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LAGRANGE INTERPOLATION AND ENTIRE FUNCTIONS

by

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Abstract. For a function f defined almost everywhere on $\mathbb{R}$. Let $\{L_n(f)\}$ be the sequence of Lagrange interpolation polynomials that approximates f, where the nodes are taken to be the zeros of a certain sequence of orthogonal polynomials. In this paper, we will give a condition on the decay of the error term $|f-L_n(f)|$, which makes f the restriction on $\mathbb{R}$ of an entire function of order one and finite type. In the case of the Hermite polynomials an estimate on the type is given.

§1. Introduction. A nondecreasing bounded function α on $\mathbb{R}$ is called *m-distribution* if it takes infinitely many values and all integrals

$$\int_{\mathbb{R}} x^n d\alpha(x), \quad n = 0,1,2,\ldots$$

converge; α generates a Lebesgue-Stieltjes measure which we shall briefly call *the m-distribution* $d\alpha$.

For any m-distribution $d\alpha$ there exist a unique sequence of orthonormal polynomials $\{p_n(d\alpha;x)\}$ (see [4, Sect. I.1]) with the properties:

a) $p_n(d\alpha;x) = \gamma_n x^n + \ldots + \gamma_0$ is a polynomial of degree n and $\gamma_n > 0$.

b) $\int_{\mathbb{R}} p_n(d\alpha;x) p_m(d\alpha;x) d\alpha(x) = \delta_{nm}$, the Kronecker symbol.

The zeros x_{kn} $(k = 0,1,\ldots,n)$ of $p_n(d\alpha;x)$ are real, and simple.

We shall assume, as usual, that $x_{1n} > x_{2n} > \ldots > x_{nn}$. If, in addition, $d\alpha$ is an absolutely continuous m-distribution, then $d\alpha(x) = \alpha'(x)dx$ and $\alpha'(x)$ is a weight function. In this case $\alpha'(x)$ will be denoted by $w(x)$ and $p_n(d\alpha)$ by $p_n(w)$.

For a given function f the *Lagrange interpolation polynomial* $L_n(f,x)$ corresponding to the m-distribution $d\alpha$ is defined to be the unique algebraic polynomial of degree at most $n-1$ which coincide with f at the nodes x_{kn}. Thus

$$L_n(f,x) = \sum_{k=1}^{n} f(x_{kn})\ell_{kn}(x)$$

where $\ell_{kn}(x)$ are the fundamental polynomials of Lagrange interpolation defined by

$$\ell_{kn}(x) = \frac{p_n(d\alpha,x)}{p_n'(d\alpha,x_{kn})(x-x_{kn})}, \quad (k = 1,2,\ldots,n).$$

§2. Main result. A weight function $w_Q(x) = \exp(-Q(x))$, $x \in \mathbb{R}$ is said *to be in the class* V, if $Q(x)$ satisfies the following conditions:

(i) $Q:\mathbb{R} \to \mathbb{R}$ is an even function in $C^2(0,\infty)$.

(ii) Q'' is positive and nondecreasing on $(0,\infty)$.

(iii) $1 \leqslant C_1 \leqslant x\dfrac{Q''(x)}{Q'(x)} \leqslant C_2$, $\quad x \in (0,\infty)$.

For example, weights of the form $\exp(-C|x|^{\alpha})$ where $C \in (0,\infty)$ are in the class V if $\alpha \geqslant 2$.

For $w_Q \in V$, let q_n be the unique positive solution of the equation $xQ'(x) = n$, i.e.

$$q_nQ'(q_n) = n, \qquad n = 1,2,\ldots$$

and define $\lfloor q_n = q_nq_{n-1}\cdots q_1$.

For a Lebesgue measurable function g on $\mathbb{R}$, we have

$$\|g\|_p = \{\int_{\mathbb{R}} |g(x)|^p dx\}^{1/p}; \text{ if } 1 \leqslant p < \infty,$$

$$\|g\|_\infty = \operatorname*{ess\,sup}_{x\in\mathbb{R}} |g(x)|,$$

and we say that $g \in L^p(\mathbb{R})$ if and only if $\|g\|_p$ exists, $p \geqslant 1$.

Finally, for $w_Q \in V$, let $\{p_k(x)\}$ be the sequence of orthonormal polynomials generated by the weight function $w_Q^2(x)$, and for $w_Q f \in L^p(\mathbb{R}), p > 1$, we consider the Fourier orthonormal expansion

$$f(x) \sim \sum_{k=0}^{\infty} b_k p_k(x)$$

where

$$b_k = \int_{\mathbb{R}} f(t) p_k(t) w_Q^2(t)\, dt.$$

We can now formulate our main theorem as follows:

THEOREM 2.1. *Suppose that* $w_Q \in V$, $w_Q f \in L^2(\mathbb{R})$, *and* $\lim_{n\to\infty}\sup \|(f - L_n(f))w_Q\|_2^{1/n} < 1$. *Then the function* f *has an extension on the complex plane* $\mathbb{C}$ *to an entire function of order one and finite type* σ, *where*

$$\sigma = \lim_{R\to\infty}\sup \frac{\operatorname{Log} M(R)}{R} < \infty$$

and $M(R) = \max_{|z|=R} |f(z)|$. *Moreover, if* $Q(x) = \frac{x^2}{2}$ *(the Hermite weight function), then* $\sigma < \sqrt{2}$.

§3. **Preliminaries.** Before we prove our main result, we need the following lemmas.

LEMMA 3.1. *There exists a constant* C_3 *such that for every* $\pi_k \in P_k$ *we have*

$$\|w_Q \pi_k'\|_2 \leq C_3 \left(\frac{k}{q_k}\right) \|w_Q \pi_k\|_2 \tag{3.1}$$

where P_k *is the set of all polynomials with real coefficients and degree at most* n.

(For $Q(x) = \frac{x^2}{2}$, in which case the π_k's are the orthonormal Hermite polynomials, one can show by straightforward computation that C_3 can be chosen to be $\sqrt{2}$ (and no less)

Proof. (see [5]).

LEMMA 3.2. *There exist constans C_4 and C_5 such that for each polynomial $\pi_n \in P_n$*

$$C_4\left(\frac{1}{q_n}\right)^{1/p-1/r}\|w_Q\pi_n\|_p < \|w_Q\pi_n\|_r \leqslant C_5\,\frac{n}{q_n}^{1/p-1/r}\|w_Q\pi_n\|_p \tag{3.2}$$

where $1 \leqslant p < r \leqslant \infty$ and $C_4, C_5 > 0$ depend only on Q, p and r.

Proof. (see [6]).

LEMMA 3.3. *Suppose that $w_Q \in V$, $w_Q f \in L^2(\mathbb{R})$, and $\limsup_{n\to\infty}\|(f-L_n(f))w_Q\|_2^{1/n} < 1$. Then there exist constants C_6 and $\beta \in (0,1)$ such that for sufficiently large n we have*

$$\sum_{k=n}^{\infty} |b_k|\,\|w_Q p_k^{(n)}\|_\infty < C_6 C_3^n \beta^n$$

where C_3 is as in Lemma 3.1.

Proof.
$$\begin{aligned} b_k &= \int_{\mathbb{R}} f(x)\,p_k(x)\,w_Q^2(x)\,dx \\ &= \int_{\mathbb{R}} (f(x) - L_n(f,x) + L_n(f,x))\,p_k(x)w_Q^2(x)\,dx \\ &= \int_{\mathbb{R}} (f(x) - L_n(f,x))\,p_k(x)w_Q^2(x)\,dx + \int_{\mathbb{R}} L_n(f,x)\,p_k(x)w_Q^2(x)\,dx, \end{aligned}$$

but, by orthogonality,

$$\int_{\mathbb{R}} L_n(f,x)\,p_k(x)\,w_Q^2(x)\,dx = 0 \quad \text{for } k \geqslant n;$$

hence

$$b_k = \int_{\mathbb{R}} (f(x) - L_n(f,x))\,p_k(x)\,w_Q^2(x)\,dx.$$

By applying the Cauchy-Schwartz inequality, we get

$$\begin{aligned} |b_k| &= \left|\int_{\mathbb{R}} (f(x) - L_n(f,x))\,w_Q(x)\,p_k(x)\,w_Q(x)\,dx\right| \\ &\leqslant \left\{\int_{\mathbb{R}} (f(x) - L_n(f,x))^2 w_Q^2(x)\,dx\right\}^{1/2} \left\{\int_{\mathbb{R}} p_k^2(x)\,w_Q^2(x)\,dx\right\}^{1/2}, \end{aligned}$$

but, by orthogonality,

$$\left\{\int_{\mathbb{R}} p_k^2(x)\,w_Q^2(x)\,dx\right\}^{1/2} = 1$$

hence

$$|b_k| \leq \{\int_{\mathbb{R}} (f(x) - L_n(f,x))^2 w_Q^2(x)\,dx\}^{1/2}$$

$$= \|(f - L_n(f))w_Q\|_2 .$$

Therefore

$$|b_k|^2 \leq \|(f - L_n(f))w_Q\|_2^2 . \tag{3.3}$$

From our assumption

$$\operatorname*{Lim\,sup}_{n\to\infty} \|(f - L_n(f))w_Q\|_2^{1/n} < 1$$

it follows that there exists a constant $\beta \in (0,1)$ such that for sufficiently large n ($n > N$, say) we have

$$\|(f - L_n(f))w_Q\|_2^{1/n} < \beta$$

or

$$\|(f - L_n(f))w_Q\|_2^2 < \beta^{2n}. \tag{3.4}$$

By combining (3.3) and (3.4), we conclude that

$$|b_k| < \beta^n \text{ for all } k \geq n \geq N. \tag{3.5}$$

Now taking $p = 2$ and $r = \infty$ in (3.2) with π_n replaced by $p_k^{(n)} \in P_{n-k}$ we obtain

$$\sum_{k=n}^{\infty} |b_k| \|w_Q p_k^{(n)}\|_\infty < c_5 \sum_{k=n}^{\infty} |b_k| \left(\frac{k-n}{q_{k-n}}\right)^{1/2} \|w_Q p_k^{(n)}\|_2 . \tag{3.6}$$

By repeated application of (3.1), we obtain

$$\|w_Q p_k^{(n)}\|_2 \leq c_3 \left(\frac{k-n+1}{q_{k-n+1}}\right) \|w_Q p_k^{(n-1)}\|_2$$

$$\leq c_3^2 \left(\frac{k-n+1}{q_{k-n+1}}\right)\left(\frac{k-n+2}{q_{k-n+2}}\right) \|w_Q p_k^{(n-2)}\|_2$$

$$\vdots$$

$$\leq c_3^n \left(\frac{k-n+1}{q_{k-n+1}}\right)\left(\frac{k-n+2}{q_{k-n+2}}\right) \cdots \left(\frac{k}{q_k}\right) \frac{(k-1)!}{\lfloor q_{k-1}} \frac{\lfloor q_{k-1}}{(k-1)!} \|w_Q p_k^{(0)}\|_2$$

$$\leq c_3^n \frac{(k-n+1)!}{\lfloor q_{k-n+1}} \frac{\lfloor q_{k-1}}{(k-1)!} ,$$

hence

$$\|w_Q p_k^{(n)}\|_2 \leq c_3^n \frac{(k-n+1)!}{\lfloor q_{k-n+1}} \frac{\lfloor q_{k-1}}{(k-1)!} . \tag{3.7}$$

From (3.6), (3.7) and (3.5) we conclude that

$$\sum_{k=n}^{\infty} |b_k| \|w_Q p_k^{(n)}\|_\infty \leq c_5 \sum_{k=n}^{\infty} |b_k| \left(\frac{k-n}{q_{k-n}}\right)^{1/2} c_3^n \frac{(k-n+1)!}{\lfloor q_{k-n+1}} \frac{\lfloor q_{k-1}}{(k-1)!}$$

$$\leq c_5 \sum_{k=n}^{\infty} \beta^k \left(\frac{k-n}{q_{k-n}}\right)^{1/2} \frac{(k-n+1)!}{\lfloor q_{k-n+1}} \frac{\lfloor q_{k-1}}{(k-1)!} c_3^n .$$

Letting $k' = k-n$, we get

$$\sum_{k=n}^{\infty} |b_k| \|w_Q p_k^{(n)}\|_\infty \leq c_5 c_3^n \sum_{k'=0}^{\infty} \beta^{k'+n} \left(\frac{k'}{q_{k'}}\right)^{1/2} \cdot \frac{(k'+1)!}{\lfloor q_{k'+1}} \frac{\lfloor q_{k'+n-1}}{(k'+n-1)!}$$

$$\leq c_5 c_3^n \beta^n \sum_{k=0}^{\infty} \beta^k \left(\frac{k}{q_k}\right)^{1/2} \cdot \frac{(k+1)!}{\lfloor q_{k+1}} \frac{\lfloor q_{k+n-1}}{(k+n-1)!} .$$

The proof of the Lemma will be complete if we show that

$$\sum_{k=0}^{\infty} \beta^k \left(\frac{k}{q_k}\right)^{1/2} \frac{(k+1)!}{\lfloor q_{k+1}} \frac{\lfloor q_{k+n-1}}{(k+n-1)!}$$

is a convergent series. But this can be easily seen by the ratio test, since

$$\lim_{k\to\infty} \left| \frac{\beta^{k+1}}{\beta^k} \left(\frac{k+1}{k}\right)^{1/2} \left(\frac{q_k}{q_{k+1}}\right)^{1/2} \frac{(k+2)!}{(k+1)!} \frac{\lfloor q_{k+1}}{\lfloor q_{k+2}} \frac{\lfloor q_{k+n}}{\lfloor q_{k+n-1}} \frac{(k+n-1)!}{(k+n)!} \right| = \beta < 1.$$

Therefore

$$\sum_{k=n}^{\infty} |b_k| \|w_Q p_k^{(n)}\|_\infty \leq c_6 c_3^n \beta^n .$$

which complets the proof.

We are ready now to prove our main result.

§ 4. Proof of Theorem 2.1. Let S be a compact subset of $\mathbb{C}$, and take $z \in S$. Then there exists a constant k such that

$$\sum_{n=0}^{\infty} \frac{1}{n!} \sum_{k=n}^{\infty} |b_k||p_k^{(n)}(0)||z|^n \leq \sum_{n=0}^{\infty} \frac{k^n}{n!} \sum_{k=n}^{\infty} |b_k||p_k^{(n)}(0)|.$$

$$= w_Q^{-1}(0) \sum_{n=0}^{\infty} \frac{k^n}{n!} \sum_{k=n}^{\infty} |b_k||p_k^{(n)}(0) w_Q(0)|.$$

$$\leq w_Q^{-1}(0) \sum_{n=0}^{\infty} \frac{k^n}{n!} \sum_{k=n}^{\infty} |b_k| \| p_k^{(n)} w_Q \|_\infty$$

$$\leq w_Q^{-1}(0) C_6 \sum_{n=0}^{\infty} \frac{k^n}{n!} C_3^n \beta^n, \text{ (Lemma 3.3)}$$

$$\leq C_7 \sum_{n=0}^{\infty} \frac{C_8^n}{n!} < \infty,$$

where C_7 and C_8 are positive constants. Therefore the series

$$\sum_{n=0}^{\infty} \frac{1}{n!} |b_k||p_k^{(n)}(0)||z|^n$$

converges uniformly in z on compact subsets of the complex plane $\mathbb{C}$. Hence we can interchange the order of summation to get

$$\sum_{n=0}^{\infty} \frac{1}{n!} \sum_{k=n}^{\infty} b_k p_k^{(n)}(0) z^n = \sum_{k=0}^{\infty} b_k \sum_{n=0}^{\infty} \frac{1}{n!} p_k^{(n)}(0) z^n = \sum_{k=0}^{\infty} b_k p_k(z).$$

Thus the series

$$\sum_{n=0}^{\infty} \frac{1}{n!} \sum_{k=n}^{\infty} b_k p_k^{(n)}(0) z^n$$

converges uniformly on compact subsets of $\mathbb{C}$ to an entire function, say $g(z)$.

It follows that the restriction of g to the real line is almost everywhere equal to f. Further, for g we have the power series.

$$g(z) = \sum_{n=0}^{\infty} a_n z^n = \sum_{n=0}^{\infty} \frac{1}{n!} \Big[\sum_{k=n}^{\infty} b_k p_k^{(n)}(0) \Big] z^n.$$

Hence

$$a_n = \frac{1}{n!}\sum_{k=n}^{\infty} b_k p_k^{(n)}(0),$$

and by using Lemma (3.3) we get

$$n!|a_n| \leqslant w_Q^{-1}(0)\, C_6 C_3^n \beta^n .$$

But, $g^{(n)}(0) = n!a_n$, therefore

$$\operatorname*{Lim}_{n\to\infty} \sup |g^{(n)}(0)|^{1/n} = \operatorname*{Lim}_{n\to\infty} \sup\{n!|a_n|\}^{1/n}$$

$$\leqslant C_3\beta < C_3 < \infty,$$

which shows (see [3], formula 2.2.12) that g is of order one and type σ where

$$\sigma = \operatorname*{Lim}_{n\to\infty} \sup\{n!|a_n|\}^{1/n} < C_3 < \infty.$$

Finally, from Lemma 3.1, we have $C_3 = \sqrt{2}$ when $Q(x) = \frac{x^2}{2}$, and this completes the proof of the theorem.

Corollary. *Suppose that* $w_Q \in V$, $w_Q f \in L^2(\mathbb{R})$, *and* $\operatorname*{Lim}_{n\to\infty} \sup |f(x) - L_n(f,x)|^{1/n} < 1$, *uniformly on* $\mathbb{R}$. *Then the function* f *has an extension on* $\mathbb{C}$ *to an entire function of order one and finite type* σ, *where* σ *is as in Theorem 2.1.*

Proof. Since $\operatorname*{Lim}_{n\to\infty} \sup |f(x) - L_n(f,x)|^{1/n} < 1$, then there exists a constant $\beta \in (0,1)$ such that for all n sufficiently large ($n \geqslant N$, say) we have

$$|f(x) - L_n(f,x)|^2 < \beta^{2n} \text{ for all } x \in \mathbb{R}. \tag{3.8}$$

Since

$$\|(f - L_n(f))w_Q\|_2^2 = \int_{\mathbb{R}} (f(x) - L_n(f,x))^2 w_Q^2(x)\,dx$$

then it follows, by using (3.8), that

$$\operatorname*{Lim}_{n\to\infty} \sup \|(f - L_n(f))w_Q\|_2^{1/n} \leqslant \operatorname*{Lim}_{n\to\infty} \sup\{\beta^n (\int_{\mathbb{R}} w_Q^2(x)\,dx)^{1/2}\}^{1/n} < \beta < 1.$$

The corollay follows now from Theorem 2.1.

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