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ON THE DEGREE OF SIMULTANEOUS APPROXIMATION BY MODIFIED BERNSTEIN POLYNOMIALS

by

S.P. SINGH, G. PRASAD, M.K. TIWARI

Abstract. Recently we proved some approximation theorems on the rth derivative of a Lebesgue integrable function by the corresponding rth derivative of modified Bernstein polynomials, Publ.Inst.Math., 87 (51) (1986) . In the present paper we improve an estimate of our earlier paper and compare it with the corresponding known results.

§1. Introduction. For a Lebesgue integrable function $f \in L^1[0,1]$, Derriennic [2] studied a new kind of positive linear operator $\{L_n\}$ of order n as defined by

$$(L_n f)(x) = (n+1)\sum_{k=0}^{n} p_{n,k}(x)\int_0^1 p_{n,k}(t) f(t)\,dt, \tag{1.1}$$

where

$$p_{n,k}(x) = \binom{n}{k} x^k (1-x)^{n-k}$$

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and modifies the Bernstein polynomials $\{B_n\}$,

$$(B_n f)(x) = \sum_{k=0}^{n} p_{n,k}(x) f(\tfrac{k}{n}), \quad f \in C[0,1]. \tag{1.2}$$

In our recent paper [3], we gave some theorems on the approximation of the rth derivative of a given function f by the corresponding rth derivative of the operators (1.1). Thr object of the present paper is to sharpen a result of that earlier paper ([8], Theorem 3.2).

§2. **Preliminary Lemmas.** In this section we give some lemmas which are useful in proving the main results of section 3.

LEMMA 2.1. *For $r < n$, one obtains that*

$$(L_n^{(r)} f)(x) = ((n+1)!n!/(n-r)!(n+r)!) \sum_{k=0}^{n-r} P_{n-r,k}(x) \int_0^1 P_{n+r,k+r}(t) \frac{d^{(r)} f(t)}{dt^r} dt \tag{2.1}$$

Proof. It follows from Derriennic ([2], page 334).

LEMMA 2.2. *For all t, $x_0 \in [a,b]$ and $\delta > 0$, one obtains*

$$\int_x^t \lceil |y-y_0|/\delta \rceil dy \leqslant ((t-x)^2/2\delta + (|t-x|)/2 + \delta/8), \tag{2.2}$$

where the symbol $\lceil \cdot \rceil$ indicates the ceiling of the number.

Proof. It follows from Anastassiou ([1], page 264).

LEMMA 2.3. *For $n > r$, let*

$$T_{n-r,m-r}(x) = m!/(m-r)!) \sum_{k=0}^{n-r} p_{n-r,k}(x) \int_0^1 p_{n\ r,k\ r}(t)(t-x)^{m-r} dt, \tag{2.3}$$

then we get the following relation,

$$((n+m+2)(m-r+1)/(m+1))T_{n-r,m-r+1}(x) \tag{2.4}$$

$$= (m+1)(1-2x)T_{n-r,m-r}(x)+2mx(1-x)T_{n-r,m-r-1}(x)+\ldots+x(1-x)T'_{n-r,m-r}(x),$$

with

$$T_{n-r,0}(x) = r!/(n+r+1). \tag{2.5}$$

Proof. It follows from the paper [3].

LEMMA 2.4. *For $n > r$, one gets*

$$L_n^{(r)}(t-x))(x) = (n!(n+1)!/(n-r)!(n+r+2)!))(1-2x), \tag{2.6}$$

and

$$(L_n^{(r)}(t-x)^2)(x)$$
$$= ((n+1)!(n+1)!/(n-r)!(n+r+3)!)\{2x(1-x)+(r+1)(r+2)(1-2x)^2/(n+1)\}. \tag{2.7}$$

Proof. It follows from (2.1), (2.3) to (2.5).

LEMMA 2.5. *For $n > r$, $0 \leqslant x \leqslant 1$, one gets*

$$(L_n^{(r)}(t-x)^2(x) \leqslant d(n,r)/(n+r+2) \tag{2.8}$$

where

$$d(n,r) = \begin{cases} (r+1)(r+2)/(n+1) & \text{if } n > 2r^2+6r^3 \\ 1/2 & \text{otherwise.} \end{cases} \tag{2.9}$$

Proof. We see from (2.7) that

$$(L_n^{(r)}(t-x)^2)(x) \leqslant (1/(n+r+2))\{2x(1-x)+(r+1)(r+2)(1-2x)^2/(n+1)\},$$

$$= (1/(n+r+2))A(n,x,r) \qquad \text{(say)}.$$

Clearly the maximum values of $A(n,x,r)$ for $0 \leqslant x \leqslant 1$ are $d(n,r)$.

§3. **Main results.**

THEOREM 3.1. *Let $f \in C^{(r+1)}[0,1]$ and let $w(f^{(r+1)};\ )$ be the moduli of continuity of $f^{(r+1)}$. Then for $n > r$, $(r = 0,1,2,\ldots)$, we have*

$$\|L_n^{(r)} f - f^{(r)}\| \leq ((r+1)/(n+r+2))\,\|f^{(r+1)}\| + w(f^{(r+1)};1/\sqrt{n+r+2})\cdot(1/\sqrt{n+r+2})\ldots \tag{3.1}$$
$$\ldots\left(\frac{d(n,r)}{2} + \frac{\sqrt{d(n,r)}}{2} + \frac{1}{8}\right),$$

where $d(n,r)$ has been defined in (2.9) and norm $\|\cdot\|$ stands for sup-norm over $[0,1]$.

Proof. Following [4], we write that

$$f^{(r)}(t) - f^{(r)}(x) = (t-x)f^{(r+1)}(x) + \int_x^t \{f^{(r+1)}(y) - f^{(r+1)}(x)\}\,dy$$

Now on applying (2.1) to the above and using the inequality of Anastassion ([1], page. 251).

$$|f^{(r+1)}(y) - f^{(r+1)}(x)| \leq w(f^{(r+1)};\delta)\lceil |y-x|/\delta \rceil$$

we get that

$$|(L_n^{(r)} f)(x) - f^{(r)}(x)|$$
$$\leq |f^{(r+1)}(x)|\cdot|(L_n^{(r)}(t-x))(x)| + w(f^{(r+1)};\delta)\,|L_n^{(r)}(\int_x^t \lceil \ldots |(y-x)/\delta| \rceil dy)(x)|$$
$$\leq |f^{(r+1)}(x)|\cdot|(L^{(r)}(t-x))(x)| + w(f^{(r+1)};\delta)\left\{\frac{(L_n^{(r)}(t-x)^2)(x)}{2\delta} + \frac{\sqrt{(L_n^{(r)}(t-x)^2)(x)}}{2} + \delta/8\right\}.$$

Be choosing $\delta = 1/\sqrt{n+r+2}$ and using the results (2.6) and (2.8), we get the required result (3.1). This completes the proof.

COROLLARY 3.2. *If $f^{(r+1)} \in \mathrm{Lip}_M$, $0 < \alpha \leq 1$; $M > 0$,* *then we have*

$$\|L_n^{(r)}f-f^{(r)}\| \leqslant ((r+1)/(r+n+2))\|f^{(r+1)}\|+\ldots \quad (3.2)$$

$$\ldots+M(n+r+2)^{-(\alpha+1)/2}\left\{\frac{d(n,r)}{2}+\ldots+\frac{\sqrt{d(n,r)}}{2}+\frac{1}{8}\right\}$$

THEOREM 3.3. *If, in addition to the hypotheses of Theorem 3.1,* $f \in C^{(r+2)}[0,1]$, *then we have*

$$\|L_n^{(r)}f-f^{(r)}\| \leqslant ((r+1)/(n+r+2))\|f^{(r+1)}\| \quad (3.3)$$

$$+ (\|f^{(r+2)}\|/(n+r+2))\left\{\frac{d(n,r)}{2}+\ldots+\frac{\sqrt{d(n,r)}}{2}+\frac{1}{8}\right\}.$$

Proof. From the definition of $w(f^{(r+1)};\delta)$, it follows that

$$w(f^{(r+1)};\delta) = \sup_{|x-y|\leqslant\delta} |f^{(r+1)}(y)-f^{(r+1)}(x)|$$

$$= \sup_{|x-y|\leqslant\delta} |y-x|\cdot|f^{(r+2)}(\xi)|, \quad x \leqslant \xi \leqslant y$$

$$\leqslant \delta \sup_{|x-y|\leqslant\delta} |f^{(r+2)}(\xi)|$$

$$\leqslant \delta \sup_{0\leqslant\xi\leqslant1} |f^{(r+2)}(\xi)| < \delta\|f^{(r+2)}\| \quad (3.4)$$

By using (3.4) in (3.1), we get the required result (3.5). This completes the proof.

§4. Remarks.

[A] The estimate (3.1) is sharper than the following corresponding estimate [3],

$$\|L_n^{(r)}f-f^{(r)}\| \leqslant ((r+1)/(n+r+2))\|f^{(r+2)}\|$$

$$+ (1/\sqrt{n})\{\sqrt{\lambda_r}+(\lambda_r/2)\}\ldots w(f^{(r+1)};1/\sqrt{n}),$$

where $\lambda = 1+(r/2)$.

[B] By putting $\gamma = 0$ in the estimate (3.1), we get

$$\|L_n f - f\| \leqslant (\|f'\|/n+2) + (1/\sqrt{+2})\{(d(n,0)/2) + (\sqrt{d(n,0)}/2) + \ldots$$
$$\ldots + \tfrac{1}{8})\}\ (d';1/\sqrt{n+2}),$$

where

$$d(n,0) = \begin{cases} 2/(n+1) & \text{if } n > 3 \\ 1/2 & \text{otherwise.} \end{cases}$$

We note here that the estimate (4.1) is sharper than the following estimate ([5], page 27),

$$\|L_n f - f\| < (\|f'\|/(n+2)) + (1/\sqrt{n})((2\sqrt{2}+1)/4)\omega(f';1\sqrt{n}).$$

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Department of Mathematics
Garhwal University, Srinagar
U.P., India 246174.

Department of Mathematics
University of Roorkee
U.P., India 247667.

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