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# ON THE RATE OF CONVERGENCE OF HERMITE-FEJÉR POLYNOMIALS TO FUNCTIONS OF BOUNDED VARIATION ON THE ZEROS OF CERTAIN JACOBI POLYNOMIALS

by

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**Abstract**. In this paper, we study the Hermite-Fejér interpolation, $H_n(f,x)$, for a function f of bounded variation on $[-1,1]$ when the interpolation is taken over the zeros of Jacobi polynomials, $P_n^{(\alpha,\beta)}(x)$ when $|\alpha|, |\beta| \leq \frac{1}{2}$.

Our main result is an estimate for the rate of convergence of $H_n(f,x)$ at points of continuity of f, and a special attention is given to the interpolation over the zeros of the Legendre polynomials.

**§1. Introduction and Results**. Let $f$ be a real-valued function defined on $[-1,1]$. The polynomial $H_n(f,x)$ of degree $2n-1$ at most, of Hermite-Fejér interpolation based on the zeros $x_{kn}$, $k = 1,2,\ldots,n$, of the Jacobi polynomials $P_n^{(\alpha,\beta)}(x)$, is defined by

$$H_n(f,x) = \sum_{k=1}^{n} f(x_{kn})H_{kn}(x), \tag{1.1}$$

where

$$H_{kn}(x) = \frac{1-x[\alpha-\beta+(\alpha+\beta+2)x_{kn}]+(\alpha-\beta)x_{kn}+(\alpha+\beta+1)x_{kn}^2}{1-x_{kn}^2} \cdot \left[\frac{P_n^{(\alpha,\beta)}(x)}{(x-x_{kn})P_n^{(\alpha,\beta)\prime}(x_{kn})}\right]^2. \tag{1.2}$$

Bojanic, R., and Cheng, F.H., [2], estimated the rate of convergence of the Hermite-Fejér interpolation based on the zeros of the Tchebyshev polynomials of the first kind, $T_n(x)$, to functions of bounded variation at points of continuity in $(-1,1)$.

**THEOREM.** (Bojanic and Cheng [2]). *Let $f$ be a function of bounded variation on $[-1,1]$, and continuous at $x \in (-1,1)$. Then for all n sufficiently large*

$$|H_n(f,x)-f(x)| \leq \frac{64|T_n^2(x)|}{n}\sum_{k=1}^{n} V_f\left[x-\frac{\pi}{k}, x+\frac{\pi}{k}\right]$$

$$+\ 2V_f\left[x-\frac{\pi|T_n(x)|}{2n},\ x+\frac{\pi|T_n(x)|}{2n}\right]$$

*where, $V_f[a,b]$ is the total variation of $f$ on $[a,b]$.*

Al-Jarrah [1], estimated this rate of convergence but with the nodes of interpolation taken to be the zeros of the Tchebyshev polynomials of the second kind, $U_n(x)$.

**THEOREM.** (Al-Jarrah [1]). *Let $f$ be a function of bounded variation on $[-1,1]$ and continuous at $x \in (-1,1)$. Then for all n sufficiently large*

$$|H_n(f,x)-f(x)| < \frac{48}{1-x^2}\left\{\frac{8}{n+1}\sum_{k=1}^{n} V_f\left[x-\frac{\pi}{k}, x+\frac{\pi}{k}\right]\right.$$

$$\left.+\ V_f\left[x-\frac{\pi(1-x^2)^{1/2}|U_n(x)|}{2(n+1)},\ x+\frac{\pi(1-x^2)^{1/2}|U_n(x)|}{2(n+1)}\right]\right\}$$

*where, $V_f[a,b]$ is the total variation of $f$ on $[a,b]$.*

Here, we estimate this rate of convergence when the nodes of interpolation are taken to be the zeros of the Jacobi polynomial, $P_n^{(\alpha,\beta)}(x)$, for $-\frac{1}{2} \leq \alpha,\beta \leq \frac{1}{2}$, and special attention will be given to this intepolation over the zeros of the Legendre polynomials $P_n(x)\left(=P_n^{(0,0)}(x)\right)$.

In our work, the above mentioned results of Bojanic and Al-Jarreh become special cases of our result corresponding to $\alpha=\beta=-\frac{1}{2}$ and $\alpha=\beta=\frac{1}{2}$, respectively. Our main results can be stated as follows:

**THEOREM (1).** *Let $f$ be a function of bounded variation on $[-1,1]$ and continuous at $\cos\theta = x \in (-1,1)$. Then for all $n$ sufficiently large, we have*

$$|H_n(f,x)-f(x)| \leqslant M_1(\theta)V_f\left[x-\frac{\pi}{2n+1},\ x+\frac{\pi}{2n+1}\right] + \frac{M_2(\theta)}{2n+1}\sum_{k=1}^{n} V_f\left[x-\frac{\pi}{k},\ x+\frac{\pi}{k}\right], \tag{1.3}$$

*where $M_1(\theta)$ and $M_2(\theta)$ are positive constants depending on $\theta$, $\alpha$, and $\beta$.*

Here, $V_f[a,b]$ is the total variation of $f$ on $[a,b]$, and we assume that $f$ is extended to the entire real line by $f(x) = f(1)$ *for* $x > 1$, and $f(x) = f(-1)$ *for* $x < -1$.

As far as the precision of (1.3) is concerned, consider the Hermite-Fejḗr polynomial of the function $f(x) = x^2$ at $x = 0$, for even $n$. On one hand, we have

$$H_n(f,0)-f(0) = \sum_{k=1}^{n} x_{kn}^2\,\frac{1+(\alpha-\beta)x_{kn}+(\alpha+\beta+1)x_{kn}^2}{1-x_{kn}^2}\left(\frac{P_n^{(\alpha,\beta)}(0)}{x_{kn}P_n^{(\alpha,\beta)'}(x_{kn})}\right)^2$$

$$= \sum_{k=1}^{n} \frac{1+(\alpha-\beta)x_{kn}+(\alpha+\beta+1)x_{kn}^2}{1-x_{kn}^2}\left(\frac{P_n^{(\alpha,\beta)}(0)}{P_n^{(\alpha,\beta)'}(x_{kn})}\right)^2$$

$$\geqslant \frac{1}{2}\sum_{k=1}^{n}\left(\frac{P_n^{(\alpha,\beta)}(0)}{P_n^{(\alpha,\beta)'}(x_{kn})}\right)^2 .$$

By using Lemma 2 and Lemma 3 of section 2, we have

$$H_n(f,0) - f(0) \geq \frac{M(\alpha,\beta)}{n^2} \sum_{k=1}^{n} 1 = \frac{M(\alpha,\beta)}{n},$$

for some positive constant $M(\alpha,\beta)$ depending on $\alpha$ and $\beta$.

On the other hand, from (1.3) it follows that

$$|H_n(f,0)-f(0)| \leq M_1(\alpha,\beta) V_f\left[-\frac{\pi}{2n+1}, \frac{\pi}{2n+1}\right]$$

$$+ \frac{M_2(\alpha,\beta)}{2n+1} \sum_{k=1}^{n} V_f\left[-\frac{\pi}{k}, \frac{\pi}{k}\right]$$

$$\leq 2M_1(\alpha,\beta) V_f\left[0, \frac{\pi}{2n+1}\right] + \frac{2M_2(\alpha,\beta)}{2n+1} \sum_{k=1}^{n} V_f\left[-\frac{\pi}{k}, \frac{\pi}{k}\right].$$

Since $V_f[0,\delta] = \delta^2$, we have

$$|H_n(f,0)-f(0)| \leq 2M_1(\alpha,\beta) \frac{\pi^2}{(2n+1)^2} + \frac{2M_2(\alpha,\beta).\pi^2}{2n+1} \sum_{k=1}^{n} \frac{1}{k^2} \leq \frac{C(\alpha,\beta)}{n}$$

for some positive constant $C(\alpha,\beta)$.

Hence for the function $f(x) = x^2$, when $n$ is an even integer, we have

$$\frac{M(\alpha,\beta)}{n} \leq |H_n(f,0)-f(0)| \leq \frac{C(\alpha,\beta)}{n},$$

which shows that (1.3) cannot be improved asymptotically.

However, for the special case of Theorem (1) when $\alpha = \beta = 0$, the polynomial $H_n(f,x)$, of degree $2n-1$ at most, of the Hermite-Fejér interpolation based on the zeros $x_{kn}$, $k = 1,2,\ldots,n$, of the Legendre polynomials is defined by

$$H_n(f,x) = \sum_{k=1}^{n} f(x_{kn}) H_{kn}(x), \tag{1.4}$$

where

$$H_{kn}(x) = \frac{1-2xx_{kn}+x_{kn}^2}{1-x_{kn}^2}\left(\frac{P_n(x)}{(x-x_{kn})P_n'(x_{kn})}\right)^2. \tag{1.5}$$

In this case, we have the following:

**THEOREM (2).** *Let $f$ be a function of bounded variation on $[-1,1]$, and continuous at $\cos\theta = x \in (-1,1)$. Then for all $n$ sufficiently large, we have*

$$|H_n(f,x)-f(x)| \leqslant \left(284+\frac{23}{\sin\theta}\right)\cdot\frac{16}{2n+1}\cdot\sum_{k=1}^{n} V_f\left[x-\frac{\pi}{k},x+\frac{\pi}{k}\right] + (42\pi)^2 V_f\left[x-\frac{\pi}{2n+1},x+\frac{\pi}{2n+1}\right]. \tag{1.6}$$

**§2. Preliminaries.** To avoid unnecessary repetition, the following assumptions will be adopted: $-1 < x = \cos\theta < 1, 0 < \theta < \pi$.

$$E_r(n,\theta) = \{k: \frac{r\pi}{2n+1} < |\theta-\theta_{kn}| \leqslant \frac{(r+1)\pi}{2n+1}\} \qquad r=0,1,\ldots,2n \tag{1.7}$$

$$u_n(\theta) = -\frac{1}{\sqrt{2}}\sin^{1/2}\theta P_n^{(\alpha,\beta)}(\cos\theta). \tag{1.8}$$

It is well known that, for $r = 0,1,\ldots,2n$, $E_r(n,\theta)$ has at most two elements.

Before we prove our main results, the following lemmas are in order.

**LEMMA (1).** *For $-\frac{1}{2} \leqslant \alpha,\beta \leqslant \frac{1}{2}$, there exists positive constants $C_1(\alpha,\beta)$, $C_2(\alpha,\beta)$ and $n_0(\alpha,\beta)$, all finite, such that for $n \geqslant n_0(\alpha,\beta)$, we have*

$$\left|\frac{u_n(\theta)}{(\theta-\theta_{kn})u_n'(\theta_{kn})}\right| \leqslant \begin{cases} 1+\dfrac{3C_1(\alpha,\beta)}{C_2(\alpha,\beta)}\cdot\dfrac{(n+1)^2}{n(4n+2)}, & \text{on } E_0(n,\theta) \\[2ex] \dfrac{C_1(\alpha,\beta)}{C_2(\alpha,\beta)}\cdot\dfrac{1}{n|\theta-\theta_{kn}|}, & \text{on } E_r(n,\theta), r=1,2,\ldots,n \end{cases}$$

*where,*

$$C_1(\alpha,\beta) = \begin{cases} \frac{3}{2\sqrt{\pi}} + \frac{1}{2|\alpha|}, & 0 < |\alpha| \leq \frac{1}{2},\ 0 < \theta \leq \frac{\pi}{2} \\ 3/2, & \alpha = 0 \text{ or } \beta = 0,\ 0 < \theta < \pi \\ \frac{3}{2\sqrt{\pi}} + \frac{1}{2|\beta|}, & 0 < |\beta| \leq \frac{1}{2}, \frac{\pi}{2} < \theta < \pi \end{cases}$$

$$C_2(\alpha,\beta) = \begin{cases} \frac{\sqrt{\pi}}{8|\rho(\alpha)|}, & 0 < \theta_{kn} \leq \frac{\pi}{2},\ n \geq n_0(\alpha) = \sqrt{\pi}|\rho(\alpha)|.C_1(\alpha,\beta) \\ \frac{\sqrt{\pi}}{8|\rho(\beta)|}, & \frac{\pi}{2} < \theta_{kn} < \pi,\ n \geq n_0(\beta) = \sqrt{\pi}\ |\rho(\beta)|.C_1(\alpha,\beta) \end{cases}$$

*and* $\rho(\alpha) = \begin{cases} \frac{\pi}{2 \sin \alpha\pi}, & \alpha \neq 0 \\ -1, & \alpha = 0 \end{cases}$

*Proof*. See [4], Theorem 1.3 .

**LEMMA (2)**. *There exists* $\psi \in (0,\pi)$, *and constants* $M_1(\alpha,\beta)$, $N(\alpha,\beta)$ *such that*

$$|P_n^{(\alpha,\beta)}(\cos\psi)| \geq \sqrt{\frac{\pi}{n}} M_1(\alpha,\beta) > 0, \text{ for } n \geq N(\alpha,\beta),$$

*where* $\alpha, \beta > -1$.

*Proof*. See ([4], Lemma 3.1).

**LEMMA (3)**. *There exists a constant* $M_2(\alpha,\beta)$ *such that*

$$|P_n^{(\alpha,\beta)'}(\cos\theta_{kn})| \leq \frac{M_2(\alpha,\beta).n^{1/2}}{(\sin\frac{\theta_{kn}}{2})^{\alpha+7/2}.(\sin\frac{\theta_{kn}}{2})^{\beta+7/2}}, \quad n \geq 2.$$

*Proof*. See [4], Lemma 3.2 .

**LEMMA (4)**. *Let* $x \in (-1,1)$ *and* $x_{kn} = \cos\theta_{kn}$ *be the zeros of the Jacobi polynomials,* $P_n^{(\alpha,\beta)}(x)$, *then for* $|\alpha| \leq \frac{1}{2}$, $|\beta| \leq \frac{1}{2}$, *we have*

$$\left|\frac{P_n^{(\alpha,\beta)}(\cos\theta)\sin^{1/2}\theta}{(\theta-\theta_{kn})\sin^{3/2}\theta_{kn}P_n^{(\alpha,\beta)'}(\cos\theta_{kn})}\right| \leq \begin{cases} 15, & \text{on } E_0(n,\theta) \\ \frac{12}{\sqrt{\pi}}\frac{1}{n|\theta-\theta_{kn}|}, & \text{on } E_r(n,\theta), r \geq 1 \end{cases}$$

*Proof*. For $U_n(\theta) = \frac{-1}{\sqrt{2}} \sin^{1/2}\theta P_n^{(\alpha,\beta)}(\cos\theta)$,

$$U_n'(\theta_{kn}) = \frac{1}{\sqrt{2}} \sin^{3/2}\theta_{kn} P_n^{(\alpha,\beta)'}(\cos\theta_{kn}).$$

From Lemma (1) we have,

$$\left|\frac{\sin^{1/2}\theta P_n^{(\alpha,\beta)}(\cos\theta)}{(\theta-\theta_{kn})\sin^{3/2}\theta_{kn}P_n^{(\alpha,\beta)'}(\cos\theta_{kn})}\right| = \left|\frac{U_n(\theta)}{(\theta-\theta_{kn})U_n'(\theta_{kn})}\right|$$

$$\leq 1 + \frac{3}{2}\,|\theta-\theta_{kn}|\,\frac{24(n+1)^2}{2\sqrt{\pi}n} \leq 1 + \frac{3}{2}\,\frac{24}{2\sqrt{\pi}}\,\frac{n+1}{n(4n+2)}$$

$$\leq 1 + \frac{36}{2\sqrt{\pi}} \leq 15 \text{ on } E_0(n,\theta) ,$$

and

$$\left|\frac{P_n^{(\alpha,\beta)}(\cos\theta)\sin^{1/2}\theta}{(\theta-\theta_{kn})\sin^{3/2}\theta_{kn}P_n^{(\alpha,\beta)'}(\cos\theta_{kn})}\right| \leq \frac{12}{\sqrt{\pi}}\,\frac{1}{n|\theta-\theta_{kn}|}$$

on $E_r(n,\theta)$, $r \geq 1$.

**LEMMA (5)**. *Let* $x \in [-1,1]$, *and* $x_{kn} = \cos\theta_{kn}$, $k = 1,2,\ldots,n$ *be the zeros of the Jacobi polyomials,* $P_n^{(\alpha,\beta)}(\cos\theta)$, *then for* $|\alpha| \leq \frac{1}{2}$, $|\beta| \leq \frac{1}{2}$, *we have*

$$\frac{(\theta-\theta_{kn})^2\sin\theta\,\sin\theta_{kn}}{(\cos\theta-\cos\theta_{kn})^2} \leq \pi^2.$$

*Proof*. Since $\cos\theta-\cos\theta_{kn} = 2\sin\left(\frac{\theta+\theta_{kn}}{2}\right)\sin\left(\frac{\theta-\theta_{kn}}{2}\right)$, and $\sin\theta \leq 2\sin\left(\frac{\theta+\theta_{kn}}{2}\right)$, we have

$$\frac{(\theta-\theta_{kn})^2 \sin\theta \sin\theta_{kn}}{(\cos\theta-\cos\theta_{kn})^2} = \frac{(\theta-\theta_{kn})\sin\theta \sin\theta_{kn}}{4\sin^2(\frac{\theta+\theta_{kn}}{2})\sin^2(\frac{\theta-\theta_{kn}}{2})}$$

$$\leqslant \frac{(\theta-\theta_{kn})^2}{\sin^2(\frac{\theta-\theta_{kn}}{2})} \leqslant \frac{(\theta-\theta_{kn})^2}{(\theta-\theta_{kn})^2}\pi^2 = \pi^2 .$$

**LEMMA (6).** *For the fundamental polynomials,* $H_{kn}(x)$, *associated with the Jacobi polynomial abscissas,* $x = \cos\theta \in [-1,1]$, *and for* $|\alpha|$, $|\beta| \leqslant \frac{1}{2}$, *we have*

$$|H_{kn}(x)| \leqslant \begin{Bmatrix} k_1(\theta), & \text{on } E_0(n,\theta) \\ \frac{k_2(\theta)}{r^2}, & \text{on } E_r(n,\theta),\ r \geqslant 1 \end{Bmatrix}.$$

*Proof.*

$$H_{kn}(x) = \frac{1-(\alpha-\beta)x-(\alpha+\beta+2)xx_{kn}+(\alpha+\beta+1)x_{kn}^2+(\alpha-\beta)x_{kn}}{1-x_{kn}^2}$$

$$\cdot\left(\frac{P_n^{(\alpha,\beta)}(x)}{(x-x_{kn})P_n^{(\alpha,\beta)'}(x_{kn})}\right)^2$$

$$= \frac{1-(\alpha+\beta+2)xx_{kn}+(\alpha-\beta)(x_{kn}-x)+(\alpha+\beta+1)x_{kn}^2}{1-x_{kn}^2}\left(\frac{P_n^{(\alpha,\beta)}(x)}{(x-x_{kn})P_n^{(\alpha,\beta)'}(x_{kn})}\right)^2$$

$$= \frac{1+(\alpha-\beta)(x_{kn}-x)+(\alpha+\beta+2)(x-x_{kn})^2-(\alpha+\beta+2)x^2+(\alpha+\beta+2)xx_{kn}-x_{kn}^2}{1-x_{kn}^2}$$

$$\cdot\left(\frac{P_n^{(\alpha,\beta)}(x)}{(x-x_{kn})P_n^{(\alpha,\beta)'}(x_{kn})}\right)^2$$

$$= \frac{1-x_{kn}^2+(\alpha+\beta+2)(x-x_{kn})^2+(\alpha-\beta)(x_{kn}-x)+(\alpha+\beta+2)xx_{kn}-(\alpha+\beta+2)x^2}{1-x_{kn}^2}$$

$$\cdot\left[\frac{P_n^{(\alpha,\beta)}(x)}{(x-x_{kn})P_n^{(\alpha,\beta)'}(x_{kn})}\right]^2$$

$$= \frac{1-x_{kn}^2}{1-x_{kn}^2}\left[\frac{P_n^{(\alpha,\beta)}(x)}{(x-x_{kn})P_n^{(\alpha,\beta)'}(x_{kn})}\right]^2 + \frac{(\alpha+\beta+2)(x-x_{kn})^2}{1-x_{kn}^2}$$

$$\cdot\left[\frac{P_n^{(\alpha,\beta)}(x)}{(x-x_{kn})P_n^{(\alpha,\beta)'}(x_{kn})}\right]^2 + \frac{((\alpha-\beta)+(\alpha+\beta+2)x)(x_{kn}-x)}{1-x_{kn}^2}$$

$$\cdot\left[\frac{P_n^{(\alpha,\beta)}(x)}{(x-x_{kn})P_n^{(\alpha,\beta)'}(x_{kn})}\right]^2$$

$$= S_1 + S_2 + S_3,$$

where

$$S_1 = \left[\frac{P_n^{(\alpha,\beta)}(x)}{(x-x_{kn})P_n^{(\alpha,\beta)'}(x_{kn})}\right]^2 = \frac{\sin^2\theta_{kn}}{\sin^2\theta}\left[\frac{(\theta-\theta_{kn})^2\sin\theta\sin\theta_{kn}}{(\cos\theta-\cos\theta_{kn})^2}\right]$$

$$\cdot\left[\frac{P_n^{(\alpha,\beta)}(x)\sin^{1/2}\theta}{(\theta-\theta_{kn})\sin^{3/2}\theta_{kn}P_n^{(\alpha,\beta)'}(x_{kn})}\right]^2,$$

using Lemma (4) and Lemma (5), we have

$$S_1 \leq \begin{cases} c_1(\theta)\pi^2(15)^2, & on\ E_0(n,\theta) \\ c_2(\theta)\pi^2\ \frac{144}{\pi}\ \frac{1}{n^2(\theta-\theta_{kn})^2}, & on\ E_r(n,\theta),\ r \geq 1 \end{cases}$$

$$\leq \begin{Bmatrix} c_1(\theta), & on\ E_o(n,\theta) \\ \dfrac{c_2(\theta)}{r^2}, & on\ E_r(n,\theta),\ r \geq 1 \end{Bmatrix};$$

$$S_2 = \frac{(\alpha+\beta+2)(x-x_{kn})^2}{1-x_{kn}^2}\left[\frac{P_n^{(\alpha,\beta)}(x)}{(x-x_{kn})P_n^{(\alpha,\beta)'}(x_{kn})}\right]^2$$

$$= \frac{(\alpha+\beta+2)}{\sin^2\theta_{kn}}\left[\frac{P_n^{(\alpha,\beta)}(x)}{P_n^{(\alpha,\beta)'}(x_{kn})}\right]^2$$

$$= \frac{(\alpha+\beta+2)(\theta-\theta_{kn})^2\sin\theta_{kn}}{\sin\theta}\left[\frac{P_n^{(\alpha,\beta)}(x)\sin^{1/2}\theta}{(\theta-\theta_{kn})\sin^{3/2}\theta_{kn}P_n^{(\alpha,\beta)'}(x_{kn})}\right]^2;$$

$$S_2 \leq \begin{Bmatrix} 3(21)^2\,\dfrac{(\theta-\theta_{kn})^2}{\sin\theta}, & on\ E_o(n,\theta) \\ 3\,\dfrac{(\theta-\theta_{kn})^2}{\sin\theta}\,\dfrac{144}{\pi}\,\dfrac{1}{n^2(\theta-\theta_{kn})^2}, & on\ E_r(n,\theta),\ r \geq 1 \end{Bmatrix}$$

$$\leq \begin{Bmatrix} \dfrac{c_3(\theta)}{n^2}, & on\ E_o(n,\theta) \\ \dfrac{c_4(\theta)}{n^2}, & on\ E_r(n,\theta),\ r \geq 1 \end{Bmatrix},$$

and

$$S_3 = \frac{(x_{kn}-x)[(\alpha-\beta)+(\alpha+\beta+2)x]}{1-x_{kn}^2}\left[\frac{P_n^{(\alpha,\beta)}(x)}{(x-x_{kn})P_n^{(\alpha,\beta)'}(x_{kn})}\right]^2$$

$$= \frac{(\alpha-\beta)+(\alpha+\beta+2)x}{(1-x_{kn}^2)(x-x_{kn})}\left[\frac{P_n^{(\alpha,\beta)}(x)}{P_n^{(\alpha,\beta)'}(x_{kn})}\right]^2$$

$$\leqslant \frac{4(\cos\theta-\cos\theta_{kn})}{\sin^2\theta}\,\frac{(\theta-\theta_{kn})^2\sin\theta\sin\theta_{kn}}{(\cos\theta-\cos\theta_{kn})^2}$$

$$\cdot\left[\frac{P_n^{(\alpha,\beta)}(x)\sin^{1/2}\theta}{(\theta-\theta_{kn})\sin^{3/2}\theta_{kn}P_n^{(\alpha,\beta)'}(x_{kn})}\right]^2$$

$$\leqslant\left\{\begin{array}{ll}\dfrac{4(\cos\theta-\cos\theta_{kn})}{\sin^2\theta}\,\pi^2(21)^2, & on\ E_0(n,\theta)\\[2ex] \dfrac{4(\cos\theta-\cos\theta_{kn})}{\sin^2\theta}\,\pi^2\,\dfrac{(12)^2}{\pi}\,\dfrac{1}{n^2(\theta-\theta_{kn})^2}, & on\ E_r(n,\theta)\end{array}\right\};$$

$$S_3\leqslant\left\{\begin{array}{ll}c_5(\theta), & on\ E_0(n,\theta)\\[1ex] \dfrac{c_6(\theta)}{r^2}, & on\ E_r(n,\theta),\ r\geqslant 1\end{array}\right\}.$$

Which completes the proof.

## §3. Proof of Theorems (1 and 2).

**Proof of Theorem (1).** For all $x \in (-1,1)$,

$$|H_n(f,x)-f(x)| \leqslant \sum_{k=1}^{n}|f(x_{kn})-f(x)|\cdot|H_{kn}(x)|$$
$$\leqslant \sum_{k=1}^{n}V_f|x-t_{kn},\ x+t_{kn}|\cdot|H_{kn}(x)|$$

where $t_{kn} = |x - x_{kn}|$,

$$|H_n(f,x)-f(x)| \leqslant \sum_{r=0}^{2n}\sum_{k\in E_r(n.\theta)}V_f[x-t_{kn},\ x+t_{kn}]\,|H_{kn}(x)|$$

where $E_r(n,\theta)$ is defined as in (1.7).

Since $t_{kn} = |x-x_{kn}| = |\cos\theta-\cos\theta_{kn}| \leqslant |\theta-\theta_{kn}| \leqslant \frac{\pi}{2n+1}$, for $k \in E_0(n,\theta)$, and $E_0(n,\theta)$ has at most two elements, and by Lemma (6), we have

$$\sum_{k\in E_0(n,\theta)} V_f[x-t_{kn},x+t_{kn}]\,|H_{kn}(x)| \leq 2k_1(\theta)V_f[x-\frac{\pi}{2n+1},\ x+\frac{\pi}{2n+1}]\ .$$

Since $E_r(n,\theta), r = 1,2,\ldots,2n$ has at most two elements and by Lemma (6), and $t_{kn} \leq \frac{(r+1)\pi}{2n+1}$, if $k \in E_r(n,\theta)$, we find for $r = 1,2,\ldots,2n$, that

$$\sum_{k\in E_r(n,\theta)} V_f[x-t_{kn},x+t_{kn}]\,|H_{kn}(x)| \leq 2\,\frac{k_2(\theta)}{r^2}V_f[x-\frac{(r+1)\pi}{2n+1},x+\frac{(r+1)\pi}{2n+1}].$$

Therefore,

$$|H_n(f,x)-f(x)| \leq 2k_1(\theta)V_f[x-\frac{\pi}{2n+1},x+\frac{\pi}{2n+1}]$$
$$+\ 2k_2(\theta)\sum_{r=1}^{2n}\frac{1}{r^2}V_f[x-\frac{(r+1)\pi}{2n+1},\ x+\frac{(r+1)\pi}{2n+1}]\ . \tag{1.9}$$

Let $P(t) = V_f[x-t,x+t]$, then

$$\sum_{r=1}^{2n}\frac{1}{r^2}V_f[x-\frac{(r+1)\pi}{2n+1},x+\frac{(r+1)\pi}{2n+1}] = \sum_{r=2}^{2n+1}\frac{1}{(r-1)^2}P(\frac{r\pi}{2n+1})$$
$$\leq 4\sum_{r=2}^{2n+1}\frac{1}{r^2}P(\frac{r\pi}{2n+1})\ . \tag{1.10}$$

Since $P(t)$ is non-decreasing function, we have

$$\int_{\frac{r\pi}{2n+1}}^{\frac{(r+1)\pi}{2n+1}}\frac{P(t)}{t^2}dt \geq P(\frac{r\pi}{2n+1})\int_{\frac{r\pi}{2n+1}}^{\frac{(r+1)\pi}{2n+1}}\frac{dt}{t^2}$$

$$\geq P(\frac{r\pi}{2n+1})\frac{2n+1}{\pi r(r+1)} \geq P(\frac{r\pi}{2n+1})\frac{2n+1}{2\pi r^2}$$

or, $$\frac{1}{r^2}P(\frac{r\pi}{2n+1}) \leq (\frac{2\pi}{2n+1})\int_{\frac{r\pi}{2n+1}}^{\frac{(r+1)\pi}{2n+1}}\frac{P(t)}{t^2}dt.$$

Hence,

$$\sum_{r=2}^{2n+1} \frac{1}{r^2} P\left(\frac{r\pi}{2n+1}\right) \leqslant \left(\frac{2\pi}{2n+1}\right) \int_{\frac{2\pi}{2n+1}}^{\frac{(2n+2)\pi}{2n+1}} \frac{P(t)}{t^2} dt$$

$$\leqslant \frac{2}{2n+1} \int_{\frac{2n+1}{2n+2}}^{\frac{2n+1}{2}} P\left(\frac{\pi}{t}\right) dt \leqslant \frac{2}{2n+1} \int_{\frac{2n+1}{2n+2}}^{n+1} P\left(\frac{\pi}{t}\right) dt$$

Since $P\left(\frac{\pi}{t}\right)$ is non-increasing, and $P\left(\frac{\pi}{t}\right) = p(\pi)$, $0 < t \leqslant 1$, we have

$$\int_{\frac{2n+1}{2n+2}}^{n+1} P\left(\frac{\pi}{t}\right) dt = \int_{\frac{2n+1}{2n+2}}^{1} P\left(\frac{\pi}{t}\right) dt + \int_{1}^{n+1} P\left(\frac{\pi}{t}\right) dt$$

$$\leqslant P(\pi) + \sum_{k=1}^{n+1} P\left(\frac{\pi}{k}\right) \leqslant 2 \sum_{k=1}^{n+1} P\left(\frac{\pi}{k}\right),$$

and so

$$\sum_{r=2}^{2n+1} \frac{1}{r^2} P\left(\frac{r\pi}{2n+1}\right) \leqslant \frac{4}{2n+1} \sum_{k=1}^{n+1} P\left(\frac{\pi}{k}\right). \tag{1.11}$$

Theorem (1) follows now from inequalities (1.9), (1.10), and (1.11).

As to the proof of Theorem 2, we need mainly to repeat the proof of Theorem 1 above when $\alpha = \beta = 0$ and determine $M_1(\theta)$ and $M_2(\theta)$ in this case. So the details of the proof will be left to the reader.

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