Revista Colombiana de Matemáticas
Vol. XXIV (1990) págs. 41-49

UNIFORM ORDERED SPECTRAL DECOMPOSITIONS

by

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Resumen. Introducimos la noción de *UOSD*-multiplicidad de una proyección P relativa a una medida espectral $E(\cdot)$ con la *CGS*-propiedad y la comparamos con la noción de multiplicidad introducida por Halmos [2]. También se dan varias caracterizaciones para que una medida espectral tenga la *CGS*-propiedad.

Abstract. We introduce the notion of *UOSD*-multiplicy of a projection P relative to a spectral measure $E(\cdot)$ with the *CGS*-property and compare it with the notion of multiplicy introduced by Halmos [2]. Also are given some characterizations for a spectral measure to have the *CGS*-property.

In our earlier work [4] we introduced the notion of ordered spectral decomposition (*OSD*, in abbreviation) of a Hilbert space relative to a spectral measure $E(\cdot)$ and defined the *OSD*-multiplicity of a projection P commuting with $E(\cdot)$. Here we introduce the concepts of uniform *OSD* and *UOSD*-multiplicity and compare the concept of multiplicity in Halmos [2] with the *OSD* and *UOSD*-multiplicities. Also we obtain various characterizations for a spectral measure to have the *CGS*-property.

Supported by the C.D.C.H. projects C-S-149, 150 of the Universidad de los Andes.

§1. **Preliminaries**. In this section we fix the terminology and notations and give some definitions and results from the literature which are needed in the sequel.

S denotes a σ-algebra of subsets of a set $X(\neq\phi)$. H is a (complex) Hilbert space and $E(\cdot)$ is a spectral measure on S with values in projections of H. The closed subspace generated by a subset X of H is denoted by $[X]$. For a vector $x \in H$, $Z(x) = [E(\sigma)x:\sigma \in S]$; $\sum_{i\in J}\oplus M_i$ denotes the orthogonal direct sum of the subspaces M_i of some Hilbert space.

W is the Von Neumann algebra generated by the range of $E(\cdot)$ and W' is the commutant of W. If $W' = \sum\oplus W'Q_n$ is the type I_n direct sum decomposition of W', then the central projections $Q_n(\neq 0)$ are unique (such that $W'Q_n$ is of type I_n) and in the sequel Q_n will denote these central projections. For $x \in H$, $[Wx] = [Ax:A \in W]$ and, sometimes, also denotes the orthogonal projection with the range $[Wx]$. For aprojection $P' \in W'$, $C_{p'}$ denotes the central support of P'. Other terminology in Von Neumann algebras is standard and we follows Dixmier [1].

As was observed in [5] a projection P' in W' is abelian if and only if P' is a row projection in the sense of [2] and the column $C(P')$ generated by P' as in [2] is the same as $C_{p'}$.

NOTATION 1.1. Let P be a projection in W. The multiplicity (respy. uniform multiplicity) of P in the sense of Halmos [2] will be referred to as its H-multiplicity (respy. UH-multiplicity) relative to $E(\cdot)$.

As was noted in [5] Theorem 64.4 of Halmos [2] can be interpreted as follows:

THEOREM 1.2. *A non-zero projection F in W has UH-multiplicity n if and only if there exists an orthogonal family $\{E'_\alpha\}_{\alpha\ J}$ of abelian projections in W' such that card $(J) = n$, $C_{E'_\alpha} = F$ and $\sum_{\alpha\in J} E'_\alpha = F$. In other words, F has UH-multiplicity n if and only if $W'F$ is of type I_n.*

Consequently, the following proposition is immediate.

PROPOSITION 1.3. *A non-zero projection P in W has UH-multiplicity n if and only if $P \leq Q_n$.*

DEFINITION 1.4. $E(\cdot)$ is said to have the *CGS-property* (*i.e. countable generating set property*) in H if there exists a countable set X in H such that $[E(\sigma)x : \sigma \in S,\ x \in X] = H$.

Let $\rho(x) = \|E(\cdot)x\|^2$. Then $\rho(x)$ is a finite measure on S. We say that $\rho(x_2)$ is absolutely continuous with respect to $\rho(x_1)$ and write $\rho(x_2) \ll \rho(x_1)$(or $\rho(x_1) \gg \rho(x_2)$) if $\rho(x_1)(\sigma) = 0$ implies $\rho(x_2)(\sigma) = 0$.

DEFINITION 1.5. Let $\{x_i\}_{i<N}$, $N \in \mathbb{N} \cup \{\omega\}$, be a countable set of non-zero vectors in H such that (i) $H = \sum_1^N \oplus Z(x_i)$ and (ii) $\rho(x_1) \gg \rho(x_2) \gg \ldots$ Then we say that $H = \sum_1^N \oplus Z(x_i)$ is an *OSD* of H relative to $E(\cdot)$

The cardinal number $N \in \mathbb{N} \cup \{\omega\}$ in the above definition is uniquely fixed by $E(\cdot)$ and is called the *OSD*-multiplicity of $E(\cdot)$. If P is a projection commuting with $E(\cdot)$ and $PE(\cdot)$ has the *CGS*-property in H, then the *OSD*-multiplicity of $PE(\cdot)$ is called the *OSD*-multiplicity of P. Besides, it has been shown in [4] that $E(\cdot)$ has the *CGS*-property in H if and only if H has an *OSD* relative to $E(\cdot)$.

§2. *UOSD*-multiplicity of projections. We introduce the concepts of *UOSD*s and *UOSD*-multiplicity relative to a spectral measure $E(\cdot)$ with the *CGS*-property in H and show that for a projection P in W the *UOSD*-multiplicity and the *UH*-multiplicity are one and the same when P is countably decomposable in W.

DEFINICION 2.1. An *OSD* $H = \sum_1^N \oplus Z(x_i)$ relative to $E(\cdot)$ is said to be a uniform *OSD* (*UOSD*, in abbreviation) of H if $\rho(x_1) \equiv \rho(x_2) \equiv \ldots$, where $\mu \equiv \nu$ if $\mu \ll \nu$ and $\nu \ll \mu$.

The following proposition is immediate from Theorem 1. (ii) of [4].

PROPOSITION 2.2. *If H has a UOSD relative to $E(\cdot)$, then all the OSDs of H relative to $E(\cdot)$ are UOSDs.*

DEFINITION 2.3. If H has a *UOSD* relative to $E(\cdot)$ then the *UOSD*-multiplicity of $E(\cdot)$ is defined to be the same as its *OSD*-multiplicity. If P is a projection of H commuting with $E(\cdot)$ and if $PE(\cdot)$ has *UOSD*-multiplicity n, then we say that P has *UOSD*-multiplicity n relative to $E(\cdot)$.

The following simple example shows that, in general, the *OSD*-multiplicity and H-multiplicity of a projection P relative to $E(\cdot)$ are not the same even though H is finite dimensional.

EXAMPLE 2.4. Let $H = \mathbb{C}^5, S = \{\phi, \{\lambda_1\}, \{\lambda_2\}, \{\lambda_1, \lambda_2\}\}, \lambda_1, \lambda_2 \in \mathbb{C}, \lambda_1 \neq \lambda_2$ and $E(\cdot)$ be a spectral measure on S given by $E(\{\lambda_1\})H = [e_1, e_2]$ and $E(\{\lambda_2\})H = [e_3, e_4, e_5]$, where $e_1 = (1,0,0,0,0,)$, $e_2 = (0,1,0,0,0,)$, etc. Since any maximal orthogonal family of row projections (in the sense of Halmos [2]) $\{E'_\alpha\}$ in W' with $C_{E'_\alpha} = I$ consists of just two members, the H-multiplicity of I is 2. On the other hand, if $x_1 = e_1 + e_3$, $x_2 = e_2 + e_4$ and $x_3 = e_5$, then $H = \sum_1^3 \oplus Z(x_i)$ is an *OSD* of H since $\rho(x_1) \equiv \rho(x_2) \gg \rho(x_3)$. Thus the *OSD*-multiplicity of I is 3.

The following result is well-known in the theory of Von Neumann algebras, and its proof is indicated also on page 108 of [2]. Using this result we compare the *UH*-multiplicity and *UOSD*-multiplicity of a projection.

LEMMA 2.5. *Let P' be an abelian projection in W'. If the central support $C_{P'}$ of P' is countably decomposable in W, then P' is cyclic.*

THEOREM 2.6. *Let P be a countably decomposable non-zero projection in W'. Then P has UH-multiplicity $N \leqslant \omega$ if* *and only if P has UOSD-multiplicity N (relative to $E(\cdot)$).*

Proof. Suppose the UH-multiplicity of P is $N \leq \omega$. Then by Theorem 1.2 there exists an orthogonal family $\{P'_j\}_{j\in J}$ of abelian projections in W' such that card. $J = N, C_{P'_j} = P$ and $P = \sum_{j\in J} P'_j$. Let $J = \{1,2,...,N\}$. By Lemma 2.5 there exists $x_j \in P'_j H$ such that $P'_j = [Wx_j]$, $j \in J$. Thus $PH = \sum_{j\in J} [Wx_j] = \sum_{j=1}^{N} \oplus Z(x_j)$. Besides, by Theorem 66.2 of [2], $C(\rho(x_j)) = C[Wx_j] = C_{P'_j} = P$ for all j. Therefore, by Theorem 65.2 of [2], $\rho(x_j) \equiv \rho(x_{j'})$ for $j,j' \in J$. Hence the condition is necessary.

Conversely, if P has UOSD-multiplicity N, then clearly $N \leq \omega$. Let $PH = \sum_1^N \oplus Z(x_i)$ be an OSD of PH relative to $PE(\cdot)$. Then by Proposition 2.2, $\rho(x_1) \equiv \rho(x_2) \equiv$ ... Consequently, by Theorem 66.2 of [2] we conclude that $C[Wx_1] = C[Wx_2] =$ ... $= Q(say)$. Clearly, $P = \sum_1^N [Wx_i] \leq Q$. As $P \in W$, $[Wx_i] \leq C[Wx_i] = P$ so that $Q = P$. Since each $[Wx_i]$ is an abelian projection in W' by Theorem 60.2 of [2], from Theorem 1.2 it follows that P has UH-multiplicity N.

§3. Some characterizations of the CGS-property. In terms of the existence of OSDs and OSRs of H the CGS-property of a spectral measure $E(\cdot)$ is characterized in [4]. The following Theorem gives some more charecterizations of this property.

THEOREM 3.1. *Let $E(\cdot)$ be a spectral measure on* **S** *with values in projections of H. Then the following statements are equivalent.*

(i) *Every projection of UH-multiplicity N in W is countably decomposable in W and $n \leq \omega$.*

(ii) *The projections Q_n are countably decomposable in W and $Q_n = 0$ for $n > \omega$.*

(iii) *Every projection in W is countably decomposable in W and has H-multiplicity $n \leq \omega$.*

(iv) *Every projection of UH-multiplicity in W is countably decomposable in W'.*

(v) *The projections Q_n are countably decomposable in W'.*

(vi) *Every projection in* W *is countably decomposable in* W'.

(vii) *Every non-zero projection of* UH*-multiplicity in* W *has* UOSD*-multiplicity (and hence they are equal).*

(viii) $E(\cdot)$ *has the* CGS*-property in* H.

Proof. (i) ⇒ (ii) Let $Q_n \neq 0$. Then by Proposition 1.3, Q_n has UH-multiplicity n. Therefore, (ii) is immediate from (i).

(ii) ⇒ (iii) If P is a non-zero projection in W, then by (ii) $P = \sum_{n \leq \omega} Q_n P$. Being Q_n countably decomposable in W, it follows that the same is true for P. Then the H-multiplicity of $P = \min\{n : PQ_n \neq 0\} \leq \omega$ by Theorem 64.2 of [2] and by Proposition 1.3.

(iii) ⇒ (iv) Let P be a non-zero projection of UH-multiplicity n. By (iii), $n \leq \omega$. By Theorem 1.2 there exists an orthogonal family $\{E_i'\}_1^n$ of abelian projections in W' such that $P = \sum_1^n E_i'$ and $C_{E_i} = P$ for all i. Now by (iii) and Lemma 2.5 there exist vectors x_i in PH such that $[Wx_i] = E_i'$. If $x = \{x_i\}_1^n$, then clearly $PH = [W_x]$ so that by Lemma 3.3.9 of [3] P is countably decomposable in W'.

(iv) ⇒ (v) This is immediate, since Q_n has UH-multiplicity n by Proposition 1.3.

(v) ⇒ (vi) Let P be a non-zero projection in W'. Then $P = \sum_{n \leq \dim H} PQ_n$ and by (v) Q_n are countably decomposable in W'. To prove that P is countably decomposable in W', it suffices to show that $Q_n = 0$ for $n > \omega_0$. If $Q_n \neq 0$, as Q_n has UH-multiplicity n by Proposition 1.3, there exists an orthogonal family $\{E_\alpha'\}_{\alpha \in J_n}$ of abelian projections in W' such that card. $J_n = n$, $Q_n = \sum_{\alpha \in J_n} E_\alpha'$ and $C_{E'} = Q_n$. As Q_n is countably decomposable in W', it follows that J_n is countable so that $n \leq \omega$. Consequently, $Q_n = 0$ for $n > \omega$.

(vi) ⇒ (vii) Let P be a non-zero projection of UH-multiplicity n. By (vi) P is countably decomposable in W' and hence in W. By Proposition 1.3, there exists a unique Q_n such that $P \leq Q_n$. As in the proof of (v) ⇒ (vi) we note that $Q_k = 0$ for $k > \omega_0$ and hence $n \leq \omega$. Consequently, by Theorem 2.6, (vii) holds.

(vii) ⇒ (viii) By Proposition 1.3 Q_n has UH-multiplicity n

if $Q_n \neq 0$. Then by (vii), $n \leqslant \omega$ if $Q_n \neq 0$ and hence $Q_n = 0$ for $n > \omega$. Again by (vii) as Q_n has *UOSD*-multiplicity n for $n \in J_o = \{n: Q_n \neq 0\}$ there exists an orthonormal set $\{x_{nj}\}_{j=1}^{n}$ in $Q_n H$ such that $Q_n H = [E(\sigma)x_{nj}: \sigma \in S, j = 1,2,\ldots,n]$. Therefore, $H = [E(\sigma)x_{nj}, n \in J_o, j = 1,2,\ldots,n]$ and hence (viii) holds.

(viii) $\Rightarrow$ (i) By (viii) and by Lemma 3.3.9 of [3] W' is countably decomposable and hence W is countably decomposable. Besides, evidently for every projection P of *UH*-multiplicity n in W, $n \leqslant \omega$. Thus (i) holds.

§4. COMPARISON BETWEEN *OSD*-MULTIPLICITY AND *H*-MULTIPLICITY.

Example 2.4 is just a particular case of the following more general result.

THEOREM 4.1. *Suppose $E(\cdot)$ has the CGS-property in H. Let P be a non-zero projection in W with the H-multiplicity n and with the OSD-multiplicity (relative to $E(\cdot)$) N. Then:*

(i) $n \leqslant N$.

(ii) *$n = N$ if and only if P has UH-multiplicity n.*

(iii) *$n = N$ if and only if P has UOSD-multiplicity n (relative to $E(\cdot)$).*

Proof. By Theorem 62.4 of [2] there exists a non-zero projection Q in W such that $Q \leqslant P$ and such that Q has *UH*-multiplicity n. Besides, by Theorem 3.1, Q is countably decomposable in W and $n \leqslant \omega$. Therefore, by Therorem 2.6 Q has the *UOSD*-multiplicity n relative to $E(\cdot)$. Consequently, by Theorem 5 of [4] the total multiplicity of Q is n and therefore, $n \leqslant N$.

(ii) Suppose $n = N$. We discuss the following two cases.

Caso 1. n is finite.

By hypothesis, there exists a maximal orthogonal family $\{E'_i\}_1^n$ of abelian projections in W' such that $C_{E'_i} = P$ for all i. If P does not have *UH*-multiplicity then by Theorem 2.2 $P \neq \sum_1^n E'_i$. Now, by Theorem 3.1 (iii) and Lemma 2.5 there exist vectors $x_i \in PH$ such that $E'_i = [Wx_i]$, $i = 1,2,\ldots,n$. Then by Theorems 66.2 and 65.2 of [2] we have $\rho(x_1) \equiv \rho(x_2) \equiv \ldots \equiv \rho(x_n)$. Let $E' = \sum_1^n E'_i$. Clearly, $E' \in W'$ and $E'H = \sum_1^n \oplus Z(x_i)$ is a *UOSD* of $E'H$ relative to $E(\cdot)E'$. If $x \in (P-E')H$ and $x \neq 0$, then $[Wx] = Z(x) \perp E'H$ and $C_{[Wx]} \leqslant P = C_{[Wx_1]}$. Consequently, by Theorem 65.2 of [2] we conclude that $\rho(x) \ll \rho(x_1)$. On the other hand, by Theorem 1 of [4] there exists an *OSD*: $(P-E')H = \sum_{n+1}^{\ell} \oplus Z(x_i)$, $\ell \in \mathbb{N} \cup \{\omega\}$, of $(P-E')H$ relative to $E(\cdot)(P-E')$ so that $PH = \sum_1^{\ell} \oplus Z(x_i)$ is an *OSD* of PH relative to $E(\cdot)P$. Thus $\ell = N$ and P has *OSD*-multiplicity $N > n$. This contradiction proves that P has *UH*-multiplicity n.

Caso 2. n is infinite.

Due to Theorem 3.1, $n = \omega$ and $Q_k = 0$ for $k > \omega$. Since $P = \sum_{\ell \leqslant \dim H} PQ = \sum_{\ell \leqslant \omega} PQ_\ell$ and since the H-multiplicity of P is ω and is given by $\min\{\ell : PQ_\ell \neq 0\}$, we have $pQ_\ell = 0$ for $\ell \neq \omega$. Thus $P \leqslant Q_\omega$ and hence P has *UH*-multiplicity ω by Proposition 1.3.

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(Recibido en Julio de 1988)