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ON ROOT SYSTEMS

by

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Abstract. Let Σ be a root system. $\Delta \subset \Sigma$ a base and Σ^+ the set of positive roots. We prove the following two propositions.

THEOREM. If $\alpha \in \Sigma^+$ *say* $\alpha = \sum_{\sigma\in\Delta} n_\sigma \cdot \sigma$ *then* $\Delta_n(\alpha) = \{\sigma : n_\sigma \geqslant n\}$ *is a connected subset of the Dynkin diagram of* Σ.

THEOREM. Let r_N *be the root in* Σ^+ *of largest height. Then for each* $\alpha \in \Sigma^+$ *such that* $\alpha \neq r_N$ *and height of* $\alpha \geqslant 2$ *there exists* $\sigma \in \Delta$ *with* $\alpha - \sigma \in \Sigma^+$*, but* $r_N - \sigma \notin \Sigma^+$.

Introduction. The purpose of this note is to prove two simple properties about root systems: Theorems 2.1 and 3.1. Theorem 3.1 is useful in the study of definability in certain nilpotent groups (in fact it was during this study that the theorem was discovered, see [Vi]). Perhaps it us useful in other situations.

§1. Preliminaries. In this section we recall the definition of root systems and state some of their properties. The references we use are [Hu, Chapter III] and [Bo, Chapter 7.8].

DEFINITION 1.1. A subset Σ of euclidean space E is called a *root system* in E (and $\alpha \in \Sigma$ is called a root) is the following axioms are satisfied:

(a) Σ is finite, spans E, and does not contain 0.

(b) If $\alpha \in \Sigma$ then the only multiplies of α in Σ are $\pm\alpha$.

(c) If $\alpha \in \Sigma$ then the reflection σ_α leaves Σ invariant (here $\sigma_\alpha(\beta) = \beta - \frac{2(\beta,\alpha)}{(\alpha,\alpha)}\alpha)$.

(d) If we let $<\beta,\alpha> = 2(\beta,\alpha)/(\alpha,\alpha)$, then for $\alpha,\beta \in \Sigma$, $<\beta,\alpha> \in \mathbb{Z}$.

The last condition puts a big constraint on the possible angles between pairs of roots. In fact all root systems can be described explicitly. From condition (d) one can construct the following table [Hu, p.45].

Let α,β, be roots, nonproportional.

$<\alpha,\beta>$	$<\beta,\alpha>$	θ	$\lVert\beta\rVert^2/\lVert\alpha\rVert^2$
0	0	$\pi/2$	undetermined
1	1	$\pi/3$	1
-1	-1	$2\pi/3$	1
1	2	$\pi/4$	2
-1	-2	$3\pi/4$	2
1	3	$\pi/6$	3
-1	-3	$5\pi/6$	3

Note that the symbol $<\beta,\alpha>$ is linear (only) in the first variable, and $<\beta,\beta> = 2$.

We will use the following lemmas constantly.

LEMMA 1.2. *Let α, β be nonproportional roots. If* $(\alpha,\beta) > 0$ *then $\alpha-\beta$ is a root. If $(\alpha,\beta) < 0$ then $\alpha+\beta$ is a root.*

Given $\alpha,\beta \in \Sigma$, nonproportional, one can look at the α-string through β: this consists of all the roots of the form $\beta+i\alpha$ ($i \in \mathbf{Z}$). Let r,q be the largest nonnegative integers for which $\beta-r\alpha \in \Sigma$ and $\beta+q\alpha \in \Sigma$. We have the following

LEMA 1.3. *The* α*-string through* β *is unbroken from* $\beta - r\alpha$ *to* $\beta + q\alpha$*. Furthermore* $\langle\beta,\alpha\rangle = r - q$.

We turn to bases. A subset $\Delta \subset \Sigma$ is called a *base of* Σ if Δ is a basis of E and each root $\beta \in \Sigma$ can be written as $\beta = \Sigma k_\alpha \alpha$ $(\alpha \in \Delta)$ with integral coefficients all nonnegative or all nonpositive. Bases exist [Hu, p.48]. The elements of Δ are called *simple roots;* $ht(\beta) \overset{\text{def}}{=} \Sigma k_\alpha$ (read: *'height of* β*'*). If all $k_\alpha \geqslant 0$ we call β *positive*. Finally, $\beta \overset{\text{def}}{\leqslant} \alpha$ if $\beta - \alpha$ is a sum of positive roots or $\beta = \alpha$.

LEMMA 1.4. *If* $\alpha, \beta \in \Delta$ *then* $(\alpha,\beta) \leqslant 0$ *for* $\alpha \neq \beta$ *and* $\alpha - \beta$ *is not a root.*

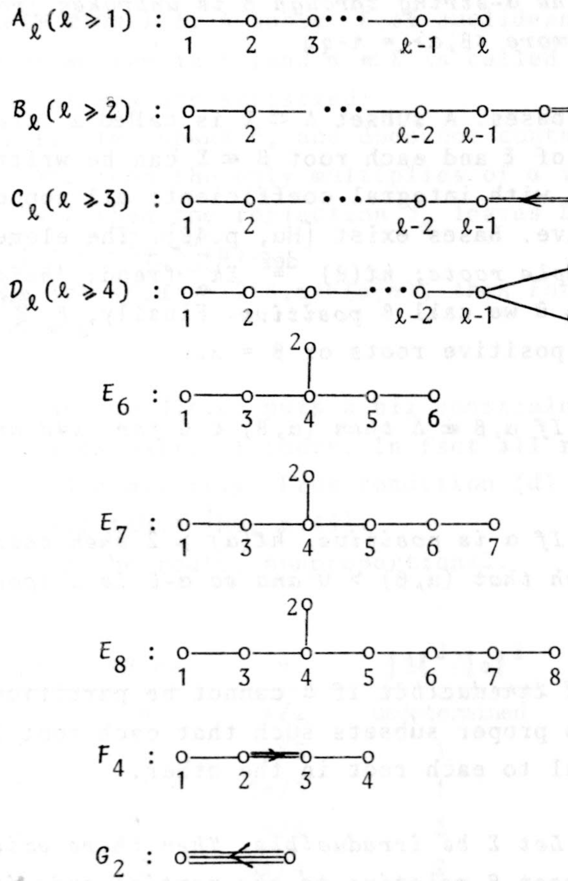
LEMMA 1.5. *If* α *is positive,* $ht(\alpha) \geqslant 2$ *then there exists* $\beta \in \Delta$ *such that* $(\alpha,\beta) > 0$ *and so* $\alpha - \beta$ *is a (positive) root.*

Σ *is called irreducible* if Δ cannot be partitioned into the union of two proper subsets such that each root in one set is orthogonal to each root in the other.

LEMMA 1.6. *Let* Σ *be irreducible. Then there exists a unique maximal root* β *relative to the partial ordering* $\leqslant$*. Furthermore if* $\beta = \Sigma k_\alpha \alpha$ $(\alpha \in \Delta)$ *then* $k_\alpha \geqslant 1$*. Hence, if* $\alpha \in \Sigma^+$*, with* $\alpha \neq \beta$*, then* $ht\alpha < ht\beta$ *and* $(\beta,\alpha) \geqslant 0$, $\forall \alpha \in \Delta$.

1.7. Dynkin Diagrams. Let the *rank* of Σ be the dimension of E (so $\#\Delta$ = rank of Σ). The *Dynkin diagram* of Σ is constructed as follows: we have a vertex for each $\alpha \in \Delta$. The vertex i is joined to the vertex j by $\langle\alpha_i,\alpha_j\rangle\langle\alpha_j,\alpha_i\rangle$ edges. When two or more edges occur then it is known that the lenghts of the corresponding simple roots are different. In this case we add an arrow pointing to the shorter of the two roots.

THEOREM. *If* Σ *is an irreducible root system its Dynkin diagram is one of the following:*



§2. The support of a root. Let $\alpha \in \Sigma^+$ with $\alpha = \Sigma n_\sigma \sigma$ $(\sigma \in \Delta)$. We define the *support* of α (written $supp(\alpha)$) to be the set of $\alpha \in \Delta$ with $n_\sigma \geqslant 1$. Similarly let $\Delta_n(\alpha) = \{\alpha \in \Delta : n_\sigma \geqslant n\}$.

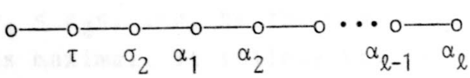
THEOREM 2.1. *$\Delta_n(\alpha)$ is a connected subset of the Dynkin diagram of Σ.*

Proof. Let $n = 1$. If $\alpha = \sigma \in \Delta$ then the results is true. Otherwise by Lemma 1.5 there exists $\sigma \in \Delta$ with $\alpha - \sigma \in \Sigma^+$. Let m be the largest integer $\geqslant 1$ such that $\gamma = \alpha - m\sigma \in \Sigma$. Since $\gamma - \sigma \notin \Sigma$ and γ is not proportional to σ it follows that $(\gamma, \beta) \leqslant 0$. From Lemma 1.3 it follows that $(\gamma, \beta) < 0$.

This means that β is joined to at least one of the simple roots which occur in γ (with a nonzero coefficient). We may assume by induction on height that $\Delta_1(\gamma)$ is connected. Hence $\Delta_1(\alpha) = \Delta_1(\gamma) \cup \{\beta\}$ is connected.

Now assume $n \geqslant 2$. Fix $\alpha \in \Sigma^+$ and by induction suppose that $\Delta_\kappa(\beta)$ is connected for all β with $ht(\beta) < ht(\alpha)$ and all $1 \leqslant \kappa \leqslant n$. Suppose $\Delta_n(\alpha)$ is disconnected. There exist simple roots $\sigma_1 < \sigma_o < \sigma_2$ with $n_{\sigma 2}, n_{\sigma 1} > n_{\sigma_o}$ and $n_{\sigma 2}, n_{\sigma 1} > n \cdot n_{\sigma_o} \not> n$. The notation $\alpha < \beta$ for simple roots means that α lies on the left of β in the Dynkin diagram. Note that since supp(α) is connected then the unique shortest arc joining σ_1 to σ_2 consists of simple roots appearing in the decomposition of α. Take the triple $(\sigma_1, \sigma_o, \sigma_2)$ of minimal length (i.e. the length of the arc from σ_1 to σ_2 is minimal). In other words, the number of roots between σ_2 and σ_1 is minimal. It follows that if σ is on the arc joining σ_1 to σ_2 (i.e. $\sigma_1 < \sigma < \sigma_2$) then $n_\sigma = n_{\sigma_o}$.

Here is our basic plan. By Lemma 1.5 there is a σ such that $(\alpha, \sigma) > 0$ (so $\alpha - \sigma$ is a root). If $\sigma \neq \sigma_1, \sigma_2$ then $\alpha - \sigma$ is a root of smaller height with $\Delta_n(\alpha - \sigma)$ disconnected. Hence we assume that $(\alpha, \sigma) \leqslant 0$ for $\sigma \neq \sigma_1, \sigma_2$ (and so either $(\alpha, \sigma_1) > 0$ or $(\alpha, \sigma_2) > 0$). The subgraph of the Dynkin diagram formed by taking the roots in supp(α) will have one end without double edges or bifurcations [see 1.7]. Without loss of generality assume it is σ_2's end. So suppose then that the support of α looks like:



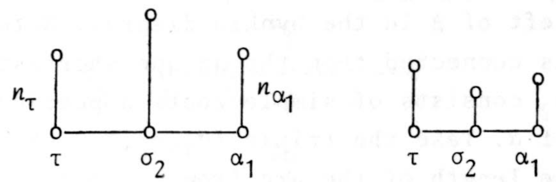
Here $\ell \geqslant 0$.

We show that:

$$n_{\sigma 2} > n_{\sigma 1} > \ldots > n_{\alpha_{\ell-1}} > 2n_{\alpha_\ell} \tag{A}$$

We start from α_ℓ. Since $\langle \alpha, \alpha_\ell \rangle = 2n_{\alpha_\ell} - n_{\alpha_{\ell-1}} \leqslant 0$ we have $n_{\alpha_\ell} - n_{\alpha_{\ell-1}} \leqslant 0$, hence $n_{\alpha_{\ell-1}} \geqslant n_{\alpha_\ell}$.

Next $\langle\alpha,\alpha_{\ell-1}\rangle = -n_{\alpha_{\ell-2}} + 2n_{\alpha_{\ell-1}} - n_{\alpha_\ell} \leqslant 0$, hence $2n_{\alpha_{\ell-1}} - n_{\alpha_\ell} = (n_{\alpha_{\ell-1}} - n_{\alpha_\ell}) + n_{\alpha_{\ell-1}} \leqslant n_{\alpha_{\ell-2}}$. Since $n_{\alpha_{\ell-1}} - n_{\alpha_\ell} \geqslant 1$ we get $n_{\alpha_{\ell-2}} > n_{\alpha_{\ell-1}}$. It is clear that we can continue and so obtain (A). To finish note that $\langle\alpha,\sigma_2\rangle = 2n_{\sigma_2} - n_\tau - n_{\alpha_1} = c$. If we let $n_\tau = n_{\sigma_2} - \kappa$ with $\kappa \geqslant 1$ then $c = (n_{\sigma_2} - n_{\alpha_1}) + \kappa \geqslant 2$. Hence $\alpha - c\sigma_2$ is a root (by Lemma 1.3) of smaller height than that of α and with $\Delta_\ell\ (\alpha - c\sigma_2)$ disconnected where $\ell = \min\{n_{\alpha_1}, n_\tau\}$ See the figure below:



If $\ell = 0$. i.e. if σ_2 is the last root in the support of α then $\langle\alpha,\sigma_2\rangle = 2n_{\sigma_2} - n_\tau = n_{\sigma_2} + \kappa \geqslant 1$. From 1.1. $\langle\alpha,\sigma_2\rangle \leqslant 3$. Since $n_{\sigma_2} \geqslant 2$ it follows that $\langle\alpha,\sigma_2\rangle = 3$. So again $\alpha - 3\sigma_2$ is a root but the coefficient of σ_2 in $\alpha - 3\sigma_2$ is negative while that of τ is positive. This is impossible. This completes the proof. ▲

There is a converse to the above theorem.

LEMMA 2.2. *Let $A \subset \Delta$ be a nonempty connected set of simple roots. Then $\alpha = \sum_{\sigma\in A} \sigma$ is a root.*

Proof. [Bo, Ch.8].

§3. Good roots and the heighest root. Let $r_N \in \Sigma^+$ be the root with largest height (with respect to the base Δ). A root $\sigma \in \Delta$ is called *good* if $r_N - \sigma \in \Sigma$. There are good roots, by Lemma 1.5, however usually there is only one. We discuss this in more detail. Using the argument in [Hu, p.60], in particular his notion of an admissible set of vectors in E applied to the sets $A_i = \{-r_N, \sigma_1, \ldots, \sigma_i, \sigma_{i+1}, \ldots, \sigma_\ell\}$, where $\Delta = \{\sigma_i . 1 \leqslant i \leqslant \ell\}$, yields that there is only one good root for all systems except, possibly A_ℓ. On the other hand $r_N = \sigma_1+\sigma_2+\ldots+\sigma_\ell$ for A_ℓ so σ_1 and σ_ℓ are good (from Lemma 2.2 it follows that $\sigma_1+\ldots+\sigma_\ell$ is a root; it is easy to see that one can substract σ_1 and σ_ℓ and a proof that $\sigma_1+\ldots+\sigma_\ell$ is indeed r_N is implicit in the first part of the proof of Theorem 3.1 below).

It therefore seems that Theorem 3.1 below is extremely likely. We note that it is possible to identify the good roots in each root system: as a matter of fact one can write down all Σ^+. We will try to avoid this but will use the following fact: if one deletes a root $\sigma \in \Delta$ and considers $\Delta-\{\sigma\} \cup \{-r_N\}$ then its Dynkin diagram must be one of the list in 1.7 (see [Hu.p.60]). This implies that the good root for the systems E_6, E_7, E_8, E_4 must be an end vertex. For C_ℓ it actually tells so that the good root is the first one (as labelled in section 1.7). For B_ℓ and D_ℓ one can conclude only that it is either the first or the second while for F_4 it must be the first root. G_2 is rather special. At this point one could not tell if it has two good roots or not, and if not which one is!

THEOREM 3.1. *Let $\alpha \in \Sigma^+$ with $ht(\alpha) \geqslant 2$, $\alpha \neq r_N$. Then there exists $\sigma \in \Delta$ such that $\alpha-\sigma \in \Sigma^+$ but $r_N-\sigma \notin \Sigma^+$.*

Proof. We consider cases on the graph formed by the support of α. From now on $\alpha = \sum_{\sigma\in\Delta} n_\sigma \cdot \sigma$.

Case I. Graph of supp(α) does not contain double edges or bifurcations.

Let $\sigma_1 < \sigma_2 < \ldots < \sigma_\ell$ be the graph of α. Let σ_{i_0} be such that $n_{\sigma_{i_0}}$ is maximal. It follows that if $\sigma_1 < \sigma_{i_0} < \sigma_\ell$ then $n_{\gamma_{i_0-1}} = n_{\gamma_{i_0+1}} = n_{\gamma_{i_0}}$. Otherwise $\alpha-\langle\alpha,\sigma_1\rangle\sigma_{i_0}$ is a positive root with Δ_ℓ for $\ell = \min\{r_{\sigma_{i_0-1}}, n_{\gamma_{i_0+1}}\}$ disconnected. Continuing in this manner, all n_j's are equal in which case $\alpha-\sigma_1$ and $\alpha-\sigma_\ell$ are roots or without loss of generality $\exists \sigma_j, \sigma_i < \sigma_j < \sigma_\ell$ with $n_{\sigma_{j-1}} > n_{\sigma_j}$. Again we get that $\alpha-\sigma_j$ is a root. So, altogether it is possible to subtract two roots from α. This proves the proposition for all types except A_ℓ. But we have seen that if Σ has two good roots then they occur at end points of the Dynkin diagram. Hence they can only be σ_1, or σ_ℓ. This implies that $n_{\sigma_j} = n_{\sigma_i} = c$ $\forall i,j$ so $\alpha = \sum_{1\leq i\leq \ell} c\cdot\alpha_i = c\sum_{1\leq i\leq\ell}\alpha_i$. Hence $c = 1$ by Lemma 2.2 and the definition of root system. If indeed σ_1 and σ_ℓ are good (otherwise we are done) then supp(α) involves all simple roots in Δ. We claim that α is the heighest root. If not then using the above argument we find a root $\beta = c\alpha$ with $c > 1$ which is impossible or $\beta > \alpha$ such that $\beta - \langle\beta,\sigma_j\rangle\sigma_j$ is disconnected for some $\sigma_j \in \Delta$. Finally if $\sigma_{i_0} = \sigma_1$ say then if $n_{\sigma_2} = n_{\sigma_1} - \kappa$ with $\kappa \geq 1$ we get $\langle\alpha,\sigma_1\rangle = 2n_{\sigma_1} - (n_{\sigma_1} - \kappa)\sigma_1$. It follows that $\alpha - (n_{\sigma_1} + \kappa)\sigma_1$ is a root which is impossible. The remaining possibilities are treated similarly.

Case II. In this case we allow double bonds and bifurcations so we are not in A_ℓ or G_2. Let α_g be the unique good root. The strategy here is to begin with a root $\alpha \neq r_N$ and argue to a contradiction assuming that α_g is the only root one can subtract from α: i.e. we assume

(1) $\langle\alpha,\alpha_g\rangle \geq 1$

(2) $\langle\alpha,\sigma\rangle \leq 0 \quad \forall\sigma \in \mathrm{Supp}(\alpha), \sigma \neq \alpha_g$

and in fact that the σ-string through α is

$$\alpha, \alpha+\sigma, \ldots, \alpha+q\sigma.$$

We have not been able to find a single argument which covers all cases. The systems F_4 and E_8 can be treated easily noting that the good root of E_8 must be the last one, i.e. the root labelled 8 in section 1.7. We omit the proofs of these. We prove it for C_ℓ; B_ℓ is similar.

Let $\Delta = \{\sigma_1, \ldots, \sigma_\ell\}$ be a base for C_ℓ. The good root is σ_1 and $\langle\sigma_{\ell-1},\sigma_\ell\rangle = -1$, $\langle\sigma_\ell,\sigma_{\ell-1}\rangle = -2$. Let σ_M be the root with n_{σ_M} maximal.

We distinguish cases:

Case A: $1 < m < \ell - 1$.

It follows that $c = n_{\sigma_i} = n_{\sigma_M}$ for $1 \leq i \leq \ell-1$. Next, $\langle\alpha,\sigma_\ell\rangle = -c + 2n_{\sigma_\ell} \leq 0$ and $\langle\alpha,\sigma_{\ell-1}\rangle = 2c - c - 2n_{\sigma_\ell} \leq 0$. Hence

$c = 2n_{\sigma_\ell}$, this implies that $r_N = \sum_{1 \leqslant i \leqslant \ell-1} (2a)\sigma_i + a\sigma_\ell$ for some $a \geqslant 1$. Note that then $\langle r_N, \sigma_1 \rangle = 2a$, and so $a = 1$. Since $\alpha \neq r_N$ we must have $n_{\sigma_\ell} = 0$ which implies $\alpha = 0$ which is impossible.

Case B: $m = \ell - 1$.

This is entirely similar.

Case C. $m = 1$.

If $n_{\sigma_2} = n_{\sigma_1}$ then it follows that $n_{\sigma_i} = n_{\sigma_1}$, $3 \leqslant i \leqslant \ell-1$. Now we can use case A. So suppose $n_{\sigma_2} < n_{\sigma_1}$. Then $\langle \alpha, \sigma_1 \rangle = 2n_{\sigma_1} - n_{\sigma_2} \leqslant 3$. This implies $n_{\sigma_1} = 2$ and $n_{\sigma_2} = 1$, in which case $\langle \alpha, \sigma_1 \rangle = 3$. Hence $\alpha - 3\sigma_1$ is a root but this is impossible. One can treat G_2, D_ℓ, E_6 and E_7 in a similar way. We do only E_6. The good root of E_6 can be identified as follows: E_6 has an automorphism which leaves 2,4 fixed and maps $3 \mapsto 5$, $1 \mapsto 6$ (see section 1.7). The good root must be fixed hence it is equal to the root 2.

We first find r_N (that is the maximal root). Let $r_N = \sum n_i \cdot i$. By symmetry $n_1 = n_6 = x$, $n_3 = n_5 = y$. Since 2 is the good root $2x = y$, $2y - x - n_4 = 0$. Hence $3x = n_4$. Next, $2n_4 - 2y - n_2 = 0$ from which we get that $2x = n_2$. Finally $2n_2 - n_4 = x$. From this last equality it follows that $x = 1$, 2,3. It is easy to eliminate the cases $x = 2,3$. Hence $r_N = 1 + 2.2 + 3.4 + 2.5 + 1.6$.

Let $\alpha = n_i 1 = 1.2 + n_3\ 3 + n_4.4 + n_5.5 + n_6.6$. Since we are arguing by contradiction (see (1), (2) after case II in the beginning of the proof) it follows that $n_4 = 1$. Also $2.n_3 - n_1 - 1 \leqslant 0$ so $(n_3 - n_1) + (n_3 - 1) \leqslant 0$. But $2n_1 - n_3 < 0$ so $(n_3 - n_1) + (n_3 - 1) > 0$ if $n_3 \geqslant 1$. Since this is not possible we must have $n_3 = 0$ and so $n_1 = 0$. The same situation applies to the roots 5 and 6. Hence we are left with $\alpha = 1.2 + 1.4$, but then we may subtract the root 4.

Final Remarks 3.2. The proof of Theorem 3.1 is unsatisfactory in the sense that it has too many cases and does not 'explain' why it is true. Another proof could be given by listing all roots and checking the statement directly (see [Bo, Ch.7]). This is not too dificult but is still unsatisfactory.

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