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COMPACT POLYNOMIALS ON NON-ARCHIMEDEAN BANACH SPACES

by

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Abstract. The object of the present note is to prove that every compact polynomial between non-Archimedean Banach spaces over a complete discretely valued field of characteristic zero is a limit of a sequence of polynomials of finite rank.

In [2] Enflo has given a counterexample to the Banach-Grothendieck approximation problem. However, Serre [6] has proved the validity of the Banach-Grothendieck approximation problem in the case of compact operators between non-Archimedean Banach spaces over a local field. Recently, Krishnamachari [3] has pointed out that Serre's result still holds when the ground field is complete. The purpose of this note is to establish a polynomial version of Serre's result. For further results concerning compact polynomials see [1] and [4].

Now let us fix some notations adopted in the text. Throughout this note $\mathbb{K}$ denotes a field of characteristic zero with a non-trivial non-Archimedean absolute value. Given E and F non-Archimedean normed spaces over $\mathbb{K}$, $p(E;F)$ (resp. $p(^mE;F)$) denotes the $\mathbb{K}$-vector space of continuous (resp. con-

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tinuous m-homogeneous) polynomials from E into F ([4],[5]). We endow $p(E;F)$ with the non-Archimedean norm

$$P \in p(E;F) \mapsto |P| = \sup_{\|x\| \leq 1} |P(x)| \in \mathbb{R}_+ .$$

DEFINITION. A polynomial $P:E \to F$ is said to be *compact* if P maps the unit ball B of E into a relatively compact subset of F. We denote by $p_c(E;F)$ (resp. $p_c({}^mE;F)$) the $\mathbb{K}$-vector space of compact (resp. compact m-homogeneous) polynomials from E into F.

PROPOSITION 1. *If $P:E \to F$ is a non-zero compact polynomial, $P = P_0+\ldots+P_m$ (P_j being j-homogeneous, $j = 0,\ldots,m$, $P_m \neq 0$), then each P_j is a compact j-homogeneous polynomial. In particular, $P_j \in p({}^jE;F)$, $j = 1,\ldots,m$, and so $P \in p(E;F)$.*

Proof. We argue by induction on m, the cases $m = 0$ and $m = 1$ being clear. Now let $m \geq 1$ and assume the proposition true for every non-zero compact polynomial of degree $\leq m-1$. If $P_0 = \ldots = P_{m-1} = 0$, the proposition is clear. Let us suppose the contrary and fix $\lambda \in \mathbb{K}$ with $0 < |\lambda| < 1$. Then

$$\lambda^m P(x) - P(\lambda x) = (\lambda^m - e)P_0(x)+\ldots+(\lambda^m-\lambda^{m-1})P_{m-1}(x)$$

for all $x \in E$ and, moreover, $\lambda^m - e \neq 0,\ldots,\lambda^m-\lambda^{m-1} \neq 0$ (here e denotes the identity element of $\mathbb{K}$). Therefore the mapping $x \in E \mapsto \lambda^m P(x) - P(\lambda x) \in F$ is a non-zero compact polynomial of degree $\leq m-1$ and the induction hypothesis ensures that $P_0,\ldots,P_{m-1}$ are compact polynomials. Thus $P_m = P - (P_0+\ldots+P_{m-1})$ is also a compact polynomial.

As in the linear case we obtain

PROPOSITION 2. *$p_c(E;F)$ is a closed vector subspace of $p(E;F)$ if F is a Banach space.*

COROLLARY 1. *If $\mathbb{K}$ is a local field, F is a Banach space, and $(P_k)_{k\in\mathbb{N}}$ is a sequence of polynomials of finite rank converging to $P \in p(E;F)$, then $P \in p_c(E;F)$ (by a polynomial of finite rank we mean a continuous polynomial whose image generates a finite dimensional vector space).*

Proof. By Proposition 2 it suffices to prove that each P_k is compact. Let G_k be the finite dimensional vector subspace of F generated by $P_k(E)$; G_k is a locally compact normed space (with the norm induced by F). Then $P_k(B)$ is contained in some λT_k, where T_k denotes the compact unit ball of G_k. Consequently, P_k is a compact polynomial.

Our purpose is to establish

PROPOSITION 3. *Let* $\mathbb{K}$ *be complete under a discrete valuation, and let* E *and* F *be Banach spaces over* $\mathbb{K}$.
If $P \in p_c(E;F)$, *there exists a sequence of polynomials of finite rank which converges to* P.

Before we prove Proposition 3 let us state an auxiliary lemma ([6], Proposition 2).

LEMMA. *Let* L *be a complete non-trivially valued ultrametric field whose absolute value is discret, and let* E *be a Banach space over* L *satisfying condition* (N) *below:*
(N) *For each* $x \in E$, $\|x\|$ *belongs to* $\{\overline{|\lambda|;\lambda \in L, \lambda \neq 0}\}$.
If V *is a closed vector subspace of* E, *there exists a continuous projection* $p:E \to E$ *having* V *as image such that* $\|p\| \leq 1$.

Proof of Proposition 3. We first claim that every compact r-homogeneous polynomial can be approximated by r-homogeneous polynomials of finite rank ($r = 1,2,\ldots,$). Indeed, let $P \in p_c({}^rE;F)$ and assume additionally that F satisfies condition (N). Given $\varepsilon > 0$ we can find $y_1,\ldots,y_n$ in F such that

$$P(B) \subset \bigcup_{i=1}^{n} \overline{B}(y_i,\varepsilon).$$

Let V be the finite dimensional vector subspace of F generated by $\{y_1,\ldots,y_n\}$. By the Lemma there exists a continuous projection $p:F \to F$ whose image is V with $|p| \leq 1$. Then $P' = p\circ P$ is an r-homogeneous polynomial of finite rank such that $|P - P'| < \varepsilon$.

Now, if F is arbitrary, there exists a non-Archimedean norm $\|\!|\ \|$ in F which is equivalent to the given norm of F and such that $F^* = (F,\|\!|\ \|\!|)$ satisfies condition (N) (it suffices to take $\|x\| = \inf\{|\lambda|; \lambda \in \mathbb{K}, \lambda \neq 0, \|x\| \leqslant |\lambda|\}$). By what we have seen above the compact r-homogeneous polynomial $Id \circ P$ can be approximated by r-homogeneous polynomials of finite rank from E into F^*, where $Id : F \to F^*$ denotes the identity mapping. As a direct consequence our claim is verified.

Finally let $P \in p_c(E;F)$, $P \neq 0$, $P = P_0 + \ldots + P_m$ (where P_j is j-homogeneous, $j = 0,\ldots,m$, and $P_m \neq 0$). The proposition being clear if $m = 0$, let us assume $m \geqslant 1$. By Proposition 1 $P_j \in p_c({}^jE;F)$ for $j = 1,\ldots,m$. Hence there exists a sequence $(Q_k^j)_{k\in\mathbb{N}}$ of j-homogeneous polynomials of finite rank such that $P_j = \lim_{k\to\infty} Q_k^j$ $(j = 1,\ldots,m)$. Let $Q_k = P_0 + \sum_{j=1}^{m} Q_k^j$ for $k \in \mathbb{N}$. Clearly $(Q_k)_{k\in\mathbb{N}}$ is a sequence of polynomials of finite rank and $P = \lim_{k\to\infty} Q_k$, as was to be shown.

COROLLARY 2. *Let $\mathbb{K}$ be a local field, and let E and F be Banach spaces over $\mathbb{K}$. If $P \in p(E;F)$, then $P \in p_c(E;F)$ if and only if P is a limit of a sequence of polynomials of finite rank.*

Proof. Immediate from Corollary 1 and Proposition 3 since every local field is complete and discretely valued.

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