T) *be a space. Then the following are equivalent:* (a) (X,T) *is semiregular,* (b) $(X_o, Q(X,T))$ *is semiregular, and* (c) $(X_S, Q_S(X,T))$ *is semiregular.*

Proof. (a) *implies* (b). Let $O \in Q(X,T)$. Let $C_x \in O$. Then $P^{-1}(O) \in T = T_S$, $x \in P^{-1}(O)$, and there exists $O \in T$ such that $x \in \text{scl } O \subset P^{-1}(O)$. Then $C_x \in P(\text{scl } O) = \text{scl } P(O) \subset O$ and $\text{scl } P(O) \in RO(X_o, Q(X,T))$, which implies $O \in Q(X,T)_S$. Thus $Q(X,T) \subset Q(X,T)_S$, which implies $Q(X,T) = Q(X,T)_S$.

(b) *implies* (c). Let $O \in Q_S(X,T)$. Let $K_x \in O$. Then $x \in P_S^{-1}(O) \in T$ and $C_x \in P(P_S^{-1}(O)) \in Q(X,T)_S$ and there eixsts $\mathcal{U} \in Q(X,T)$ such that $C_x \in \text{scl } \mathcal{U} \subset P(P_S^{-1}(O))$. Then $x \in P^{-1}(\text{scl } \mathcal{U}) = \text{scl } P^{-1}(\mathcal{U}) \subset P_S^{-1}(O)$ and $K_x \in P_S(\text{scl } P^{-1}(\mathcal{U})) = P_S(\text{scl } P_S^{-1}(P_S(P^{-1}(\mathcal{U})))) = \text{scl } P_S(P^{-1}(\mathcal{U})) \subset O$, where $\text{scl } P_S(P^{-1}(\mathcal{U})) \in RO(X_S, Q_S(X,T))$, which implies $O \in Q_S(X,T)_S$. Thus $Q_S(X,T) \subset Q_S(X,T)_S$, which implies $Q_S(X,T) = Q_S(X,T)_S$.

(c) *implies* (a). Let $O \in T$. Let $x \in O$. Then $K_x \in P_S(O) \in Q_S(X,T)_S$ and there exists $\mathcal{V} \in Q_S(X,T)$ such that

$K_x \in \mathrm{scl}\, \mathcal{V} \subset P_S(0)$. Then $x \in P_S^{-1}(\mathrm{scl}\, \mathcal{V}) = \mathrm{scl}\, P_S^{-1}(\mathcal{V}) \subset 0$, where $\mathrm{scl}\, P_S^{-1}(\mathcal{V}) \in T_S$, which implies $0 \in T_S$. Thus $T \subset T_S$ and $T = T_S$. ▲

THEOREM 3.6. *The following are equivalent:* (a) *every Urysohn space is semiregular,* (b) *every weakly Urysohn space is semiregular, and* (c) *every s-essentially Urysohn space is semiregular.*

Proof. (a) *implies* (b). Let (X,T) be weakly Urysohn. Then $(X_0, \mathcal{Q}(X,T))$ is Urysohn and $(X_0, \mathcal{Q}(X,T))$ is semiregular, which implies (X,T) is semiregular. Clearly, from the results above, (b) implies (c) and (c) implies (a). ▲

Since not every Urysohn space is semiregular [21], then not every weakly Urysohn or s-essentially Urysohn space is semiregular, which implies not every weakly Urysohn or s-essentially Urysohn space is regular.

THEOREM 3.7. *The following are equivalent:* (a) *every semiregular T_2 space is Urysohn,* (b) *every semiregular R_1 space is weakly Urysohn, and* (c) *every semiregular s-essentially T_2 space is s-essentially Urysohn.*

Proof. (a) *implies* (b). Let (X,T) be semiregular R_1. Then $(X_0, \mathcal{Q}(X,T))$ is semiregular and T_2 [4], which implies $(X_0, \mathcal{Q}(X,T))$ is Urysohn and (X,T) is weakly Urysohn.

(b) *implies* (c). Since every s-essentially T_2 space is R_1 [8], then every semiregular s-essentially T_2 space is weakly Urysohn and s-essentially T_2, which implies every semiregular s-essentially T_2 space is s-essentially Urysohn. Similarly, (c) implies (a). ▲

Since not every semiregular T_2 space is Urysohn [21], then not every semiregular R_1 space is weakly Urysohn and not every semiregular s-essentially T_2 space is s-essentially Urysohn. In [10] it was shown that for rim-compact spaces, regular and R_1 are equivalent. Combining this result with those above give the following corollaries.

COROLLARY 3.4. *If* (X,T) *is rim-compact, then the following are equivalent:* (a) (X,T) *is* T_3, (b) (X,T) *is Urysohn* (c) (X,T) *is* T_2, *and* (d) (X,T) *is semiregular* T_2.

COROLLARY 3.5. *If* (X,T) *is rim-compact, then the following are equivalent:* (a) (X,T) *is regular,* (b) (X,T) *is weakly Urysohn,* (c) (X,T) *is* R_1, *and* (d) (X,T) *is semiregular* R_1.

COROLLARY 3.6. *If* (X,T) *is rim-compact, then the following are equivalent:* (a) (X,T) *is regular s-essentially* T_0, (b) (X,T) *is s-essentially Urysohn,* (c) (X,T) *is s-essentially* T_2, *and* (d) (X,T) *is s-essentially* T_2 *and semiregular.*

There are examples of compact, regular, non *s*-essentially T_0 spaces.

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Department of Mathematics and Statistics
Louisiana Tech University
P.O. Box 3189
Ruston Louisiana 71272-0001
U. S. A.

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