Revista Colombiana de Matemáticas
Vol. XXII (1988), págs. 193-202

GENERIC PROPERTY OF TONELLI'S METHOD FOR ORDINARY DIFFERENTIAL EQUATIONS

by

Jozef BANAS(*) and Jesús RIVERO

Introduction. In the theory of ordinary differential equations there are many methods allowing to obtain solutions of the considered equations, provided such solutions do exist. Let us mention such methods as Peano-Picard successive approximations, that of Euler-Cauchy polygonals, and the method of upper and lower functions associated with the notion of Perron integral (cf. [4,6,9,11,13], for example).

One of the most frequently used method turns out to be the Tonelli's method [12] (see also [3,4,11]), since it is very handy and convenient in many situations.

The goal of this paper is to show that Tonelli's method is generic, what means, roughly speaking, that most differential equations have such property, that is: Tonelli's sequences constructed for them do converge to solutions of those equations. It is worthwhile to mention that the methods involving Peano-Picard successive approximations or Euler-Cauchy polygonals share this property [1,2,7,8]. More-

(*) The results contained in this paper were obtained while the first author was a visitor at the Universidad de los Andes, in Mérida, Venezuela.

over, the oldest known theorem connected with the notion of a generic property, due to Orlicz [10], asserts that most of the differential equations have unique solutions.

Let us mention also that many results concerning generic properties may be found in the survey work of Myjak [8].

1. Preliminaries. We begin by stating some basic notions used in the theory of generic properties (compare [8]).

Let (M,ρ) be a given complete metric space. Recall that a subset X of M is said to be *of the first category in* M if it is the union of a countable family of sets each of which is nowhere dense in M. A subset of M is said to be *of the second category in* M if it is not of the first category. Finally, a set X, M is called a *residual set in* M if its complement is of the first category in M.

Actually, every residual subset of M is of the second category in M, but not conversely. Moreover, every residual set in M is dense in M. Obviously, the reversedassertion is no longer true.

We will say that a property is generic in M if the subset on which it is true is a residual set in M.

In what follows we will use the following lemma due to Lasota and Yorke [7].

LEMMA 1. *Let X be a dense subset of a complete metric space M. Suppose that $\Phi: M \to \langle 0,+\infty\rangle$ is a function having the following property: for each $x \subset X$ and any sequence $\{x_n\} \subset M$ which converges to x, we have $\lim_{n\to\infty} \Phi(x_n) = 0$. Then, the set $\{x \in M: \Phi(x) = 0\}$ is residual in M.*

For the proof we refer to [8].

Let us state yet some notations which will be used in the sequel.

For given complete metric spaces U and V, the symbol

$C(U,V)$ will denote the space of all mappings $x:U \to V$, continuous and bounded on U, equipped with the distance

$$d(x,y) = \sup[\rho_V(x(t),y(t)) : t \in U].$$

Of course, $C(U,V)$ with this distance forms a complete metric space.

The subset of $C(U,V)$ consisting of all mappings which are locally lipschitzian on U, will be denoted by $L_{loc}(U,V)$ and the symbol $L(U,V)$ is reserved for the set of lipschitzian mappings from U to V. Obviously, in case U is a compact metric space we have $L_{loc}(U,V) = L(U,V)$.

Finally let us state that the closed ball centered at x and with radius r will be denoted by $B(x,r)$, and the closed interval with endpoints α,β, by $\langle\alpha,\beta\rangle$.

2. Generic property of Tonelli's method in the one-dimensional case. Let us consider an ordinary differential equation

$$x' = f(t,x), \tag{1}$$

with the initial value condition

$$x(0) = x_o . \tag{2}$$

(Throughout this section we will always assume that

$$f:\langle 0,1\rangle \times \langle x -r, x +r\rangle \to \langle -r,r\rangle$$

is continuous. Obviously, the problem (1)-(2) is equivalent to the following integral equation

$$x(t) = x_o + \int_o^t f(s,x(s))ds, \quad t \in \langle 0,1\rangle \tag{3}$$

Further, assume that $\delta > 0$ is a given number and

$$\phi:\langle -\delta,0\rangle \to \langle -r,r\rangle$$

is a given function, continuously differentiable on $\langle -\delta,0\rangle$

and such that $\phi(0) = x_0$, $\phi'(0) = f(0,x_0)$ and $|\phi'(t)| \leqslant r$ in $(-\delta,0)$. Such class of function will be called an *initial function* (for the problem (1)-(2)). Next, choose a number $\varepsilon \in \langle 0,\delta\rangle$ and define a function $x_\varepsilon(t)$ on the interval $\langle -\delta,1\rangle$ in the following way:

$$x_\varepsilon(t) = \begin{cases} \phi(t) \text{ for } t \in \langle -\delta,0\rangle \\ x_0 + \int_0^t f(s,x_\varepsilon(s-\varepsilon))ds \text{ for } t \in \langle 0,1\rangle. \end{cases} \quad (4)$$

It is easy to check (c.f. [4]) that the function $x_\varepsilon(t)$ is well defined, continuously differentiable on $\langle -\delta,1\rangle$, and $x_\varepsilon : \langle -\delta,1\rangle \to \langle -r,r\rangle$. Moreover, $|x'_\varepsilon(t)| \leqslant r$. Now take an arbitrary sequence $\{\varepsilon_n\}$ converging to 0. Without loss of generality we may assume that $\{\varepsilon_n\}$ is non-increasing so that it suffices to assume that $\varepsilon_1 \leqslant \delta$.

The sequence $\{x\varepsilon_n(t)\}$ associated with the sequence $\{\varepsilon_n\}$ via (4), will be called *the Tonelli sequence for the problem (3) (or (1)-(2))*.

Assume that $\{x\varepsilon_n(t)\}$ is an arbitrary given Tonelli sequence. Thus, by virtue of what has been said above, we conclude that this sequence is equicontinuous and bounded, so it contains a subsequence convergent to a solution of the equation (3) [4]. In the case when (3) posseses exactly one solution, all Tonelli sequences are convergent to this one [4]. Such a situation occurs in the case we assume hypotheses guaranteeing the existence of a unique solution of (3), for example, when f is Lipschitzian. On the other hand, the subset consisting of all Lipschitzian mappings does not form a residual set in the space of all continuous mappings [8]. Nevertheless, we show that the convergence of a Tonelli sequence to a unique solution of (3) is a generic property.

In order to do this let us denote by L the space $L(\langle 0,1\rangle \times \langle x_0-r,x_0+r\rangle,\langle -r,r\rangle)$ and by C the space $C(\langle 0,1\rangle \times \langle x_0-r,x_0+r\rangle,\langle -r,r\rangle)$. Both of these spaces are equipped with the usual maximum norm which will be denoted by $\|\cdot\|$. The symbol $C\langle 0,1\rangle$ will be reserved for the space

$C(\langle 0,1\rangle,\mathbf{R})$, and its norm will be denoted by $\|\cdot\|_{C\langle 0,1\rangle}$.

If we work in the Cartesian product $C\times\mathbf{R}$, then for $(f,x)\subset C\times\mathbf{R}$ we define the norm $(f,x) = \|f\| + |x|$.

We begin with the following lemma.

LEMMA 2. *Let $g\in L$ and let x^g denote the unique solution of the equation (3) (with g instead of f). Then, for every $\xi>0$ there are $\sigma>0$ and $\rho>0$ such that, if $(f,y)\in B((g,x_0),\sigma)\subseteq C\times\mathbf{R}$ and $\{\varepsilon_n\}$ is an arbitrary sequence as described above, while ϕ is an arbitrary initial function, then we have*

$$\|x_{\varepsilon_n}^{f,y} - x^g\|_{C\langle 0,1\rangle} \leqslant \xi$$

for $n \geqslant n_0 = \min\{k:\varepsilon_K \leqslant \rho\}$.

Proof. Let K denote a Lipschitz constant for the function g. Taking $n\in N$ and $t\in\langle 0,1\rangle$ we get

$$|x_{\varepsilon_n}^{f,y}(t)-x^g(t)| \leqslant |y-x_0|+\int_0^t |f(s,x_{\varepsilon_n}^{f,y}(s-\varepsilon_n)-g(s,x^g(s))|ds$$

$$\leqslant \sigma+\int_0^t\{|f(s,x_{\varepsilon_n}^{f,y}(s-\varepsilon_n))-g(s,x_{\varepsilon_n}^{f,y}(s-\varepsilon_n))|$$

$$+|g(s,x_{\varepsilon_n}^{f,y}(s-\varepsilon_n))-g(s,x^g(s))|\}ds$$

$$\leqslant \sigma+\int_0^t(\sigma+K|x_{\varepsilon_n}^{f,y}(s-\varepsilon_n)-x^g(s)|)ds$$

$$\leqslant 2\sigma+K\int_0^t\{|x_{\varepsilon_n}^{f,y}(s-\varepsilon_n)-x^g(s-\varepsilon_n)|+|x^g(s-\varepsilon_n)-x^g(s)|\}ds$$

$$\leqslant 2\sigma+K\int_0^t\{[\max_{\tau\in\langle 0,s\rangle}|x_{\varepsilon_n}^{f,y}(\tau)-x^g(\tau)|]+r\varepsilon_n+w(\phi,\varepsilon_n)\}ds\ ,$$

where $w(\phi,\varepsilon_n)$ denotes the modulus of continuity of the function ϕ on the interval $\langle -\delta,0\rangle$. Obviously, in virtue of the assumptions made at the beginning, we have $w(\phi,\varepsilon_n)\leqslant r\varepsilon_n$. Further, using the above estimate, we can proceed as follows

$$|x_{\varepsilon n}^{f,y}(t)-x^g(t)| \leqslant 2\sigma + 2K r\rho + \int_0^t [\max_{\tau\in\langle o,s\rangle} |x_{\varepsilon n}^{f,y}(\tau)-x^g(\tau)|]ds.$$

Hence, owing to the fact that the function on the right hand side of the above inequality is nondecreasing, we obtain

$$\max_{\tau\in\langle o,t\rangle} |x_{\varepsilon n}^{f,y}(\tau)-x^g(\tau)| < 2\sigma + 2K r\rho + K\int^t [\max_{\tau\in\langle o,s\rangle} |x_{\varepsilon}^{f,g}(\tau)-x^g(\tau)|]ds .$$

Next, applying the well-known Gronwall lemma [13], we get

$$\max_{\tau\in\langle o,t\rangle} |x_{\varepsilon n}^{f,g}(\tau)-x^g(\tau)| \leqslant 2(\sigma+K r\rho)e^K, \qquad t \in \langle 0,r\rangle$$

and consequently

$$\| x_{\varepsilon n}^{f,y} - x^g\|_{C\langle o,1\rangle} \leqslant 2(\sigma+K r\rho)e^K.$$

Thus, in view of the above inequality, we can put

$$\sigma = \xi e^{-K}/4 , \quad \rho = \xi e^{-K}/4Kr$$

which completes the proof.

REMARK. As an inmediate consquence of Lemma 2 we obtain that, for each initial function ϕ and each $(g,y) \in L\times R$ the Tonelli sequence $\{x_{\varepsilon_n}^{g,y}\}$ converges uniformly on $\langle 0,1\rangle$ to a solution of the equation (3) (with f replaced by g and x_o by y). Thus corollary may be obtained with the help of a direct argumentation as the one we sketched before (cf. also [4]).

THEOREM 1. *Let T be the set of all $(f,y) \in C\times\mathbf{R}$ such that, for any sequence $\{\varepsilon_n\}$, the corresponding Tonelli sequence $\{x_{\varepsilon n}^{f,y}\}$ converges uniformly on $\langle 0,1\rangle$ to a solution of the problem (1)-(2) (or (3)), where $x_o = y$. Then T is a residual set in $C\times\mathbf{R}$.*

Proof. It is easily seen that the limit of $\{x_{\varepsilon n}^{f,y}\}$, if it exists, is a solution of the equation (3), where $x_o = y$.

Define now $\Phi: C\times\mathbb{R} \to \langle 0,+\infty\rangle$ by

$$\Phi((f,y)) = \limsup_{n,m\to\infty} \| x_{\varepsilon_n}^{f,y} - x_{\varepsilon_n}^{f,y} \| .$$

We show that Φ satisfies the assumptions of Lemma 1 with $M = C\times\mathbb{R}$ and $X = L\times\mathbb{R}$. In fact, let $\varepsilon > 0$. Let $(g,z) \in L\times\mathbb{R}$ and $x^{g,z}$ be the unique solution of the problem (3) (where $f = g$, $x_o = z$). Further let $\sigma = \sigma(\varepsilon/2)$ and $\rho = \rho(\varepsilon/2)$ correspond to $(g,y) \in C\times\mathbb{R}$, according to Lemma 2. Then, for $(f,y) \in B((g,z),\sigma(\varepsilon/2))$ and $m,n \geq n_o$, we infer

$$\| x_{\varepsilon_n}^{f,y} - x_{\varepsilon_m}^{f,y} \| \leq \| x_{\varepsilon_n}^{f,y} - x^{g,z} \| + \| x^{g,z} - x_{\varepsilon_m}^{f,y} \|$$

$$\leq \varepsilon/2 + \varepsilon/2 = \varepsilon .$$

Hence

$$\limsup_{n,m\to\infty} \| x_{\varepsilon_n}^{f,y} - x_{\varepsilon_m}^{f,y} \| \leq \varepsilon$$

and, consequently,

$$\Phi((f,y)) \leq \varepsilon$$

for each $(f,y) \in B((g,z),\sigma(\varepsilon/2))$. Now, taking into account the arbitrariness of ε and Lemma 1 we infer that the set T is residual and the theorem is proved.

3. The case of Banach space. Now let us consider the general case when the set $\mathbb{R}$ (in the consideration in the precedent section) is replaced by an arbitrary Banach space E. Actually, the formulation of the problem is the same: one should only take $B(x_o,r)$ and $B(0,r)$ instead of the intervals $\langle x_o-r, x_o+r\rangle$ and $\langle -r,r\rangle$, respectively. Thus we will also denote $C = C(\langle 0,1\rangle\times B(x,r),\ (0,r))$ and analogously L, L_{loc}. But on the other hand, the equality $L_{loc} = L$ is no longer true and, moreover, the set L is not generally dense in C.

Nevertheless, all this may be overcome via the following lemma [7].

LEMMA 3. *Let E and F be Banach spaces with norms $\|\cdot\|_E$ and $\|\cdot\|_F$ and let $\Omega \subset E$ be open. Further, let $f:\Omega \to F$ be continuous. Then, for each $\varepsilon > 0$ there exists a locally Lipschitzian function $g:\Omega \to F$ such that*

$$\|f-g\| \leqslant \varepsilon .$$

In view of this lemma we have that the set L_{loc} is dense in C.

Now, in order to preserve all the argumentation from the previous section, one only has to take into account the following simple, but very useful result [2].

LEMMA 4. *For each $g \in L_{loc}$ and every continuous function $w:<0,1> \to B(x_o,r)$, there exist $K > 0$ and $\eta > 0$ such that*

$$\|g(t,x)-g(s,y)\| \leqslant K(|t-s| + \|x-y\|)$$

for all (t,x), $(s,y) \subset B((t,w(t)),\eta) \subset <0,1>\times E$.

Thus we can formulate the following.

THEOREM 2. *Let T be the set of all $(f,y) \in C\times E$ such that, for any initial function ϕ, the corresponding Tonelli sequence $\{x^{f,y}_{\varepsilon_n}\}$ converges uniformly on $<0,1>$ to a solution of the problem (3), where $x_o = y$. Then T is a residual set in $C\times E$.*

Finally let us notice that combining the above result and a theorem of Kuratowski and Ulam, [5], we infer the following result.

THEOREM 3. *If E is a separable Banach space then there exist a set C^o, residual in C, such that, to each $f \in C^o$ corresponds a set E_f, residual in E, with the following property: if $y \in E_f$ and ϕ is an arbitrary initial function, then the Tonelli sequence $\{x^{f,y}_{\varepsilon_n}\}$ converges uniformly on $<0,1>$ to a solution of (3), where $x_o = y$.*

*

REFERENCES

[1] Benavides, T.D., *Generic existence of a solution for a differential equation in a scale of Banach space*, Proc. Amer. Soc. 86 (1982), 477-484.

[2] DeBlasi, F.S. and Myjak, J., *Generic Flows generated by continuous vector fields in Banach spaces*, Advances in Math. 50 (1983), 266-280.

[3] Deimling, K., *Ordinary differential equations in Banach spaces*, Lect. Notes in Math. 596, Springer-Verlag 1977.

[4] Hartman, P., *Ordinary differential equations*, J. Wiley, 1964.

[5] Kuratowski, J. and Ulam, S., *Quelques propietes topologigues du produit combinatoire*, Fund. Math. 19 (1932), 248-251.

[6] Lakshmikanthan, V. and Leel, S., *Differential and integral inequalities*, Academic Press, New York, 1969.

[7] Lasota, A. and Yorke, J., *The generic property of existence of solutions of differential equations in Banach space*, J. Differential Equations, 13 (1973), 1-12.

[8] Myjak, J., *Orlicz type category theorems for functional and differential equations*, Dissertationes Math. CCVI, 1983.

[9] Olech, C., *Remarks concerning criteria for uniqueness of solutions of ordinary differential equations*, Bull. Acad. Polon. Sci. Ser. Sci. Math. Astronom. Phys. 8 (1960), 661-666.

[10] Orlicz, W., *Zur theorie der Differentialgleichung* $y' = f(x,y)$, Bull. Acad. Polon. (1932), 221-228.

[11] Szarski, J., *Differential inequalities*, Monografie Matematyczne 43, Warsaw 1965.

[12] Tonelli, L., *Sulle equazioni funzionali del tipo de Volterra*, Bull. Calcutta Math. Soc. 20 (1928), 31-48.

[13] Walter, W., *Differential and integral inequalities*, Springer-Verlag, 1970.

*

J. Banas.
Department of Mathematics
I. Lukasiewicz Technical University
35-084 Rzeszow, Poznanska 2
Poland.

J. Rivero
Departamento de Matemáticas
Facultad de Ciencias
Universidad de los Andes
Mérida, 5101, Venezuela.

(Recibido en junio de 1987)