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TOTAL COMPACTNESS OF A-INDUCTIVE PARTIALLY ORDERED SETS

by

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A poset (i.e., partially ordered set) P is called *A-inductive* iff every nonempty well ordered subset W of P has a supremum (i.e., least upper bound) which need not be an element of W.

In this paper we prove the following Total Compactness Theorem of A-inductive posets:

THEOREM 1. *Let P be an A-inductive poset and S be a subset of P such that the supremum of every nonempty finite subset F of S exists in P (sup F need not be an element of S.) Then the supremum of every nonempty subset of S exists in P.*

Clearly, letting $S = P$ in Theorem 1, we immediately obtain the following Compactness Theorem of A-inductive posets:

THEOREM 2. *Let P be a nonempty A-inductive poset such that the supremum of every nonempty finite subset of P exists (in P). Then the supremum of P exists of every nonempty subset of P exists (in P)).*

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The proof of Theorem 1 is based on the following Lemmas.

We recall that a subset $\mathcal{D}$ of a poset P is called *directed* iff every finite subset of $\mathcal{D}$ has an upper bound which is an element of $\mathcal{D}$. Obviously, $\mathcal{D}$ is never empty

LEMMA 1. *Let* H_r *be an infinite subset of a directed set* $(\mathcal{D}, \leqslant)$ *such that*

$$\bar{\bar{H}}_r \leqslant \bar{\bar{\mathcal{D}}} \tag{1}$$

Then there exists a subset E_r *of* $\mathcal{D}$ *such that*

$$H_r \subseteq E_r \quad \text{and} \quad \bar{\bar{H}}_r = \bar{\bar{E}}_r \quad \text{and} \quad (E_r, \leqslant) \ \text{ is directed} \tag{2}$$

Proof. If $(H_r, \leqslant)$ is a directed set then we choose $E_r = H_r$. Otherwise, let H_0 be an extension (superset) of H_r which is obtained by supplying at most one upper bound for each finite subset of H_r that does not have an upper bound in H_r. Since the set of all finite subsets of an infinite set is obviously of the cardinality of the set, we see that $\bar{\bar{H}}_r = \bar{\bar{H}}_0$. Now, if $(H_0, \leqslant)$ is a directed set then we choose $E_r = H_0$. Otherwise, we let H_1 be an extension of H_0 the way H_0 is an extension of H_r. Clearly, again $\bar{\bar{H}}_r = \bar{\bar{H}}_0 = \bar{\bar{H}}_1$ and again if $(H_1, \leqslant)$ is a directed set then we choose $E_r = H_1$. Otherwise we continue the process obtaining successive extensions H_n for every positive integer n. Oviously, $\bar{\bar{H}}_r = \bar{\bar{H}}_n$ for every n. Moreover, it can be readily verified that $(E_r, \leqslant)$ is a directed set where E_r is given by

$$E_r = H_r \cup H_0 \cup H_1 \cup H_2 \cup \ldots \cup H_n \cup \ldots \tag{3}$$

From (3) it follows that E_r is a countable union of infinite sets each of cardianlity $\bar{\bar{H}}_r$. Consequently, $\bar{\bar{H}}_r = \bar{\bar{E}}_r$ and since $H_r \subseteq E_r$ by (3), we see that E_r satisfies (2), as desired. ▲

In connection with Lemma 1, we call E_r "*a directed closure*" of H_r. Thus,

E_r *is a directed closure of* H_r *where* $\omega \leq \bar{\bar{H}}_r = \bar{\bar{E}}_r$ (4)

LEMMA 2. *Let* D *be a directed subset of an A-inductive poset* P*. Then* D *has a supremum in* P *(which need not be an element of* D*).*

Proof. If D is finite then the upper bound of D is the supremum of D. Next, let D be countably infinite, i.e., $D = (d_i)_{i<\omega}$. Let $d_{s(0)} = d_0$ and for every $n \in \omega$ let $d_{s(n+1)}$ be the upper bound of $\{d_{s(n)}, d_{n+1}\}$ of the smallest index $s(n+1)$. Clearly, $V = (d_{s(n)})_{n\in\omega}$ is a nonempty well ordered subset and clearly, sup V = sup D. Finally, let us assume to the contrary and let

$$D = (d_i)_{i<\omega_u} \quad \textit{with} \quad \bar{\bar{D}} = \omega_u \tag{5}$$

be a directed subset of P of the smallest uncountable cardinality ω_u such that sup D does not exist. For every ordinal r with $\omega \leq r < \omega_u$, we define by induction a directed subset E_r of $D = (d_i)_{i<\omega_u}$ as follows:

E_ω *is a directed closure of* $H_\omega = \{d_i | i < \omega\}$ (6)

E_{r+1} *is a directed closure of* $E_r \cup \{d_t\}$ *where* d_t (7)

is the element of the smallest index t *of* $D - E_r$

$$E_r = \bigcup_{k<r} E_k \quad \textit{if} \quad r < \omega_u \quad \textit{is a limit ordinal.} \tag{8}$$

From (4), (5), (6), (7), (8) it follows that

$(E_r)_{\omega\leq r<\omega_u}$ *is a well ordered by* $\subseteq$ *sequence of*

directed subsets E_r *of* D (9)

and

$$\bigcup_{\omega<r<\omega_u} E_r = D \tag{10}$$

Moreover. from (4), (6), (7), (8) for every infinite ordinal $r < \omega_u$ we see that $\bar{\bar{E}}_r = \bar{\bar{r}} < \omega_u$. Thus, in view of our assumption, sup E_r exists (in P) for every infinite ordinal $r < \omega_u$. But then by (9), barring repetitions, we see that $(\sup E_r)_{\omega \leq r < \omega_u}$ is a well ordered subset of the A-inductive poset P. Hence, $h = \sup(\sup E_r)_{\omega \leq r < \omega_u}$ exists (in P). But then from (10) it readily follows that $h = \sup \mathcal{D}$ which contradicts our assumption. Thus, Lemma 2 is proved. ▲

Finally we give:

Proof of Theorem 1. Let H be a nonempty subset of S. Clearly, H and the sups of all the nonempty finite subsets of H form a directed subset of P. But then the conclusion of Theorem 1 follows from the conclusion of Lemma 2.

REMARK. If the partial order P in Theorem 1 is a partial order with respect to the set-theoretical inclusion $\subseteq$, where the supreum of a subset S of P is the sumset $\bigcup S$ of S, then the result of Lemma 2 and hence, Theorem 1 can be proved as suggested in Exercise 3.1.10 of [1,p.121]. However, the relevance of the latter dependes on the additional fact that every partially ordered set is isomophic to a partially ordered set with respect to $\subseteq$ where supremums correspond to unions (we made no use of this fact in our proofs).

REFERENCE

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