Revista Colombiana de Matemáticas
Vol. XXII (1988), págs. 203-208

ON THE INTERPOLATION BETWEEN CERTAIN THEOREMS ON FOURIER TRANSFORMS

by

M. S. YOUNIS

1. Introduction. It is well known that if $f(x)$ belongs to $L^p(R)$, $1 < p \leqslant 2$, then the Hausdorff-Young inequality ([1], Theorem 74) asserts that its Fourier transform $\hat{f}(u)$ belongs to $L^{p'}(R)$, where

$$\frac{1}{p'} + \frac{1}{p} = 1$$

G.H. Hardy and J.E. Littlewood ([1] Theorems 79,80) proved two variations of the Hausdorff-Young inequality by employing weight functions in $L^p(R)$ and $L^{p'}(R)$, respectively.

Our aim in this paper is to apply the Stein-Weiss interpolation theorem ([2], Corollary 5.5.4, p.120), to these three theorems and show the effect of this interpolation on the exponents of f and $\hat{f}$ respectively.

2. Definitions and Notations. All the definitions and notations used in this work are standard and well established in the literature. We follow [1] and [2] for the basic results that will be needed in due course. Thus $L^p(R)$ denotes the space of equivalence classes of functions whose pth power is Lebesgue integrable on the real line R.

Let w be a positive non decreasing function. Then $w(x)$ is called a *weight function*. The measure space $L_p(U, w\,du)$ with respect to the weight function w and the measure du on the domain U is the L^p space of those functions $f(x)$ such that

$$\int |f|^p w\,du < \infty .$$

The *Fourier-transform* of the function $f(x) \in L^p(R)$ is the function

$$\hat{f}(u) = \frac{1}{2\pi} \int_R f(x) e^{-ixu} dx$$

with the usual extension of this definition to functions of several variables in R^n.

It would be convenient for further reference to state the following four theorems.

THEOREM 2.A (Hausdorff-Young). *Let $f(x)$ belong to $L^p(R)$, $1 < p \leqslant 2$. Then its Fourier transform $\hat{f}(u)$ belongs to $L^{p'}$, $2 \leqslant p' < \infty$ where $\frac{1}{p} + \frac{1}{p'} = 1$.*

THEOREM 2.B ([1], p.108). *Let $f(x)^q x^{q-2}$ $(q > 2)$ belong to $L^1(R)$. Then $f(u)$ exists and belongs to $L^q(R)$. Furthermore*

$$\int_R |\hat{f}(u)|^q du \leqslant k(q) \int_R |f(x)|^q |x|^{q-2} dx$$

$k(q)$ being constant.

THEOREM 2.C ([1], p.110). *Let $f(x)$ belong to $L^p(R)$, $1 < p < 2$. Then $\hat{f}(u)$ exists and*

$$\int_R |\hat{f}(u)|^p |u|^{p-2} du \leqslant k(p) \int_R |f(x)|^p dx$$

where $k(p)$ is constant.

THEOREM 2.D (Stein-Weiss Theorem [2] p.120). *Assume that*

$1 < p_0, p_1, q_0, q_1 < \infty$ *and that*

$$T: L_{p_0}(U, w_0 d\mu) \rightarrow L_{q_0}(V, \bar{w}_0 d\gamma)$$
$$T: L_{p_1}(U, w_1 d\mu) \rightarrow L_{q_1}(V, \bar{w}_1 d\gamma)$$

with norms M_0 *and* M_1 *respectively. Then*

$$T: L_p(U, w d\mu) \rightarrow L_q(V, \bar{w} d\gamma)$$

with norm M *satisfying*

$$M \leqslant M_0^{1-\theta} M_1^{\theta}$$

where

$$\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}, \qquad \frac{1}{q} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}$$

and

$$w = w_0^{p(1-\theta)/p_0} w_1^{p\theta/p_1}$$

$$\bar{w} = \bar{w}_0^{q(1-\theta)/q_0} \bar{w}_1^{q\theta/q_1}, \qquad 0 < \theta < 1.$$

3. Main Results. In this section we shall apply the Stein-Weiss Theorem (2.D) to Theorems 2.A and 2.B, then we will apply it to (2.A) and (2.C). Thus for the first phase of application we have (for Hausdorff-Young theorem)

$$u = V = R, \quad d\mu = dx, \quad d\gamma = du,$$

$$p_0 = p, \quad q_0 = p' = \frac{p}{p-1}, \quad w_0 = \bar{w}_0 = 1.$$

For Theorem (2.B) we have

$$p_1 = q_1 = q = p' = \frac{p}{p-1} = q,$$

$$w_1 = x^{q-2} = x^{p'-2}, \quad \bar{w}_1 = 1.$$

With these notations Theorem 2.D asserts that the Fourier transform $\hat{f}(u)$ is a linear operator T from $L_p(\mathbb{R}, w dx)$ into $L_Q(\mathbb{R}, \bar{w} du)$, where

$$\frac{1}{P} = \frac{1-\theta}{p} + \frac{\theta}{p'}, \quad \frac{1}{Q} = \frac{1-\theta}{p'} + \frac{\theta}{p'}, \quad w = |x^{p'-2}| \frac{p\theta}{p'}$$

and

$$\bar{w} = 1 .$$

These relations yield

$$Q = p'$$

and hence we arrive at

$$\int |\hat{f}(u)|^{p'} du \leqslant A \int |f(x)|^P |x^{p'-2}|^{P\theta/p'} dx$$

$$= A \int |f(x)|^P |x|^{(2-p)P\theta/p'} dx .$$

To see the relation between p and P we notice that

$$\frac{1}{P} = \frac{1-\theta}{p} + \frac{\theta}{p'} = \frac{1-\theta}{p} + \frac{\theta(p-1)}{p} = \frac{1}{p} + \frac{(p-2)\theta}{p} .$$

Since θ is always positive and $p < 2$, then the second part on the right side of the last equation is negative, which indicates that

$$\frac{1}{P} < \frac{1}{p} ,$$

and hence $P > p$. Thus we have proved the following theorem.

THEOREM 3.1. *Let $f(x)$ belong to $L^P(R)$, $1 < p < 2$. Then for $P > p$, the Fourier transform $\hat{f}(u)$ belongs to $L^{p'}(R)$ and*

$$\int |\hat{f}(u)|^{p'} du \leqslant A \int |f(u)|^P |x|^{(2-p)P\theta/p} dx .$$

This theorem shows the interplay between the power of $f(x)$ and that of the weight function $w(x)$. Our next step is to show in a similar manner the interplay between the power of the Fourier transform $\hat{f}(u)$ and the weight function associated with it. For this purpose we apply the Stein-Weiss Theorem to (2.A) and (2.C). In this case we have $p_1 = p = q_1$, $w_1 = 1$, and $\bar{w}_1 = (u^{p-2})$. Then

$$\frac{1}{P} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1} = \frac{1-\theta}{p} + \frac{\theta}{p}$$

Thus $P = p$, and

$$\frac{1}{Q} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1} = \frac{1-\theta}{p'} + \frac{\theta}{p} = \frac{1-\theta}{p'} + \frac{\theta(p'-1)}{p'} = \frac{1}{p'} + \frac{(p'-2)\theta}{p'}$$

But $p' > 2$, and $\theta > 0$, hence

$$\frac{1}{Q} > \frac{1}{p'}, \qquad \text{i.e.} \quad Q < p' .$$

Now in this case, $w = 1$ and $\bar{w} = |u^{p-2}|^{Q\theta/p}$. Thus we have proved.

THEOREM 3.2. *Let $f(x)$ belong to $L^p(R)$, $1 < p \leq 2$. Then*

$$\int |\hat{f}|^Q |u|^{(p-2)Q\theta/p} du \leq A \int |f|^p dx$$

where $p' > Q$.

One can also see in this case the effect of the Q-th power of $\hat{f}(u)$ on that of the weight function $\bar{w}$.

REMARK 3.3. If we employ Parseval's identity instead of the Hausdorff-Young inequality in Theorem 3.1 we get

$$\frac{1}{P} = \frac{1-\theta}{2} + \frac{\theta}{p'}, \quad \frac{1}{Q} = \frac{1-\theta}{2} + \frac{\theta}{p'}$$

which shows that P and Q are equal in this case. But

$$\frac{1}{P} + \frac{1}{Q} = 1-\theta + \frac{2\theta}{p'}$$

and hence P, Q are conjugate only if $\theta = 2/p'$, i.e. for $p = 2$, which is quite reasonable an expectation. On the other hand if we use Parseval's identity in Theorem 3.2 we arrive at

$$\frac{1}{p} = \frac{1}{Q} = \frac{1-\theta}{2} + \frac{\theta}{p},$$

and thus

$$\frac{1}{p} + \frac{1}{Q} = 1-\theta + \frac{2\theta}{p},$$

which shows that P and Q are not conjugate in this case except for the special value $P = Q = 2$.

Finally we would like to point out that the Setin-Weisse interpolation theorem is valid for vector-valued functions in a general Banach space and this suggests the task of generalizing Hardy and Littlewood theorems ([1], Theorem 79, 80) in that direction.

*

REFERENCES

[1] Titchmarsh, E.C., *Theory of Fourier Integrals*, 2nd Ed., Oxford University Press, 1948.

[2] Bergh, J. and Löfström, J., *Interpolation Spaces*, Springer Verlag, 1976.

*

Department of Mathematics

Yarmouk University

Irbid - Jordan.

(Recibido en julio de 1984, la versión revisada en octubre de 1987).