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AN EXCHANGE PROPERTY FOR THE SUBMODULAR SYSTEMS

by

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Abstract. The purpose of this note is to point out that the subset exchange property of matroid bases is a special case of a general vector exchange property, satisfied by submodular systems.

Introduction. That the Steinitz exchange property for bases of a matroid is really a special case of a subset exchange property for pairs of bases was proved, independently, by Woodall [9], Greene [5] and Brylawski [1]. Whereas [5] and [1] use a lengthy set-theoretic approach, the argument in [8] is based on Edmonds' matroid intersection theorem [2]. Employing the matroid sum theorem, the dual of the intersection theorem, instead, McDiarmid [6] was the able to further simplify the proof.

The purpose of this note is to point out that the subset exchange property of matroid bases in turn is a special case of a general vector exchange property, satisfied by submodular systems. This exchange property rests on the fact that a polyhedral analogue of the matroid sum theorem exists for general submodular systems.

The submodular systems, considered here, are convex polytopes determined by normalized submodular functions on distributive lattices of subsets. Passing to submodular functions on crossing families of subsets would not result in greater generality, as Fujishige [4] has shown that the collection of bases of a submodular polytope with respect to a crossing family containing the ground set coincides with the collection of bases of an unique submodular system as defined here, unless the former collection was empty.

Preliminaries. In this section, we list some fundamental properties of submodular systems as introduced by Fujishige [3]. The properties are straightforward generalizations of well-known properties of polymatroids (e.g., see [2] or [7]).

Let E be a finite set and $\mathcal{D} \subseteq 2^E$ a family of subsets with $\emptyset$, $E \in \mathcal{D}$, so that $\mathcal{D}$ is closed under taking unions and intersections. A function $f:\mathcal{D} \to \mathbf{R}$ is *(normalized) submodular*, if

(1) $f(\emptyset) = 0$,

(2) $f(A \cup B) + f(A \cap B) \leqslant f(A) + f(B)$, for all $A,B \in \mathcal{D}$.

The pair $(\mathcal{D}, f)$ gives rise to a *submodular system*, i.e., to the (unbounded) convex polytope

(3) $\mathbf{P}(f) = \{x \in \mathbf{R}^E : x(A) \leqslant f(A), \text{ for all } A \in \mathcal{D}\}$,

where $y(S)$ denotes the sum of those components of the vector $y \in \mathbf{R}^E$ with indices in $S \subseteq E$. If f is non-decreasing and $\mathcal{D} = 2^E$, we obtain the *polymatroid*

(4) $\mathbf{P}^+(f) = \{x \in \mathbf{P}(f) : x \geqslant 0\}$.

Given the submodular systems $(\mathcal{D}_1, f_1)$ and $(\mathcal{D}_2, f_2)$, we may consider the submodular function $f_1 + f_2 : \mathcal{D}_1 \cap \mathcal{D}_2 \to \mathbf{R}$. The *sum theorem* [3,6] states that

(5) $\mathbf{P}(f_1 + f_2) = \mathbf{P}(f_1) + \mathbf{P}(f_2)$.

Moreover, for polymatroids, we obtain

(6) $\mathbf{P}^+(f_1 + f_2) = \mathbf{P}^+(f_1) + \mathbf{P}^+(f_2)$.

If $\mathbf{P}(f)$ is a submodular system and $v \in \mathbf{R}^E$, we define the *left translation* of $\mathbf{P}(f)$ by v as

(7) $\mathbf{P}(f-v) = \{x \in \mathbf{R}^E : x+v \in \mathbf{P}(f)\}$.

In view of (3), $\mathbf{P}(f-v)$ is the sum of the submodular polytopes determined by f and $(-v)$. Furthermore, if f is non-decreasing and $v \in \mathbf{P}^+(f)$, $f-v$ again gives rise to a polymatroid.

A submodular system or polymatroid is *integral*, if it is derived from an integer-valued function. It is well-known that the sum theorem remains true if attention is restricted to the integral vectors of integral submodular systems or polymatroids.

The exchange property. Using the sum theorem, we are now in the position to prove the vector exchange property.

THEOREM. *Let* x *and* y *be arbitrary vectors in the submodular system* $\mathbf{P}(f)$ *and* $x_1, x_2 \in \mathbf{R}^E$, *so that* $x = x_1+x_2$. *Then, there are vectors* $y_1, y_2 \in \mathbf{R}^E$, *so that* $y = y_1+y_2$ *and both* x_1+y_1 *and* $x_2+y_2 \in \mathbf{P}(f)$. *Moreover, if* f *is non-decreasing and* $x_1, x_2, y \in \mathbf{P}^+(f)$, *then one may choose* $y_1, y_2 \in \mathbf{P}^+(f)$. *The exchange property remains satisfied if attention is restricted to integral vectors of integral submodular systems or polymatroids.*

Proof. We must show $y \in \mathbf{P}(f-x_1)+\mathbf{P}(f-x_2)$, i.e., $y \in \mathbf{P}(2f-x)$, by the sum theorem. But this follows immediately from the hypothesis $y(A) \leqslant f(A)$ and $(f-x)(A) \geqslant 0$. Moreover, in the case of polymatroids, note that $y_i \in \mathbf{P}^+(f-x_i)$ and $x_i \geqslant 0$ implies $y_i \in \mathbf{P}^+(f)$. ▲

A vector $x \in \mathbf{P}(f)$ is a *basis* of the submodular system $\mathbf{P}(f)$, if $x(E) = f(E)$. It is clear that if the vectors x and y in theorem are bases, x_1+y_1 and x_2+y_2 are bases as well. Since the integral vectors of the integral polymatroid defined by a matroid rank function are exactly the 0-1 incidence vectors of independent sets, the theorem therefore yields

COROLLARY ([1,5,8]). *Let* X *and* Y *be bases of a matroid on the set* E *and* $X = X_1 \cup X_2$ *a partition of* X. *Then, there exists a partition* $Y = Y_1 \cup Y_2$ *of* Y, *so that* $X_1 \cap Y_1 = X_2 \cap Y_2 = \emptyset$ *and both* $X_1 \cup Y_1$ *and* $X_2 \cup Y_2$ *are bases.*

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