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TANGENT MAPPINGS AND CONVERGENT SEQUENCES IN THE LIPSCHITZ CATEGORY

por

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Abstract. The standard definition of a derivative in linear spaces is extended to a definition of tangency in the Lipschitz category, without any assumed algebraic structure on the underlying spaces. Tangency is characterized topologically, that is, solely in terms of continuity, without using any algebraic concepts or other analytical concepts. The mappings in the Lipschitz category are characterized as the class of functions that preserve topologically convergent sequences of finite variation.

1. Preliminaries. In this paper all spaces will be metric spaces, and d will usually denote the metric. If two spaces are present simultaneously, the context will resolve any ambiguity resulting from a multiple usage of d. $\mathbb{R}$ will denote the real line, and TOP will denote the topological category, restricted to metric spaces.

We recall that a function $f:X \to Y$ satisfies a *Lipschitz condition* (called by some authors a *uniform Lipschitz condition*) if there exists a real number M such that

$$d(f(x),f(y)) \leq Md(x,y) \quad \text{for all } x,y \in X.$$

The function f is said to satisfy a *local Lipschitz condition* if there exists an open cover $\mathcal{U}$ of X such that $f|U$ satisfies a Lipschitz condition for each $U \in \mathcal{U}$. The Lipschitz category LIP consists of all metric spaces and mappings that satisfy a local Lipschitz condition. The set of all mappings in LIP from X into Y will be denoted by LIP$(X;Y)$. If $X = Y$, this will often be abbreviated to LIP(X).

Sequences will be denoted by lower-case boldface letters; individual terms of a sequence will be denoted by the corresponding subscripted plain letters, so that $\mathbf{x}$ is a sequence whose general term is x_n, $\mathbf{y}$ is a sequence whose first term is y_1, and so on.

2. Tangents in LIP. The usual definition of the derivative in linear spaces extends readily to a notion of tangency between any two mappings in LIP.

2.1. DEFINITION. $f, g \in$ LIP$(X;Y)$ are *tangent* at $p \in X$ (in symbols, $f \sim g$) if for every $\varepsilon > 0$ there exists a neighborhood U of p such that

$$d(f(x), g(x)) \leqslant \varepsilon d(x,p) \quad \text{for all } x \in U. \tag{2.1.1}$$

We observe that (2.1.1) implies that tangent functions agree at p, and that tangency at p is an equivalence relation in LIP$(X;Y)$.

Although this definition can be applied to arbitrary (discontinuous) functions to give a definition of tangency for all functions from X into Y, some of the elementary propositions, such as the Chain Rule, will fail outside of LIP. For example, let $f(x) = \sqrt{|x|}$ and $g(x) = |x|^{3/2}$ in $\mathbf{R}$. Then g is tangent to the constant function 0 at $x = 0$, but the composite function gf is not tangent to $0f = 0$.

2.2. THE CHAIN RULE. *If* $f_1, f_2 \in$ LIP$(X;Y)$ *are tangent at* $x \in X$ *and if* $g_1, g_2 \in$ LIP$(Y;Z)$ *are tangent at* $y = f_1(x) = f_2(x)$, *then the composite mappings*

$$g_1 f_1, g_2 f_2 : X \to Z$$

are tangent at x.

Proof. By transitivity, it is sufficient to verify the two relations $g_1 f_1 \sim g_1 f_2$ and $g_1 f_2 \sim g_2 f_2$ separately. In each case, we assume that $\varepsilon > 0$ is given. To verify the first relation, since $g_1 \in$ LIP there exists a neighborhood V of y in Y and a real number $M > 0$ such that

$$d(g_1(y'), g_1(y'')) \leqslant M d(y', y'') \text{ for all } y', y'' \in V,$$

and since $f_1 \sim f_2$ at x, there exists a neighborhood U of x in X such that

$$d(f_1(u), f_2(u)) \leqslant (\varepsilon/M) d(u, x) \text{ for all } u \in U.$$

Let $W = U \cap f_1^{-1}(V) \cap f_2^{-1}(V)$. Then W is a neighborhood of x, and for any $u \in W$

$$d(g_1 f_1(u), g_1 f_2(u)) \leqslant M d(f_1(u), f_2(u)) \leqslant \varepsilon d(u, x),$$

which establishes the relation $g_1 f_1 \sim g_1 f_2$.

To verify that $g_1 f_2 \sim g_2 f_2$, since $f_2 \in$ LIP there exists a neighborhood U of x in X and a real number $M > 0$ such that

$$d(f_2(x'), f_2(x'')) \leqslant M d(x', x'') \text{ for all } x', x'' \in U,$$

and since $g_1 \sim g_2$ at $y = f_2(x)$, there exists a neighborhood V of y in Y such that

$$d(g_1(v), g_2(v)) \leqslant (\varepsilon/M) d(v, y) \text{ for all } v \in V.$$

Then for any point u in the neighborhood $U \cap f_2^{-1}(V)$ of x,

$$d(g_1 f_2(u), g_2 f_2(u)) \leqslant (\varepsilon/M) d(f_2(u), y) \leqslant \varepsilon d(u, x). \quad \blacktriangle$$

Our first result (Theorem 2.5) will be a characterization of tangency in terms of convergent sequences. The theorem states intuitively that tangent functions map sequences which converge "slowly" to the point of tangency onto sequences which converge "rapidly" towards each other.

If a sequence $\mathbf{x}$ converges to $x \in X$, we will say that

$\mathbf{x}$ is *summable at* x if $\sum d(x_n,x) < \infty$, and *essential at* x if $\sum d(x_n,x) = \infty$. If x is an isolated point of X, there are no essential sequences at x. But if x is an accumulation point of X, essential sequences at x always exist. For if $\mathbf{x}$ converges to x and $x_n \neq p$ for all n, let K_n be the smallest integer such that $K_n \geq 1/d(x_n,p)$ and let $\mathbf{w}$ be the sequence

$$\mathbf{w} = x_1,\ldots,x_1,x_2,\ldots,x_2,x_3,\ldots,x_3,\ldots,x_n,\ldots,x_n,\ldots \qquad (2.3)$$

where x_n appears K_n times. Then $\mathbf{w}$ is essential at x.

2.4. LEMMA. *Let* $\mathbf{x}$ *be an essential sequence at* $p \in X$. *Then for any real numbers* $0 \leq s < t$, *there exists a finite subsequence* $y_1,\ldots,y_k$ *of* $\mathbf{x}$ *such that* $s \leq \sum_{j\leq k} d(y_j,p) \leq t$.

Proof. Remove from $\mathbf{x}$ all terms x_n such that $d(x_n,p) > t-s$, and call the resulting subsequence $\mathbf{y}$. Since only finitely many terms of $\mathbf{x}$ have been removed, $\mathbf{y}$ is essential; this implies the existence of an index k such that $s \leq \sum_{j\leq k} d(y_j,p)$. If k is chosen to be the smallest such index, then, since $d(y_k,p) \leq t-s$, this sum is bounded above by t. ▲

2.5. THEOREM. $f, g \in \mathrm{LIP}(X,Y)$ *are tangent at an accumulation point* $p \in X$ *if and only if every essential sequence* $\mathbf{x}$ *at* p *has an essential subsequence* $\mathbf{y}$ *such that*

$$\sum d(f(y_n),g(y_n)) < \infty . \qquad (2.5.1)$$

Proof. Suppose first that $f \sim g$, and let $\mathbf{x}$ be an essential sequence at p. For each n there exists a neighbordhood U_n of p such that

$$d(f(x),g(x)) \leq (1/n)d(x,p) \text{ for all } x \in U_n. \qquad (2.5.2)$$

Let $\mathbf{u}_n$ be the subsequence of $\mathbf{x}$ of elements in U_n. Since $\mathbf{u}_n$ contains all but finitely many terms of $\mathbf{x}$, $\mathbf{u}_n$ is essential. By 2.4 we can extract from $\mathbf{u}_1$ a finite sequence

$$\mathbf{y}_1 = y_{11}, y_{12}, \ldots, y_{1m_1}$$

such that

$$1 \leq \sum_{j \leqslant m_1} d(y_{1j}, p) \leqslant 2.$$

Inductively, for $n > 1$, extract from u_n a finite sequence

$$y_n = y_{n1}, y_{n2}, \ldots, y_{nm_n}$$

such that y_{n1} follows $y_{n-1, m_{n-1}}$ in x, and such that

$$1/n \leqslant \sum_{j \leqslant m_n} d(y_{nj}, p) \leqslant 2/n \tag{2.5.3}$$

From 2.5.2,

$$d(f(y_{nj}), g(y_{nj})) \leqslant (1/n) d(y_{nj}, p),$$

so that

$$\sum_{j \leqslant m_n} d(f(y_{nj}), g(y_{nj})) \leqslant 2/n^2. \tag{2.5.4}$$

The juxtaposed sequence $y = y_1 y_2 y_3 \ldots$ is then a subsequence of x. 2.5.3 implies that y is essential, and 2.5.4 implies 2.5.1.

Conversely, suppose that f and g are not tangent. If $f(p) \neq g(p)$, the result holds trivially, so assume $f(p) = g(p)$. For some $m > 0$ there exists a convergent sequence x at p such that

$$d(f(x_n)), g(x_n)) > m d(x_n, p). \tag{2.5.5}$$

The strict inequality in 2.5.5 implies that $x_n \neq p$ for all n. Let w be the sequence defined in 2.3. Then w is essential, and 2.5.5 implies that 2.5.1 does not hold for any essential subsequence of w. ▲

As an application of this theorem, we will obtain the following result. In TOP, it is possible for a function to be invertible despite a vanishing derivative at an accumulation point (for example, $y = x^3$). This cannot happen in LIP.

2.6. THEOREM. *If $f: X \to Y$ is tangent to a constant function at an accumulation point $p \in X$, then f is not invertible.*

Proof. If f is tangent at p to a constant function c, then there exists an essential sequence $\mathbf{x}$ at p such that $f(\mathbf{x})$ is summable at $c(p) = f(p)$ in Y. Since mappings in LIP preserve summable sequences, if f^{-1} were to exist it would map $f(\mathbf{x})$ onto a summable sequence. But $f^{-1}f(\mathbf{x}) = \mathbf{x}$, so f^{-1} does not exist. ▲

Let us say that a function $f:\mathbf{R} \to \mathbf{R}$ is *smooth* in LIP if its pointwise derivative $f'(x)$ exists for all $x \in \mathbf{R}$ and $f' \in \mathrm{LIP}(\mathbf{R})$. There is a particularly simple characterization (2.9) of tangency for the class of smooth functions. First, we say that a sequence $\mathbf{x}$ that converges to $x \in X$ is *square-summable at* x if $\sum d^2(x_n, x) < \infty$.

2.7. LEMMA. *Every essential sequence in a metric space X contains an essential square-summable subsequence.*

Proof. Let $\mathbf{x}$ be an essential sequence at $x \in X$. By 2.4, we can extract from $\mathbf{x}$ a finite subsequence

$$\mathbf{y}_1 = y_{11}, y_{12}, \ldots, y_{1m_1}$$

such that

$$1 \leqslant \sum_{j \leqslant m_1} d(y_{1j}, x) \leqslant 2.$$

Inductively, for $n > 1$, extract from $\mathbf{x}$ a finite sequence

$$\mathbf{y}_n = y_{n1}, y_{n2}, \ldots, y_{nm_n}$$

such that y_{n1} follows $y_{n-1,m_{n-1}}$ in $\mathbf{x}$, and such that

$$1/n \leqslant \sum_{j \leqslant m_n} d(y_{nj}, x) \leqslant 2/n. \qquad (2.7.1)$$

We then have

$$\sum_{j \leqslant m_n} d^2(y_{nj}, x) \leqslant \left(\sum_{j \leqslant m_n} d(y_{nj}, x)\right)^2 \leqslant 4/n^2. \qquad (2.7.2)$$

The juxtaposed sequence $\mathbf{y} = \mathbf{y}_1\mathbf{y}_2\mathbf{y}_3\ldots$ is then a subsequence of $\mathbf{x}$. The first inequality in 2.7.1 implies that $\mathbf{y}$ is essential, and 2.7.2 implies that $\mathbf{y}$ is square-summable. ▲

2.8. LEMMA. *Suppose that* $f:\mathbf{R} \to \mathbf{R}$ *is smooth in* LIP. *Then for any point* $a \in \mathbf{R}$ *there exist a real number* M *and a neighborhood* U *of* a *in* $\mathbf{R}$ *such that*

$$|f(x)-f(a)-f'(a)(x-a)| \leq M(x-a)^2 \text{ for all } x \in U \tag{2.8.1}$$

Proof. Since $f' \in \text{LIP}(\mathbf{R})$, there exist a real number $M > 0$ and an open interval $U \ni a$ such that $|f'(u)-f'(v)| \leq M|u-v|$ for all $u,v \in U$. Let x be any point in U. If $x = a$, then both sides of 2.8.1 reduce to 0, so assume $x \neq a$. The Mean Value Theorem yields a point $z \in U$ with $|z-a| < |x-a|$ such that $f(x)-f(a) = f'(z)(x-a)$. Now write

$$\begin{aligned}|f(x)-f(a)-f'(a)(x-a)| &= |f'(z)(x-a)-f'(a)(x-a)| \\ &= |f'(z)-f'(a)||x-a| \\ &\leq M|z-a||x-a| \\ &< M(x-a)^2. \quad \blacktriangle\end{aligned}$$

We can now characterize tangency within the class of smooth functions in LIP(R).

2.9. THEOREM. *If* $f,g:\mathbf{R} \to \mathbf{R}$ *are smooth in* LIP, *then* $f \sim g$ *at* $p \in \mathbf{R}$ *if and only if*

$$\sum d(f(x_n),g(x_n)) < \infty \text{ for every square-summable sequence } \mathbf{x} \text{ at } p. \tag{2.9.1}$$

Proof. If f and g satisfy 2.9.1, then 2.7 implies that $f \sim g$. Conversely, assume that $f \sim g$. It is sufficient to verify that 2.9.1 holds in the case where g is the linear tangent of f. But this follows from 2.8. ▲

REMARK. For arbitrary (non-smooth) functions in LIP(R), the existence of a linear tangent does not by itself imply that 2.9.1 holds. For example, the function $f(x) = |x|^{3/2}$ is tangent to the constant function $g = 0$ at $x = 0$, but f and g do not satisfy 2.9.1. Notice that f is not smooth in LIP.

3. Convergence in LIP. In the proof of Theorem 2.6, we noted that mappings in LIP preserve summable sequences. Although the converse is not true, it is possible to relax the condition of summability to obtain a class of sequences whose preservation does characterize the mappings in LIP. We recall that the *variation* of a sequence $\mathbf{x}$ is $V(\mathbf{x}) = \sum d(x_{n+1}, x_n)$, and say that a sequence $\mathbf{x}$ in X *converges* in LIP to x, or is *LIP-convergent* to x, if $\mathbf{x}$ has finite variation and converges topogically to x. The main result of this section is

3.1. THEOREM. $f \in \mathrm{LIP}(X,Y)$ *if and only if for every* LIP-*convergent sequence* $\mathbf{x}$ *at* x, $f(\mathbf{x})$ *is* LIP-*convergent at* $f(x)$.

With the inclusion of the class of LIP-convergent sequences, LIP takes on an identity of its own, instead of being just a subcollection of topological mappings. For example, if $f \in$ LIP and s is a sequence associated with f, it is possible to consider the convergence of s in LIP. In particular, if $f:\mathbf{R} \to \mathbf{R}$ is periodic with period 2π and $s = s(x)$ is the Fourier series of f at x, then it is well known that s converges topologically to $f(x)$. But it is also true that s converges in LIP. For if a_n, b_n are the Fourier coefficients of f, then

$$V(s) = \sum|s_n(x)-s_{n-1}(x)| = \sum|a_n\cos(nx)+b_n\sin(nx)| \leq \sum(|a_n|+|b_n|),$$

which is known to be finite ([1,p.243]).

Sequences of finite variation correspond in LIP to Cauchy sequences in TOP, in that both are "preconvergent" in their respective categories. Every sequence of finite variation is a Cauchy sequence. Conversely, every Cauchy sequence contains a subsequence of finite variation; consequently every topologically convergent sequence contains a LIP-convergent subsequence. As a result, the two categories share a common notion of an accumulation point and a common notion of completeness. (X is defined to be complete in LIP if every sequence of finite variation converges).

It is not difficult to show that the results on tangency in the previous section hold if topological convergence is replaced by LIP-convergence. One must be careful, however, about recasting 2.1 into its familiar limit-of-ratios form: $f \sim g$ does not imply that for every LIP-convergent sequence $\mathbf{x}$ at p with $x_n \neq p$ for all n, the sequence with general term $d(f(x_n), g(x_n))/d(x_n, p)$ is LIP-convergent to 0 in $\mathbb{R}$. The functions

$$f(x) = x^2 \sin(1/x) \quad \text{if} \quad x \neq 0$$
$$= 0 \quad \text{if} \quad x = 0$$

and $g = 0$ provide a counterexample. The reason for this is that the concept of tangency under study here is only a measure of proximity, not a measure of variation. It would be interesting to find a concept of tangency that takes variation into account.

Proof of Theorem 3.1. It is easy to see that all mappings in LIP preserve LIP-convergent sequences. The converse is harder. Assume that f does not satisfy a local Lipschitz condition. Then there exists a point $x \in X$ such that f does not satisfy a Lipschitz condition on any neighborhood U of x. We will construct a LIP-convergent sequence $\mathbf{x}$ at x in X such that $f(\mathbf{x})$ is LIP-divergent in Y. For each positive integer n, let $M_n = 2^{n+2}$ and let

$$U_n = \{y \in X \mid d(y, x) < 1/2^{n+3}\}$$

Then there exist $y_n, z_n \in U_n$ such that

$$d(f(y_n), f(z_n)) > M_n d(y_n, z_n) = 2^{n+2} d(y_n, z_n). \tag{3.1.1}$$

Since y_n and z_n are in U_n, the Triangle Inequality yields

$$d(y_n, z_n) < 1/2^{n+2}$$

Let K_n be the smallest *even* integer such that

$$K_n d(y_n, z_n) \geqslant 1/2^{n+1}. \tag{3.1.2}$$

K_n is finite since $y_n \neq z_n$ (by the strict inequality in 3.1.1), and the minimality of K ensures that

$$(K_n - 2)d(y_n, z_n) < 1/2^{n+1},$$

so that

$$\begin{aligned} K_n d(y_n, z_n) &= (K_n - 2)d(y_n, z_n) + 2d(y_n, z_n) \\ &< 1/2^{n+1} + 2(1/2^{n+2}) \\ &= 1/2^n. \end{aligned} \tag{3.1.3}$$

Now define the finite sequence

$$s_n = y_n, z_n, y_n, z_n, y_n, z_n, \ldots, y_n, z_n$$

where y_n and z_n alternate, each appearing $K_n/2$ times, so that s_n has exactly K_n terms. The distance between any two adjacent terms of s_n is $d(y_n, z_n)$, so that the variation of s_n is

$$V(s_n) = (K_n - 1)d(y_n, z_n) < K_n d(y_n, z_n) < 1/2^n \quad \text{(by 3.1.3)}. \tag{3.1.4}$$

The juxtaposed sequence $s = s_1 s_2 s_3 \ldots$ converges topologically to x, by the definition of U_n. To verify that s has finite variation, that is,

$$\sum d(s_{k+1}, s_k) < \infty \tag{3.1.5}$$

we divide the subscripts k into two classes:

(1) the set of all k such that s_k and s_{k+1} both lie in some s_n, and

(2) the set of all k such that $s_k \in s_n$ but $s_{k+1} \in s_{n+1}$.

The inequality 3.1.4 implies that 3.1.5, when restricted to the subscripts k in class (1), is finite. For the subscripts k in class (2), we have

$$d(s_{k+1}, s_k) = d(y_{n+1}, z_n) < 1/2^{n+2},$$

since y_{n+1} and z_n both lie in U_n. Therefore 3.1.5, when restricted to the subscripts k in class (2), is also finite, and it follows that s converges in LIP.

The image of s under f, however, diverges in LIP. To see this, consider the finite sequence

$$f(s_n) = f(y_n), f(z_n), f(y_n), f(z_n), \ldots, f(y_n), f(z_n).$$

The terms $f(y_n)$ and $f(z_n)$ each appear $K_n/2$ times; therefore the variation of this sequence is

$$\begin{aligned} V(f(s_n)) &\geqslant (K_n/2)\, d(f(y_n), f(z_n)) \\ &> (K_n/2)\, 2^{n+2} d(y_n, z_n) && \text{(by 3.1.1)} \\ &= K_n 2^{n+1} d(y_n, z_n) \\ &\geqslant 1 && \text{(by 3.1.2)}, \end{aligned}$$

and since $V(f(s)) \geqslant \sum V(f(s_n))$, $f(s)$ diverges in LIP. ▲

4. Tangency and Non-Retractability.

Theorem 2.5 characterized tangency in terms of summability. In this section, we will characterize tangency topologically, that is, solely in terms of continuity, without using any algebraic or other analytical concepts.

In TOP, the following result is an immediate consequence of the pertinent definitions.

4.1. If X is the union of two closed subsets A and B, and $f: X \to Y$ is any function such that both $f|A$ and $f|B$ are continuous, then f if continuous.

In LIP, however, 4.1 is false.

4.2. EXAMPLE. In the plane $\mathbf{R}^2$, let $X = A \cup B$, where

$$A = \{(x, y) \mid y = x^2\}$$

and

$$B = \{(x, y) \mid y = 0\},$$

and let $f: X \to X$ be the function

$$f(x,y) = (0,0) \text{ if } (x,y) \in A,$$
$$= (x,y) \text{ if } (x,y) \in B.$$

Then f is continuous in LIP on A and B separately, but f is not continuous on X, since there is no M such that

$$|x| = d((0,0),(x,0)) = d(f(x,x^2),f(x,0)) \leq Md((x,x^2),(x,0)) = Mx^2$$

as $x \to 0$. ▲

This example shows, intuitively, that the curve $y = x^2$ is so close to its tangent line that it cannot be pulled away onto the point of tangency. This "closeness of fit" of tangent curves is generally true throughout LIP and will be shown to characterize tangency (Theorem 4.6). This theorem, which has no counterpart in TOP (as shown by 4.2), demostrates a fundamental difference between the two categories.

We recall the standard definition that a *retraction* of a space X onto a subset A is a mapping $r: X \to A$ such that $r|A$ is the identity on A. Given $f, g \in \text{LIP}(X;Y)$, denote their graphs in $X \times Y$ by G_f and G_g. If $p \in X$, we say that G_f is *retractable from* G_g *at* p if there exists (in LIP) a retraction $r: G_f \cup G_g \to G_g$ such that $r(G_f) = (p, f(p))$. If G_f is retractable from G_g at p, then the retraction $r: G_f \cup G_g \to G_g$ is unique and is given by

$$r(x,y) = (p, f(p)) \text{ if } (x,y) \in G_f,$$
$$= (x,y) \quad \text{if } (x,y) \in G_g.$$

A necessary condition for r to exist is that $f(x) \neq g(x)$ if $x \neq p$. This condition is also sufficient in TOP (by 4.1), but it is not sufficient in LIP, as shown by examples 4.2 and 4.4.

Since LIP is metric-sensitive, care must be taken in choosing a metric on $X \times Y$. It will be computationally convenient to use the "sum" metric:

$$d((x_1,y_1),(x_2,y_2)) = d(x_1,y_1)+d(x_2,y_2)$$

which is compatible in LIP to the usual "Pythagorean" metric.

4.3. LEMMA. *If f and g are tangent at an accumulation point $p \in X$, then G_f is not retractable from G_g at p.*

Proof. For any $M > 0$ and neighborhood U of p in X, there exists a point $x \in U$ such that $d(f(x),g(x)) < (1/M)d(p,x)$. If a retraction $r \in$ LIP were to exist, then

$$\begin{aligned} d(r(x,f(x)),r(x,g(x))) &= d((p,f(p)),(x,g(x))) \\ &\geq d(p,x) \\ &> Md(f(x),g(x)) \\ &= Md((x,f(x),(x,g(x)). \end{aligned}$$

Since M is arbitrary, $r \notin$ LIP. ▲

If f and g are not tangent at p, it is still possible that G_f is not retractable from G_g at p. As already noted, this will happen if $f(x) = g(x)$ for some $x \neq p$. But even if $f(x) \neq g(x)$ for all $x \neq p$, it is still possible that G_f is not retractable from G_g.

4.4. EXAMPLE. Let X be the space obtained by removing from $\mathbb{R}$ the points $1/n\pi$ for all nonzero integer n. Let $Y = \mathbb{R}$, and define $f: X \to Y$ by

$$\begin{aligned} f(x) &= x\sin(1/x) \quad \text{if } x \neq 0 \\ &= 0 \qquad\qquad\quad \text{if } x = 0. \end{aligned}$$

Then f is not tangent to the constant function $g = 0$ at $x = 0$ and $f(x) \neq 0$ for all $x \neq 0$, but G_f is not retractable from G_g at 0.

The following lemma gives a sufficient condition for G_f to be retractable from G_g.

4.5. LEMMA. *Let $f, g \in$ LIP(X,Y) and let p be an accumulation point of X. If there exists $m > 0$ such that*

$d(f(x),g(x)) \geq md(x,p)$ *for all* $x \in X$, *then* G_f *is retractable from* G_g *at* p.

Proof. The result is trivial if $f(p) \neq g(p)$, since in this case G_f and G_g are disjoint closed (and therefore open) subsets of $G_f \cup G_g$. So assume that $f(p) = g(p)$. Let $G = G_f \cup G_g$, and let $q = (p,f(p)) = (p,g(p)) \in G$. We must show that the function r cn G defined by

$$r(x,y) = q \quad \text{if } (x,y) \in G_f,$$
$$= (x,y) \text{ if } (x,y) \in G_g,$$

belongs to LIP. Since $G_f - \{q\}$ and $G_g - \{q\}$ are open in $G_f \cup G_g - \{q\}$, this reduces to showing that r satisfies a Lipschitz condition on some neighborhood of q in G.

Since $f,g \in$ LIP, there exists a neighborhood U of p in X such that

$$d(f(x'),f(x'')) \leq M_f d(x',x'') \quad \text{and} \quad d(g(x'),g(x'')) \leq M_g d(x',x'')$$

for all $x',x'' \in U$. We can assume that $M_f \leq 1$, since if $M_f > 1$ we can replace the original metric d on X by the compatible metric $d' = M_f d$.

We will show that r satisfies a Lipschitz condition on the set $V = G \cap (U \times Y)$. Let $a,b \in V$. It is sufficient to consider the case in which $a \in G_f$ and $b \in G_g$. Then there exist $x,y \in U$ such that $a = (x,f(x))$ and $b = (y,g(y))$. To reduce the number of parentheses in the following calculation, let us write $fx = f(x)$ and $gy = g(y)$. Recall also that we are using the "sum" metric in $X \times Y$. Then

$$\begin{aligned}
d(r(b),r(a)) &= d(r(y,gy),r(x,fx)) \\
&= d((y,gy),(p,gp)) \\
&= d(y,p) + d(gy,gp) \\
&\leq (1+M_g)d(y,p) \\
&\leq ((1+M_g)/m)d(fy,gy) \\
&\leq ((1+M_g)/m)(d(fy,fx)+d(gy,fx)) \\
&\leq ((1+M_g)/m)(d(y,x)+d(gy,fx)) \qquad \text{(since } M_f \leq 1) \\
&= ((1+M_g)/m)d((y,gy),(x,fx)) \\
&= ((1+M_g)/m)d(b,a). \quad \blacktriangle
\end{aligned}$$

We can now characterize tangency in terms of continuity or, more specifically, in terms of non-retractability. Let us call a set $S \subset X$ an *accumulation set* of $p \in X$ if $p \in S$ and p is an accumulation point of S.

4.6. THEOREM. *f and g are tangent at an accumulation point $p \in X$ if and only if for every accumulation set S of p, $G_{f|S}$ is not retractable from $G_{g|S}$ at p.*

Proof. If f and g are tangent at p in X, then they are tangent at p on any subset S of X. If S is an accumulation set of p, then 4.3, applied to S, implies that $G_{f|S}$ is not retractable from $G_{g|S}$ at p.

Conversely, if f and g are not tangent at p, there exists $m > 0$ such that the set

$$S = \{x \in X \mid d(f(x), g(x)) \geqslant m d(x,p)\}$$

is an accumulation set of p. Then 4.5, applied to S, implies that $G_{f|S}$ is retractable from $G_{g|S}$ at p. ▲

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