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SEPARATION PROPERTIES AND n-POINT TOPOLOGICAL EXTENSIONS

by

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§0. Introduction. A topological extension of a topological space $(X,\mathcal{T})$ is a topological space $(X^*,\mathcal{T}^*)$ containing $(X,\mathcal{T})$ as a dense subspace. Two topological extensions $(X^*,\mathcal{T}^*)$, and $(X_1^*,\mathcal{T}_1^*)$ of $(X,\mathcal{T})$ are said to be equivalent if there is a homeomorphism $h:(X^*,\mathcal{T}^*) \to (X_1^*,\mathcal{T}_1^*)$ such that $h|X = \mathrm{id}_X$. Given a positive integer n, we say that $(X^*,\mathcal{T}^*)$ is an *n-point extension* of $(X,\mathcal{T})$ if $(X^*,\mathcal{T}^*)$ is a topological extension of $(X,\mathcal{T})$, and $X^* \setminus X$ has exactly n elements. If, furthermore $(X^*,\mathcal{T}^*)$ is a T_i-space (i.e. it satisfies the separation axiom T_i) we talk about an n-point T_i-extension. The statements of the separation axioms that we will use here are those appearing in [8, 91-103]. A topological extension is not necessarily a compact space; however, most of the research on n-point extensions has dealt, up to now, with compactifications ([2],[6],[7],[9]), maybe because "compactness" is an essential ingredient in the proofs of many important theorems in mathematics. Not less important is the fact of being Hausdorff, since it guarantees, for example, the uniqueness of the limit point of a convergent filter. The principal aim of this paper is to explore the n-point Hausdorff extensions (independently of their compactness), in particular, the

n-point T_i-extensiones ($i = 3,4$) of a topological space.

In Section §1, we make some precisions on notation, and introduce a new concept, that of clustering set of a filter, which will be of some use in this paper. In the following sections, we establish necessary and sufficient conditions for the existence of n-point T_i-extensions of T_i-spaces. On the way, we remark some connections between our results, compactness, and T_i-closed spaces. In theorem 3.3, we show the existence of at least $\mathfrak{c}$ non-equivalent η-point T_2-extensions of $\mathbb{R}$, which we think is an interesting result by itself.

§1. Preliminaries. If $\mathcal{B}$ is a filter base on a set X, we shall denote by $\overline{\mathcal{B}}^X$ the filter generated by $\mathcal{B}$. If $(X,\mathcal{T})$ is a topological space and x is a point in X, we shall denote by $\mathcal{U}_x$ the filter of neighborhoods of x with respect to the topology $\mathcal{T}$.

If $(X^*,\mathcal{T}^*)$ is a topological extension of $(X,\mathcal{T})$, and $\mathcal{F}^*$ is a filter on X^*, *the trace of* $\mathcal{F}^*$ (on X) is the set $Tr(\mathcal{F}^*) = \{F^* \cap X;\ F^* \in \mathcal{F}^*\}$.

If $\mathcal{F}$ and $\mathcal{G}$ are filters on a set X, we say that $\mathcal{F}$ is *finer* than $\mathcal{G}$ if $\mathcal{G} \subseteq \mathcal{F}$. It is an easy matter to verify that this is an order relation on the set of all filters on the set X. Moreover, given two filters $\mathcal{G}$ and $\mathcal{F}$ on X, we shall denote by $\mathcal{F} \vee \mathcal{G}$ the l.u.b. of these filters with respect to the order given above. It is not difficult to show that $\mathcal{F} \vee \mathcal{G} = \{F \cap G;\ F \in \mathcal{F},\ G \in \mathcal{G}\}$. Of course this l.u.b. may be the trivial filter $\mathcal{P}(X)$, consisting of all subsets of X.

The following definition will prove to be useful in our discussion:

DEFINITION 1.1. Let $\mathcal{F}$ be a filter on a topological space $(X,\mathcal{T})$, and let A be subset of X. If

(1) $A \notin \mathcal{F}$, and

(2) for each neighborhood B of A (i.e. such that there is an open set G of $(X,\mathcal{T})$ verifying $A \subseteq G \subseteq B$), and each

$F \in \mathcal{F}$, then $B \cap F \neq \emptyset$, we say that A is a *clustering set of the filter* $\mathcal{F}$.

In [1] we have shown that the notions of cluster point and clustering set are quite independent. Indeed, there are filters which admit clustering sets but no cluster points, and viceversa, and filters which admit either both or none of them.

LEMMA 1.1. *Let* $(X,\mathcal{T})$ *be a topological space, and let* $\mathcal{F}_1$ *and* $\mathcal{F}_2$ *be two distinct filters on* X. *If none of them admits clustering sets then* $\mathcal{F}_1 \vee \mathcal{F}_2 = \mathcal{P}(X)$.

Proof. Since $\mathcal{F}_1 \neq \mathcal{F}_2$ we suppose, without loss of generality, the existence of a set A in $\mathcal{F}_1 \setminus \mathcal{F}_2$. Thus, since $\mathcal{F}_2$ has no clustering sets and $A \notin \mathcal{F}_2$, it follows that there is a neighborhood B of A, and a set $F \in \mathcal{F}_2$ such that $B \cap F \neq \emptyset$. But this means that $\emptyset \in \mathcal{F}_1 \vee \mathcal{F}_2$, whence the claim. ▲

From now on, all filters considered will be non-trivial filters.

§2. T_0 and T_1 n-point extensions. From now on, X^* will denote the set $X \cup \{w_1,\dots,w_n\}$, where $w_j \notin X$, $1 \leqslant j \leqslant n$. Since each finite set in a T_1-space is closed, each cofinite set is open. In particular, if $(X^*,\mathcal{T}^*)$ is an n-point T_1-extension of $(X,\mathcal{T})$, X is open in $\mathcal{T}^*$ and so $\mathcal{T} \subseteq \mathcal{T}^*$. Therefore, when $(X^*,\mathcal{T}^*)$ is a T_1-extension of $(X,\mathcal{T})$ for $x \in X$, U^* is a neighborhood of x in $\mathcal{T}^*$ if, and only if, $U^* \cap X$ is a neighborhood of x in $\mathcal{T}$. This may be expressed as follows:

$$\mathcal{U}_x^* = \{U \cup A;\ U \in \mathcal{U}_x^{\mathcal{T}},\ A \subseteq \{w_1,\dots,w_n\}\}. \tag{1}$$

From this it follows that the topology $\mathcal{T}^*$ of an n-point T_i-extension $(1 \leqslant i \leqslant 5)$ is completely determined once we know $\mathrm{Tr}(\mathcal{U}_{w_k}^*)$, $k = 1,\dots,n$. The following lemma expresses this fact more precisely.

LEMMA 2.1. *Two n-point T_1-extensions* $(X^*,\mathcal{T}^*)$ *and*

$(X_1, \mathcal{T}_1^*)$ *of a topological space* $(X,\mathcal{T})$, *where* $X^* = X \cup \{w_1, \ldots w_n\}$, *and* $X_1^* = X \cup \{a_1,\ldots,a_n\}$, *are equivalent if, and only if,* $Tr(\mathcal{U}^*_{w_j}) = Tr(\mathcal{U}^*_{a_j})$, $1 \leq j \leq n$, *in some order.*

Proof. If there is a homeomorphism $h:(X^*,\mathcal{T}^*) \to (X_1^*,\mathcal{T}_1^*)$ such that $h|X = id_X$, we may assume without loss of generality that $h(w_j) = a_j$, for $j = 1,\ldots,n$. Then it is an easy matter to show that $Tr(\mathcal{U}^*_{w_j}) = Tr(\mathcal{U}^*_{a_j})$. Conversely, let us suppose that the preceding equality holds for all $j = 1,\ldots,n$. Defining $h:X^* \to X_1^*$ by $h(x) = x$ if $x \in X$, $h(w_j) = a_j$, for $j = 1,\ldots n$, to show that h is bicontinuous for the given topologies, it suffices to prove that for every $x \in X^*$ (resp. $y \in X_1^*$), $\mathcal{U}^*_{h(x)} = h(\mathcal{U}^*_x)$ (resp. $\mathcal{U}^*_{h^{-1}(y)} = h^{-1}(\mathcal{U}^*_y)$). But this is clear from (1) is case $x \in X$ (resp. $y \in X_1$). In order to prove that $h(\mathcal{U}^*_{w_j}) = \mathcal{U}^*_{a_j}$, we remark that for any neighborhood $U^*_{w_j}$ of w_j in $(X^*,\mathcal{T}^*)$ we get $h(U^*_{w_j}) \supseteq h((U^*_{w_j} \cap X \cup \{w_j\}) = h(U^*_{w_j} \cap X) \cup h(w_j) = (U^*_{a_j} \cap X) \cup \{a_j\}$, where $U^*_{a_j}$ is a neighborhood of a_j in $(X^*,\mathcal{T}^*)$, since, by hypothesis, $Tr\mathcal{U}^*_{w_j} = Tr\mathcal{U}^*_{a_j}$. But clearly $(U^*_{a_j} \cap X) \cup \{a_j\} \in \mathcal{U}^*_{a_j}$, which proves the relation $h(\mathcal{U}^*_{w_j}) \subseteq \mathcal{U}^*_{a_j}$. Conversely, if $U^*_{a_j} \in \mathcal{U}^*_{a_j}$, we get $U^*_{a_j} \supseteq (U^*_{a_j} \cap X) \cup \{a_j\} = h(U^*_{a_j} \cap X) \cup \{h(w_j)\} = h((U^*_{w_j} \cap X) \cap \{w_j\} \in h(\mathcal{U}^*_{w_j})$. In a complete similar way we prove that $h^{-1}(U^*_{a_j}) = \mathcal{U}^*_{w_j}$. ▲

The following example shows that the above lemma is no longer valid if $i = 0$.

EXAMPLE 2.1. If $X = \{1,2,3\}$, $X_1^* = \{1,2,3,4\}$, $X_2^* = \{1,2,3,4,5\}$, and $\mathcal{T} = \{\emptyset, X, \{1\},\{1,3\}\}$, $\mathcal{T}_1^* = \{\emptyset, X_1^*, \{1\},\{1,3\},\{1,4\},\{1,3,4\}\}$, and $\mathcal{T}_2^* = \{\emptyset, X_2^*, \{1\},\{1,5\},\{1,3,5\}\}$, it is readily verified that $(X_1,\mathcal{T}_1^*)$ and $(X_2,\mathcal{T}_2^*)$ are both 1-point T_0-extensions of $(X,\mathcal{T})$. Also, $Tr(\mathcal{U}^*_4) = Tr(\mathcal{U}^*_5)$. But these two extensiones cannot be equivalent, since the only possible bijection h between X_1^* and X_2^* such that $h|X = id_X$, is given by $h(x) = x$, for $x = 1,2,3$, and $h(4) = 5$. But, on the other hand, for this bijection we have $h(\{1,3\}) = \{1,3\} \notin \mathcal{T}_2^*$, and $\{1,3\} \in \mathcal{T}_1^*$.

LEMMA 2.2. *Let $(X,\mathcal{T})$ be a topological space, and let $\mathcal{F}_1, \mathcal{F}_2,\ldots,\mathcal{F}_n$ be open filters on X. Let*

$$(2) \qquad \mathcal{T}^* = \mathcal{T} \cup \{A \cup \{w_{k_1},\ldots,w_{k_s}\};\ \ A \in \mathcal{T} \cap \mathcal{F}_{k_1} \cap \ldots \cap \mathcal{F}_{k_s}$$

$$1 \leqslant s \leqslant n,\quad k_j \in \{1,\ldots,n\},\quad i \leqslant j \leqslant k\}.$$

Then $(X^*,\mathcal{T}^*)$ is an n-point extension of $(X,\mathcal{T})$. Furthermore, $Tr(\mathcal{U}^*_{w_k}) = \mathcal{F}_k$ $(k=1,\ldots,n)$, and if $(X,\mathcal{T})$ is a T_0-space, then $(X^*,\mathcal{T}^*)$ is a T_0-space.

Proof. In the first place, it is easy to verify that $\mathcal{T}^*$ is indeed a topology on X^*, satisfying $\mathcal{T}^*|X := \{G^* \cap X;\ G^* \in \mathcal{T}^*\} = \mathcal{T}$. Moreover, X is dense in $(X^*,\mathcal{T}^*)$. Thus $(X^*,\mathcal{T}^*)$ is an n-point extension of $(X,\mathcal{T})$. Let now $U^*_{w_s} \in \mathcal{U}^*_{w_s}$, $1 \leqslant s \leqslant n$. Then, for some t, there is $A \in \mathcal{T} \cap \mathcal{F}_{k_1} \cap \cdots \cap \mathcal{F}_{k_s} \cap \cdots \cap \mathcal{F}_{k_t}$ such that $w_s \in A \cup \{w_k,\ldots,w_s,\ldots,w_{k_t}\} \subseteq U_{w_s}$. Thus $A \subseteq U^*_{w_s} \cap X$, and $A \in \mathcal{F}_s$. Therefore, $U^*_{w_s} \cap X \in \mathcal{F}_s$, and $Tr(\mathcal{U}^*_{w_s}) \subseteq \mathcal{F}_s$. Conversely, if $F_s \in \mathcal{F}_s$, there is $G_s \in \mathcal{T} \cap \mathcal{F}_s$ such that $G_s \subseteq \mathcal{F}_s$, since $\mathcal{F}_s$ is open. Hence $G_s \cup \{w_s\} \subseteq F_s \cup \{w_s\} \in \mathcal{U}^*_{w_s}$, and consequently $F_s = (F_s \cup \{w_s\}) \cap X \in Tr(\mathcal{U}_{w_s})$. Finally, if $(X,\mathcal{T})$ is a T_0-space, let us take two distinct points x and y in X^*. If both of them lie in X, they are clearly separated as in any T_0-space. If it happens that $x \in X$, but $y = w_s$, for some $s = 1,\ldots,n$, then, for any $G \in \mathcal{T} \subseteq \mathcal{T}^*$ such that $x \in G$, we have $w_s \notin G$. If $x = w_s$, $y = w_j$ $(s \neq j)$, by taking any $A \in \mathcal{T} \cap \mathcal{F}_s$, we obtain an open neighborhood $A \cup \{w_s\}$ of w_s not containing w_j. ▲

As the following example shows, the openness of the filters $\mathcal{F}_1,\ldots,\mathcal{F}_n$ is not sufficient to guarantee that $(X^*,\mathcal{T}^*)$ is a T_1-extension.

EXAMPLE 2.2. Let $X = (0,1]$, with the usual topology $\mathcal{T}$, $X^* = [0,1]$, and $\mathcal{F}$ the filter of neighborhoods of 1 in $(X,\mathcal{T})$. Clearly, $(X,\mathcal{T})$ is a T_1-space, and $\mathcal{F}$ is an open filter on X. However, if $\mathcal{T}^* = \mathcal{T} \cup \{G \cup \{0\};\ G \in \mathcal{T} \cap \mathcal{F}\}$, then $(X^*,\mathcal{T}^*)$ is, by virtue of lemma 2.2. a 1-point T_0-extension of $(X,\mathcal{T})$, which is not a T_1-extension. Indeed, in $(X^*,\mathcal{T}^*)$

the point 1 belong to every neighborhood of 0.

THEOREM 2.1. *Let $(X,\mathcal{J})$ be a T_1-space, and $\mathcal{F}_1,\ldots,\mathcal{F}_n$ be open filters on X. Then there is an n-point T_1-extension $(X^*,\mathcal{J}^*)$ with $Tr(\mathcal{U}^*_{w_k}) = \mathcal{F}_k$ $(1 \leq k \leq n)$ if, and only if, no $\mathcal{F}_k$ consists of neighborhoods of a single point $x \in X$. Moreover, this extension is unique.*

Proof. Because of Lemma 2.2, it suffices to show that $(X^*,\mathcal{J}^*)$, ($\mathcal{J}^*$ defined by (2)) is, under the conditions of the theorem, a T_1-space, since the uniqueness follows from Lema 2.1. Let, thus, x and y be two distinct points in X^*. If these points are both in X, they can be separated in X^* as in any T_1-space, since $\mathcal{J} \subseteq \mathcal{J}^*$. If $x \in X$, but $y = w_k$ (for some $k = 1,\ldots,n$), then any neighborhood $G_x \in \mathcal{J}$ of x satisfies $w_k \notin G_x$. On the other hand, since by hypothesis, not all the elements of F_k are neighborhoods of $x \in X$, there exists an element $F_k \in \mathcal{J} \cap \mathcal{F}_k$ such that $x \notin F_k$. Therefore, $F_k \cup \{w_k\}$ is an open neighborhood of w_k in X^*, not containing x, whence the result in this case. Finally, if $x = w_k$, and $y = w_j$ $(k \neq j)$, then, for any $A \in \mathcal{J} \cap \mathcal{F}_k$, and any $B \in \mathcal{J} \cap \mathcal{F}_j$, we clearly have $w_j \notin A \cup \{w_k\}$, and $w_k \notin B \cup \{w_j\}$. Thus $(X^*,\mathcal{J}^*)$ is indeed a T_1-space.

Conversely, if $(X^*,\mathcal{J}^*)$ is an n-point T_1-extension of $(X,\mathcal{J})$ satisfying $Tr(\mathcal{U}^*_{w_k}) = F_k$ $(1 \leq k \leq n)$, and $\mathcal{F}_k = \mathcal{U}_x$, for some k $(1 \leq k \leq n)$, and some $x \in X$, it is clear that then $(X^*,\mathcal{J}^*)$ cannot be a T_1-space. ▲

§3. n-point T_2-extensions. In this section we characterize the T_2-extensions of T_2-spaces. More precisely, we have the following.

THEOREM 3.1. *Let $(X,\mathcal{J})$ be a T_2-space and let $(X^*,\mathcal{J}^*)$ be an n-point T_1-extension of $(X,\mathcal{J})$. Then $(X^*,\mathcal{J}^*)$ is a T_2-extension if, and only if, for each $i \in \{1,\ldots,n\}$, $Tr(\mathcal{U}^*_{w_i})$ is an open filter on X, without cluster points in*

X, and $Tr(\mathcal{U}^*_{w_i}) \vee Tr(\mathcal{U}^*_{w_j}) = \mathcal{P}(X)$, *for each pair* i, j, $1 \leq i, j \leq n$, $i \neq j$.

Proof. Let us suppose that $(X^*,\mathcal{T}^*)$ is a T_2-space. Each $Tr(\mathcal{U}^*_{w_k})$ is an open filter on X, without cluster points in X, since $\mathcal{U}^*_{w_k}$ is open ([4, Obs.2, p.553]), and obviously it converges to w_k, so that its only cluster point is w_k, because $(X^*,\mathcal{T}^*)$ is T_2. Now, if $i \neq j$, so that $w_i \neq w_j$, it is easlily seen that $Tr(\mathcal{U}^*_{w_i}) \vee Tr(\mathcal{U}^*_{w_j}) = \mathcal{P}(X)$, since $(X^*,\mathcal{T}^*)$ is supposed to be a T_2-space. Conversely, if the $Tr(\mathcal{U}^*_{w_k}) = F_k$, $(1 \leq k \leq n)$ satisfy the given conditions, we define $\mathcal{T}^*_1$ by (2). As in the proof of Theorem 2.1, it is not difficult to show that $(X^*,\mathcal{T}^*)$ is T_2-extension of $(X,\mathcal{T})$ such that the traces of the neighborhoods of the w_k in $(X^*,\mathcal{T}^*)$ are precisely the $\mathcal{F}_k$. But then Theorem 2.1 also implies that $\mathcal{T}^*_1 = \mathcal{T}^*$. ▲

Using Lemma 1.1 we obtain the following

COROLLARY 3.1.1. *If* $\mathcal{F}_1, \mathcal{F}_2, \ldots, \mathcal{F}_n$ *are open filters on X, without cluster points nor clustering sets in X, then* $(X^*,\mathcal{T}^*)$, *defined by* (2), *is an n-point* T_2*-extension verifying* $Tr(\mathcal{U}^*_{w_k}) = F_k$, $k = 1, \ldots, n$. ▲

The next two examples show that Theorem 3.1 is no longer valid if $i = 0$, or $i = 1$:

EXAMPLE 3.1. The spaces $(X,\mathcal{T})$ and $(X^*_1,\mathcal{T}^*_1)$ in Example 2.1 are both T_0-spaces, the second one being a 1-point extension of the first. Clearly the filter of neighborhoods of 4 in $(X^*,\mathcal{T}^*_1)$ is the set

$$\{X^*_1, \{1,4\}, \{1,2,4\}, \{1,3,4\}\},$$

but its trace, $\{X, \{1\}, \{1,2\}, \{1,3\}\}$, admits 1 as a cluster point (moreover, as a limit point).

EXAMPLE 3.2. Let $X = (0,1]$, $\mathcal{T}$ its usual topology, $X^* = [0,1]$ and $\mathcal{J}^* = \mathcal{J} \cup \{A \cup \{0\} : A \in X,\ X \setminus A \text{ finite}\}$. Then it is easy to verify that $(X^*,\mathcal{J}^*)$ is a 1-point T_1-extension of $(X,\mathcal{T})$, and that $x = 1/2$ is a cluster point of the filter $Tr(\mathcal{U}_0^*)$.

In order to state our next theorem we will suppose that $X^* := X \cup (w_1,\ldots,w_n)$, where $(w_1,\ldots,w_n)$ represents a linearly ordered set if $n \geqslant 1$.

THEOREM 3.2. *Let $(X,\mathcal{T})$ be a T_2-space and let Φ be the family of all open filters on X, without cluster points in X. Let $\Xi_2^{(n)}$ be the family of all equivalence classes of n-point T_2-extensions of $(X,\mathcal{T})$. Then the mapping ϕ from $\Xi_2^{(n)}$ into $\Phi^n = \Phi\times\ldots\times\Phi$ (n times), which sends the class of $(X^*,\mathcal{T}^*)$ into $(Tr(\mathcal{U}_{w_1}^*),\ldots,Tr(\mathcal{U}_{w_n}^*))$ is a one-to-one mapping which is not onto if $n > 1$. If $n = 1$, Φ is a bijection.*

Proof. By Lemma 2.1 and Theorem 3.1, ϕ is a well-defined one-to-one mapping. This mapping is onto if $n = 1$ (Easy!). In order to verify that ϕ is not an onto mapping, it suffices to take $\mathbf{F} = (\mathcal{F},\ldots,\mathcal{F})$, where $\mathcal{F}$ is a proper open filter on X, without cluster points in X.

Indeed, if there were an n-point T_2-extension $(X^*,\mathcal{T}^*)$ of $(X,\mathcal{T})$, such that $\phi(X^*,\mathcal{T}^*) = \mathbf{F}$, we would have $Tr(\mathcal{U}_{w_k}^*) = \mathcal{F}_k$, for every $k \in \{1,\ldots,n\}$, which in turn would imply that $\mathcal{F} = \mathcal{F} \vee \mathcal{F} = \mathcal{P}(X)$, by Theorem 3.1. But this is impossible. ▲

THEOREM 3.3. *For each positive integer n, the topological space $\mathbb{R}$ of the real numbers, with its usual topology, admits at least $\mathfrak{c}$ mutually non-equivalent n-point T_i-extensions ($i = 1,2$).*

Proof. Let $\mathcal{T}$ stand for the usual topology on $\mathbb{R}$. Let $\mathcal{A} = (A_n)_{n\in\mathbb{N}}$; where $A_n = \{x \in \mathbb{R}; x > n\}$. Clearly, $\mathcal{A}$ is an open filter base, and $\mathcal{F} = \overline{\mathcal{A}}^{\mathbb{R}}$ is an open filter on $\mathbb{R}$, which is not an open ultrafilter, since neither the open set $A = \bigcup_{n\in\mathbb{N}} (n,n+1)$ nor its complement, $\mathbb{Z} = \mathbb{R} \setminus A$, belong to $\mathcal{F}$ [3, Prop. 1,2, p.640]. Now, if $x \in \mathbb{R}$, there is always a positive integer n such that $n > x$; hence taking $\beta = (n-x)/2$ we obtain an open neighborhood $G = (x-\beta, x+\beta)$ of x satisfying $G \cap A_n = \emptyset$. This means that $\mathcal{F}$ has no cluster points.

Let us consider next the open set $\mathcal{D} = \bigcup_{n\in\mathbb{N}}(a_n, b_n)$, where $0 < a_n < b_n$, $b_n \leq a_{n+1}$, and $\lim_{n\to\infty} a_n = +\infty$. Under these condition we have $\mathcal{D} \cap A_n \neq \emptyset$ for each $A_n \in \mathcal{A}$, and consequently $\overline{\mathcal{D}}^{\mathbb{R}} \vee \mathcal{F} \neq \mathcal{P}(\mathbb{R})$. Furthermore, it is easy to prove that $\overline{\mathcal{D}}^{\mathbb{R}} \vee \mathcal{F}$ is an open filter, finer than $\mathcal{F}$. By Zorn's Lemma, there is at least one open ultrafilter $\mathcal{G}_{\mathcal{D}}$ containing $\overline{\mathcal{D}}^{\mathbb{R}} \vee \mathcal{F}$, without cluster points in $\mathbb{R}$, since $\mathcal{F}$ has none. Of course, $\mathcal{G}_{\mathcal{D}}$ has no limit points. Let now $\mathcal{D}'$ be another open set of $\mathbb{R}$, constructed the same way as $\mathcal{D}$: $\mathcal{D}' = \bigcup_{n\in\mathbb{N}}(a'_n, b'_n)$, $0 < a'_n < b'_n$, $b'_n \leq a'_{n+1}$, and $\lim_{n\to\infty} a'_n = +\infty$. For this $\mathcal{D}'$ there is another open ultrafilter $\mathcal{G}_{\mathcal{D}'}$, without limit or cluster points. Further, $\mathcal{G}_{\mathcal{D}} \neq \mathcal{G}_{\mathcal{D}'}$, since otherwise we would have $(\overline{\mathcal{D}}^{\mathbb{R}} \vee \mathcal{F}) \vee (\overline{\mathcal{D}'}^{\mathbb{R}} \vee \mathcal{F}) = \mathcal{P}(\mathbb{R})$, a contradiction. Now, the family of all distinct sequences intervals (a_n, b_n) satisfying the above conditions has cardinality $\mathfrak{c}$. Moreover, $\mathcal{G}_{\mathcal{D}} \vee \mathcal{G}_{\mathcal{D}'} = \mathcal{P}(\mathbb{R})$ since both are open ultrafilters [3,Prop.1.1, 640]. The theorem now follows from Theorem 3.1. ▲

It is worth to remark, using results due to K.D. Magill [6], that $\mathbb{R}$ admits at least $\mathfrak{c}$ n-point T_2-extensions, of which, up to equivalence, just one is compact, if $n = 1,2$. If $n > 2$, none of these n-point T_2-extensions is compact.

An interesting question is the following. When is the family Φ in Theorem 3.2 non-empty? To answer this question, let us recall first that a T_i-space is *T_i-closed* if it is closed in any T_i-space containing it as a subspace. In particular, it is known that a T_2-space is T_2-closed if and only if, every open filter on the space has at least one cluster point [8,Theo. 17.29, 145]. Thus, *Φ is non-empty if, and only if, $(X,\mathcal{J})$ is not T_2-closed.* Hence, from Theorem 3.2, we get the following.

THEOREM 3.4. *Let (X,J) be a T_2-space. Then (X,J) is T_2-closed if, and only if, (X,J) does not admit 1-point T_2-extensions.* ▲

This also means that a topological space $(X,\mathcal{J})$ admits n-point T_i-extensions $(2 \leq i \leq 5)$ only when it is not T_2-closed.

Since a space is compact if, and only if, every filter on the space has a cluster point [8, Theo. 16.9, 122], we have the following result:

COROLLARY 3.4.1. *If a topological space admits an n-point T_i-extension ($i \geqslant 2$), then it is not compact.* ▲

Moreover, since the Alexandroff compactificacion of a non-compact space $(X,\mathcal{T})$ is a T_1-space (resp. a T_2-space) if, and only if, it is a T_1-space (resp. a locally compact T_2-space) [8,Theo. 18.3 and 18.6, 148-149], we may conclude, using Theorem 3.1 ($n = 1$), the following:

COROLLARY 3.4.2. *A non-compact locally compact T_2-space cannot be T_2-closed.* ▲

COROLLARY 3.4.3. *If $(X,\mathcal{T})$ is a non-compact T_2-space, then the following statements are equivalent:*

(a) *The Alexandroff compactification $(X^*,\mathcal{T}^*)$ of $(X,\mathcal{T})$ is a T_2-space.*

(b) *$(X,\mathcal{T})$ is locally compact.*

(c) *$Tr(\mathcal{U}_w^*)$ has no cluster points in X.* ▲

§4. T_3 and T_4 n-point extensions. Let us pass now to examine the n-point T_i-extensions of T_i-spaces, for $i = 3,4$. To begin with, the following example shows that for $n = 1$, the construction given by (2), where $\mathcal{F}$ is an open filter on X, without cluster points in X, does not necessarily produces T_i-extensions from T_i-spaces, $i = 3,4$.

EXAMPLE 4.1. Let $X = (0,1]$ and $\mathcal{T}$ its usual topology. Also let $\mathcal{F} = \mathcal{B}^X$, where $\mathcal{B} = \{(0,\varepsilon) \setminus \{1/n;\ n \in \mathbb{N}\};\ 0 < \varepsilon \leqslant 1\}$. Then $(X,\mathcal{T})$ is a T_4-space, and it is an easy matter to verify that $\mathcal{F}$ is an open filter on X, without cluster points in X. Let us next take $(X^*,\mathcal{T}^*)$, where $X^* = [0,1]$, and $\mathcal{T}^* = \mathcal{T} \cup \{G \cup \{0\};\ G \in \mathcal{T} \cap \mathcal{F}\}$. Since $(X,\mathcal{T})$ is a T_2-space, we already know that $(X^*,\mathcal{T}^*)$ is a 1-point T_2-extension. However, it is not a T_3-space, because the point 0 and the set $F = \{1/n;\ n \in \mathbb{N}\}$ cannot be separated by open sets in $(X^*, \mathcal{T}^*)$.

Let us recall next that a filter on a topological space is said to be *regular* if it is both an open and a closed filter (i.e. if it admits an open base, and a closed base). This notion allows us to prove the following result, analogous to Theorem 3.1.

THEOREM 4.1. *Let $(X,\mathcal{T})$ be a T_3-space, and let $\mathcal{F}_1, \mathcal{F}_2, \ldots, \mathcal{F}_n$, be regular filters on X, without cluster points in X, and such that $\mathcal{F}_i \vee \mathcal{F}_j = \mathcal{P}(X)$, for every pair i, j, $1 \leq i, j \leq n$, $i \neq j$. Then there is an n-point T_3-extension $(X^*,\mathcal{T}^*)$ of $(X,\mathcal{T})$ such that $Tr(\mathcal{U}^*_{w_k}) = \mathcal{F}_k$, $1 \leq k \leq n$. This extension is unique up to equivalence.*

Proof. By Theorem 2.1, we already know that $(X^*,\mathcal{T}^*)$, where $\mathcal{T}^*$ is defined by (2), is, up to equivalence, the unique n-point T_1-extension of $(X,\mathcal{T})$ satisfying $Tr(\mathcal{U}^*_{w_k}) = F_k$; $1 \leq k \leq n$. Thus to prove the theorem it suffices to show that $(X^*,\mathcal{T}^*)$ is a regular space. Indeed, let $x \in X^*$, and F^* be a closed set in $(X^*,\mathcal{T}^*)$ not containing x. Then $X \setminus (F^* \cap X) = (X \setminus F^*) \cap X \in \mathcal{T}^*_X = \mathcal{T}$, which means that $F^* \cap X$ is closed in $(X,\mathcal{T})$. By hypothesis, $(X,\mathcal{T})$ is a regular space. Thus, if $x \in X$ we may find an open neighborhood G_x of x, and an open set $A \in \mathcal{T}$, satisfying $F^* \cap X \subseteq A$, and $A \cap G_x \neq \emptyset$. If $F^* \subseteq X$, the result is clear. If $F^* \nsubseteq X$, we may write $F^* = (F^* \cap X) \cup \{w_{k_1}, \ldots, w_{k_s}\}$, for some s. Since x is not a cluster point of none of the filters $\mathcal{F}_{k_i}$, there exist open neighborhoods $G_{x,i}$ of x, and open sets $A_i \subset \mathcal{T} \cap \mathcal{F}_{k_i}$ such that $G_{x,i} \cap A_i = \emptyset$. Hence the disjoint open sets $H = G_x \cap G_{x,1} \cap \ldots \cap G_{x,s}$ and $J = (A \cup A_1 \cup \ldots \cup A_s) \cup \{w_{k_1}, \ldots, w_{k_s}\}$ separate x and F^*. Next, let us suppose that $x = w_j$, for some j. If $F^* \subseteq X$, we have $X^* \setminus F^* = (X \setminus F^*) \cup \{w_1, \ldots, w_n\} \in \mathcal{T}^*$. But this implies that $X \setminus F^* \in \mathcal{T} \cap \mathcal{F}_1 \cap \ldots \cap \mathcal{F}_n$. In particular, $X \setminus F^* \in \mathcal{F}_j$. Therefore, there is a closed set $F_j \in \mathcal{F}_j$ (since $\mathcal{F}_j$ is regular), such that $F_j \subseteq X \setminus F^*$; whence $F^* \subseteq X \setminus F_j \in \mathcal{T} \subset \mathcal{T}^*$. Again, since $\mathcal{F}_j$ is regular, there exist $A_j \in \mathcal{T} \cap \mathcal{F}_j$ such that $A_j \subseteq F_j$. Now, it is easy to verify that $A_j \cup \{w_j\}$ and $X \setminus F_j$

are disjoint open sets in $(X^*, \mathcal{J}^*)$ separating w_j and F^*. Finally, if $F^* \not\subseteq X$, we may write $F^* = (F^* \cap X) \cup \{w_{k_1}, \ldots, w_{k_s}\}$, for some s, and $k_i \neq j$ for all $i \in \{1, \ldots, s\}$. But then $X^* \setminus F^* = (X \setminus F^*) \cup \{w_j, w_{\beta_1}, \ldots, w_{\beta_t}\} \in \mathcal{J}^*$, where $\{w_j, w_{\beta_1}, \ldots, w_{\beta_t}\} = \{w_1, \ldots, w_n\} \setminus \{w_{k_1}, \ldots, w_{k_s}\}$.

Therefore $X \setminus F^* \in \mathcal{J} \cap \mathcal{F}_j \cap \mathcal{F}_{\beta_1} \cap \ldots \cap \mathcal{F}_{\beta_t}$. In particular, $X \setminus F^* \in \mathcal{F}_j$. Arquing as in the last case, we may find a closed set F_j in $(X, \mathcal{J})$, belonging to $\mathcal{F}_j$, and an open set $A_j \in \mathcal{J} \cap \mathcal{F}_j$, such that $F_j \subseteq X \setminus F^*$, and $A_j \subseteq F_j$. Hence $F^* \cap X \subseteq X \setminus F_j \in \mathcal{J} \subseteq \mathcal{J}^*$, and $A_j \cap (X \setminus F_j) = \emptyset$. On the other hand, since $j \neq k_i$ for all $i \in \{1, \ldots, s\}$, we have $\mathcal{F}_j \vee \mathcal{F}_{k_i} = \mathcal{P}(X)$. This implies the existence, for every $i \in \{1, \ldots, s\}$, of open sets $G_{j,i} \in \mathcal{J} \cap \mathcal{F}_{k_i}$, and $G_i \in \mathcal{J} \cap \mathcal{F}_{k_i}$ such that $G_{j,i} \cap G_i = \emptyset$. Taking $H = (A_j \cap G_{j,1} \cap \ldots \cap G_{j,s}) \cup \{w_j\}$, and $J = (X \setminus F_j) \cup G_1 \cup \ldots \cup G_s \cup \{w_{k_1}, \ldots, w_{k_s}\}$, we obtain two disjoint open sets in $(X^*, \mathcal{J}^*)$ separating x and F^*. ▲

The foregoing theorem is also valid if we replace in it T_3 for T_4. This follows from the following

LEMMA 4.1. *If $(X^*, \mathcal{J}^*)$ is an n-point T_3-extension of a T_4-space $(X, \mathcal{J})$, then $(X^*, \mathcal{J}^*)$ is a T_4-space.*

Proof. Let us first consider the case of a 1-point extension. Given two disjoint closed sets P^* and Q^*, in $(X^*, \mathcal{J}^*)$, there are two possible cases. The first case occurs when both P^* and Q^* are contained in X, in which case the result follows from the fact that $(X, \mathcal{J})$ is a T_4-space, by hypothesis. In the second case, without loss of generalitym we may assume that w_1 is in P^*, but $Q^* \subseteq X$. Thus the sets $P^* \cap X$ and Q^* are disjoint closed sets in $(X, \mathcal{J})$, for which there are disjoint open sets $A, B \in \mathcal{J}$ satisfying $P^* \cap X \subseteq A$, and $Q^* \subseteq B$. However, by hypothesis, $(X^*, \mathcal{J}^*)$ is a T_3-space, and since w_1 belongs to P^*, but not to Q^*, it is possible to find open sets A^* and B^* in $(X^*, \mathcal{J}^*)$ such that $w_1 \in A^*$, $Q^* \subseteq B^*$, and $A^* \cap B^* = \emptyset$. It follows now that the disjoint open sets $A \cup A^*$ and $B \cap B^*$ separate, in $(X^*, \mathcal{J}^*)$, the closed sets P^* and Q^*. The general case will follow readily be recurrence on n. ▲

The proofs of the next theorems follows the pattern of those of Theorems 3.1 and 3.2.

THEOREM 4.2. *Let* $(X,\mathcal{T})$ *be a* T_i*-space* $(i = 3,4)$ *and let* $(X^*,\mathcal{T}^*)$ *be an n-point* T_1*-extension of* $(X,\mathcal{T})$. *Then* $(X^*,\mathcal{T}^*)$ *is a* T_i*-extension if, and only if, for each* $k \in \{1,\dots,n\}$, *the filter* $Tr(\mathcal{U}^*_{w_k})$ *on* X *is regular, without cluster points in* X, *and* $Tr(\mathcal{U}^*_{w_j}) \vee Tr(\mathcal{U}^*_{w_k}) = \mathcal{P}(X)$, *for each pair* j,k, $1 \leq j, k \leq n$, $j \neq k$. ▲

THEOREM 4.3. *Let* $(X,\mathcal{T})$ *be a* T_3*-space,* Ω *the family of all regular filters on* X, *without cluster points in* X. *Let* $\Xi_3^{(n)}$ *the family of all equivalence classes of n-point* T_3*-extensions of* $(X,\mathcal{T})$. *Then the mapping from* $\Xi_3^{(n)}$ *into* $\Omega^n = \Omega\times\dots\times\Omega$, *which sends the class of* $(X^*,\mathcal{T}^*)$ *into* $(Tr(\mathcal{U}^*_{w_1}),\dots,Tr(\mathcal{U}^*_{w_n}))$ *is a one-to-one mapping which is not onto if* $n > 1$. *If* $n = 1$, *this mapping is a bijection.* ▲

Since a T_3-space is T_3-closed if, and only if, every regular filter on the space has at least one cluster point [5, Satz 2, 285], we are able to state the following result

THEOREM 4.4. *A* T_3*-space is* T_3*-closed if, and only if, it does not admit 1-point* T_3*-extensions.* ▲

Next we give another sufficient condition, based on the notion of filters without clustering sets, in order to get 1-point T_3-extensiones of T_3-spaces:

THEOREM 4.5. *Let* $(X,\mathcal{T})$ *be a* T_3*-space, and let* $\mathcal{F}$ *be an open filter on* X, *without cluster points nor clustering sets. Then there is a 1-point* T_3*-extension of* $(X,\mathcal{T})$ *such that* $Tr(\mathcal{U}^*_w) = \mathcal{F}$. *This extension is unique up to equivalence.*

Proof. Since $(X^*,\mathcal{T}^*)$, where $\mathcal{T}^*$ is defined by (2) satisfies $Tr(\mathcal{U}^*_w) = \mathcal{F}$, it suffices to prove that under the conditions of the theorem, this extension is regular. Indeed, let $x \in X^*$, and let F^* be a closed set in $(X^*,\mathcal{T}^*)$ such that

$x \notin F^*$. If $F^* \subseteq X$, the result follows readily. If $F^* \not\subseteq X$, then $F^* = (F^* \cap X) \cup \{\omega\}$, and we can find open sets G_x and A in $(X,\mathcal{J})$ such that $x \in G_x$, $F^* \cap X \subseteq A$, and $A \cap G_x = \emptyset$. On the other hand, since x is not a cluster point of $\mathcal{F}$, there are G'_x, and B in $(X,\mathcal{J})$ satisfying $x \in G'_x$, $B \in \mathcal{F}$, and $G'_x \cap B = \emptyset$. Therefore, the open sets $G_x \cap G'_x$, and $A \cup B \cup \{\omega\}$ separate x and F^* in $(X^*,\mathcal{J}^*)$. Let us supose now that $x = \omega$, so that $F^* \subseteq X$ is closed in $(X,\mathcal{J})$. Under these circumstances, $F^* \notin \mathcal{F}$. Otherwise, we may find $G \in \mathcal{J} \cap \mathcal{F}$ such that $G \subseteq F^*$, and consequently $G \cap (X^* \setminus F^*) = \emptyset$, and $(G \cup \{\omega\}) \cap (X^* \setminus F^*) = \{\omega\}$; but this implies that $\{\omega\} \in \mathcal{J}^*$, which cannot be, since X is dense in $(X^*,\mathcal{J}^*)$. Now, by hypothesis, F^* is not a clustering set of $\mathcal{F}$, so there are open sets B and G in $(X,\mathcal{J})$ such that $G \in \mathcal{F}$, $F^* \subseteq B$, and $B \cap G = \emptyset$. But then B and $G \cup \{\omega\}$ separate x and F^* in $(X^*,\mathcal{J}^*)$. ▲

Because of Lemma 4.1, we have an analogous result for T_4-extensions of T_4-spaces. On the other hand, the following example shows that the converse of the above theorem does not hold in general.

EXAMPLE 4.2. Let $X = (0,1]$, $\mathcal{J}$ its usual topology, $X^* = [0,1]$, $\mathcal{J}^*$ its usual topology. Clearly, $(X^*,\mathcal{J}^*)$ is a 1-point T_3-extension of $(X,\mathcal{J})$, but the set $A = \{1/n;\ n = 1,2,\dots\}$ is a clustering set of the filter $Tr(\mathcal{U}^*_0)$.

As a curious consequence of the above, we obtain the following:

COROLLARY 4.5.1. *Let $(X,\mathcal{J})$ be a T_3-space. If $\mathcal{F}$ is an open filter on X, without cluster points nor clustering sets in X, then $\mathcal{F}$ is a regular filter without cluster point points in X.* ▲

Finally, Theorem 4.5 can be extended to n-point T_3-extensions of T_3-spaces. Indeed, if $\mathcal{F}_1,\dots,\mathcal{F}_n$ are distinct open filters on X, without cluster points nor clustering sets, then they are regular filters on X, because of Corollary 4.5.1. But also Lemma 1.1 tells us that $F_i \vee F_j = \mathcal{P}(X)$, since they have no clustering sets. Therefore we are in the conditions of Theorem 4.1, and the proposed generalization follows.

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