Revista Colombiana de Matemáticas
Vol. XXIV (1990), págs. 153-177

ON SYSTEMS OF ORTHOGONAL POLYNOMIALS WITH INNER AND END POINT MASSES

by

Jairo A. CHARRIS and Guillermo RODRIGUEZ-BLANCO

Abstract. A system of orthogonal polynomials related to the sieved ultraspherical polynomials is presented. It is shown that their orthogonality measure carries end point masses. It is also proved that the orthogonality measure of their first associated (numerator) polynomials has an embedded masspoint.

Sumario. Se describe un sistema de polinomios ortogonales, relacionado con el de los ultraesféricos cribados, cuya medida de ortogonalidad tiene masas en los extremos del intervalo de ortogonalidad. Se demuestra también que la medida de ortogonalidad de los polinomios numeradores correspondientes presenta una masa sumergida.

§1. Introduction. We study in this paper two systems of orthogonal polynomials defined by recurrence relations of the form

---

* 1980 *Mathematics Subject Classification* (1985 Revision) Primary 33A65. Secondary 33A50-33A70.
Key words and Phrases. Orthogonal polynomials, recurrence relations, continued fractions, orthogonality measures, mass points.

The first author acknowledges partial support of the N.S.F. under grant Int. 8803099.

$$xP_{2n}(x) = P_{2n+1}(x) + a_{0,n}P_{2n-1}(x) \quad (1.1)$$

$$xP_{2n+1}(x) = P_{2n+2}(x) + a_{1,n}P_{2n}(x), \quad n \geq 0,$$

and initial conditions

$$P_{-1}(x) = 0, \quad P_0(x) = 1. \quad (1.2)$$

We take

$$a_{0,0} = a_{1,0} = \alpha/2 \quad (1.3)$$

$$a_{0,n} = n/(4(n+\tau)), a_{1,n} = (n+2\tau)/(4(n+\tau)), \quad n \geq 1$$

and

$$a_{0,0} = \alpha/2, a_{1,0} = 1/(4(1+\tau)) \quad (1.4)$$

$$a_{0,n} = (n+2\tau)/(4(n+\tau)), a_{1,n} = (n+1)/(4(n+\tau+1)), \quad n \geq 1.$$

In both cases $\alpha,\tau > 0$. We observe that the polynomials determined by (1.4) are the first associated of those defined by (1.3). Hence, we will denote with $\{P_n(x)\}$ the polynomials given by (1.3) and with $\{P_n^{(1)}(x)\}$ those by (1.4).

Recurrence relations of the form (1.1) have frequently appeared in the literature of the last few years (Chihara [9], Charris and Gómez [6]) and are related to spectral problems in quantum chemistry (Slim [23], Wheeler [27]). They are also special cases of sieved orthogonal polynomials (Al-Salam, Allaway and Askey [2], Charris and Ismail [7], [8], Ismail [12], [13], [14]). As a matter of fact, the system $\{P_n(x)\}$ described above reduces to sieved ultraspherical polynomials of the first kind when $\alpha = 1$ (see [2]), and $\{P_n^{(1)}(x)\}$ is then the system of their first associated (observe that $\{P_n^{(1)}(x)\}$ is independent of α). In [2] it is shown that, when $\alpha = 1$, $\{P_n(x)\}$ is a system of orthogonal polynomials with respect to the absolutely continuous measure

$$d\mu(x) = |x|^{2\tau}(1-x^2)^{\tau-\frac{1}{2}}\Phi(x)\,dx, \quad (1.5)$$

where $\Phi(x)$ is the characteristic function of the interval $(-1,1)$, but the case $\alpha \neq 1$ was not considered. Neither was considered the system $\{P_n^{(1)}(x)\}$ of the first associated polynomials (the sieved ultraspherical polynomials of the second kind of Al-Salam, Allaway and Askey [2] are first associated polynomials of a system of orthogonal polynomials but not of the sieved ultraspherical of the first kind). The point of view adopted in [2] did not allow for the treatment of these cases. The purpose of the present paper is to establish some properties of $\{P_n(x)\}$ and $\{P_n^{(1)}(x)\}$ that seem to have gone unnoticed in the literature, perhaps because of the above reason.

First, $\{P_n(x)\}$ *has, for special values of* α *and* τ, *masses at the end points of the interval of orthogonality.* This phenomenum has been explored by Chihara [10], Koornwinder [18] and Koekoek [15], [16], [17], but their examples of this situation are rather complicated and artificial. Second, and even more remarkable, is the fact that $\{P_n^{(1)}(x)\}$ *may have masses which are embedded in the continuous spectrum,* i.e., which are interior to the support of the orthogonality measure. Examples of this situation were, as far as we know, absent from the literature prior to Geronimo and Van Assche [11]. However, the examples in [11] are more sophisticated and not so explicit.

In Section 2 we review the basic notions about orthogonal polynomials and in Section 3 we give a brief account of ultraspherical polynomials and their associated. Section 4 contains some basic formulae, and Section 5 the main results. Finally, in Section 6 we study the connection with spectral theory and examine futher generalizations.

This paper grew out of talks given by the first author at the *Classical Analysis and Aproximation Theory Seminar* held at The University of South Florida during the spring semester of 1989 and at the *Fourth Annual Southeast Aproximators Conference*, Auburn University, Alabama, in April of the same year; and by the second author at the *Second Colom-*

bian Symposium in functional Analysis, Medellín, July 1989. It originated in the research of this author for his Masters thesis at the National University of Colombia in Bogotá.

§2. Background. A system $\{P_n(x)/n \geq 0\}$ of polynomials with real coefficients is called a *system of orthogonal poynomials* (O.P.S.) if

1) $P_n(x)$ has degree n and $P_o(x) = 1$.

2) There is a positive measure μ supported by the real line such that

$$\int_{-\infty}^{+\infty} P_n(x)P_m(x)d\mu = k_n\delta_{nm}; m,n \geq 0, \quad k_n > 0. \tag{2.1}$$

It can be shown (See [6],[9]) that a system $\{P_n(x)\}$ is an O.P.S if and only if there are real numbers A_n, B_n, C_n, $n \geq 0$, such that

$$A_nA_{n+1}C_n > 0, \qquad n \geq 0, \tag{2.2}$$

$$P_{n+1}(x) = (A_nx+B_n)P_n(x) - C_nP_{n-1}(x), \qquad n \geq 0, \tag{2.3}$$

and

$$P_{-1}(x) = 0, \qquad P_o(x) = 1. \tag{2.4}$$

If that is the case μ can be so chosen that

$$k_o = 1; k_n = (A_o/A_n)C_1\ldots C_n, \qquad n \geq 1. \tag{2.5}$$

The numbers A_n, B_n, C_n are uniquely determined by $\{P_n(x)\}$, except for C_o, which is arbitrary (Favard's Theorem).

The measure μ satisfying (2.1) has bounded support, Suppμ, if and only if there is a constant $M > 0$ such that

$$|B_n/A_n| \leq M/3, \sqrt{C_{n+1}/A_nA_{n+1}} \leq M/3, \qquad n \geq 0, \tag{2.6}$$

in which case Supp$\mu \subseteq [-M,M]$ and μ is the only measure for which (2.1) holds.

Let A_n, B_n, C_n be as in (2.3). The system $\{P_n^{(i)}(x)\}, i = 0,1,2,3,\ldots$, defined by

$$P_{n+1}^{(i)}(x) = (A_{n+i}x + B_{n+i})P_n^{(i)}(x) - C_{n+i}P_{n-1}^{(i)}(x) \quad (2.7)$$

for $n \geq 0$ and

$$P_{-1}^{(i)}(x) = 0, \quad P_0^{(i)}(x) = 1 \quad (2.8)$$

is also orthogonal with respect to some positive measure $\mu_{(i)}$. It is called *the system of i^{th}-associated polynomials of* $\{P_n(x)\}$. Clearly $P_n^{(0)}(x) = P_n(x)$. In general $\mu_{(i)} \neq \mu_{(j)}$ if $i \neq j$.

Let $\{Q_n(x)\}$ be a system of polynomials with real coefficients such that (2.3) holds for $\{Q_n(x)\}$ with $n \geq 2$. Then

$$Q_n(x) = (A - C/C_1)P_n(x) + (B + C/C_1P_1(x))P_{n-1}^{(1)}(x), n \geq 1, \quad (2.9)$$

where

$$A = Q_0(x); B = Q_1(x) - Q_0(x)P_1(x), \quad (2.10)$$
$$C = Q_2(x) - P_1^{(1)}(x)Q_1(x) + C_1Q_0(x).$$

If (2.3) holds for $\{Q_n(x)\}$ with $n \geq 1$, then $C = 0$; if it hols for $n \geq 0$, then $B = C = 0$; if in the latter case $Q_0(x) = 1$, then $Q_n(x) = P_n(x)$ for all n.

Assume $\{P_n(x)\}$ is an O.P.S. and let μ be an orthogonality measure for $\{P_n(x)\}$. Let

$$p_n(x) = P_n(x)/\sqrt{k_n}, \quad n \geq 0; \quad (2.11)$$

then

$$\int_{-\infty}^{+\infty} p_n(x)p_m(x)d\mu = \delta_{mn}; m, n \geq 0. \quad (2.12)$$

The system $\{p_n(x)\}$ is called the *orthonormal polynomial system* (O.N.P.S.) of $\{P_n(x)\}$. If $\{P_n(x)\}$ satisfies (2.3) and (2.4) and k_n is given by (2.5), then $\{p_n(x)\}$ satisfies the recurrence relation

$$xp_n(x) = b_{n+1}p_{n+1}(x) + a_np_n(x) + b_np_{n-1}(x), \quad n \geq 1, \tag{2.13}$$

and the initial conditions

$$p_0(x) = 1, p_1(x) = (1/b_1)(x-a_0), \tag{2.14}$$

where

$$a_n = -B_n/A_n, b_{n+1} = \sqrt{C_{n+1}/A_nA_{n+1}}, \quad n \geq 0. \tag{2.15}$$

Condition (2.6) translates into

$$|a_n| \leq M/3, 0 < b_{n+1} \leq M/3, \quad n \geq 0, \tag{2.16}$$

in which case Supp$\mu \subseteq [-M,M]$.

In what follows we will assume (2.6) (or, (2.16)) to hold. With μ satisfying (2.1) we then have

$$\lim_{n\to\infty} A_0 \frac{p_{n-1}^{(1)}(x)}{p_n(x)} = \int_{-\infty}^{+\infty} \frac{d\mu(t)}{x-t}, \quad x \in \mathbb{C}-[-M,M]. \tag{2.17}$$

Relation (2.17) is known as *Markov's Theorem*. The left hand side of (2.17) is known as the *continued fraction of* $\{P_n(x)\}$. It is, actually, the limit of the continued fraction

$$\frac{A_0}{\lfloor A_0x+B_0} - \frac{C_1}{\lfloor A_1x+B_1} - \frac{C_2}{\lfloor A_2x+B_2} - \ldots \tag{2.18}$$

The right hand side of (2.17) is $X(x) = -2\pi i\hat{\mu}(x)$, where $\hat{\mu}(x)$, called the *Cauchy-Stieltjes transform of* μ, is an analytic function of x for $x \notin$ Suppμ. The *Stieltjes inversion formula*

$$\int_{-\infty}^{+\infty} f(x)d\mu(x) = \lim_{\varepsilon\to 0+} \frac{1}{2\pi i}\int_{-\infty}^{+\infty} \{X(x-i\varepsilon)-X(x+i\varepsilon)\}f(x)dx, \quad x \in \mathbb{R}, \tag{2.19}$$

which holds for any continuous function f on $\mathbb{R}$, then allows to recover μ from $X(x)$. Let

$$\sigma(\tau) = \int_{-\infty}^{\tau} d\mu, \quad \tau \in \mathbb{R}. \tag{2.20}$$

The right continuous, non-decreasing function σ is called *the distribution function of* μ. From (2.19), we get

$$\sigma(\tau) = \lim_{\varepsilon\to 0+} \frac{1}{2\pi i} \int_{-\infty}^{\tau} \{X(x-i\varepsilon) - X(x+i\varepsilon)\}dx, \quad \tau \in \mathbb{R}. \tag{2.21}$$

If $\mathrm{Supp}\mu \subseteq (a,b)$, $-\infty < a < b < +\infty$, then

$$\int_a^b f d\mu = \int_a^b f(x)\, d\sigma(x) \tag{2.22}$$

where the right hand side is an ordinary Riemann-Stieltjes integral. Futhermore, $\mathrm{Supp}\mu$ coincides with the set S_μ of points of increase of σ, i.e., with the set of points $x \in \mathbb{R}$ such that $\sigma(x_1) < \sigma(x_2)$ if $x_1 < x < x_2$. We have $S_\mu = C_\mu \cup P_\mu$, where C_μ is the set of points x in S_μ such that $\sigma(x) = \sigma(x-0)$, and P_μ, of those with $\sigma(x) \neq \sigma(x-0)$ (this is a countable set). Clearly $C_\mu \cap P_\mu = \phi$. The set C_μ is called the *continuous support of* μ; and P_μ, the *point support*. We observe that they are not, in general, closed sets.

The following characterization of points in P_μ is useful: $x \in P_\mu$ *if and only if*

$$\sum_{n=0}^{\infty} \frac{P_n^2(x)}{k_n} < +\infty. \tag{2.23}$$

In such case,

$$\mu(\{x\}) = \sigma(x)-\sigma(x-0) = 1/\Big(\sum_{n=0}^{\infty} \frac{P_n^2(x)}{k_n}\Big). \tag{2.24}$$

Let D_μ be the set of isolated points in $\mathrm{Supp}\mu$. Clearly $D_\mu \subseteq P_\mu$. *The points of* S_μ *in* D_μ *are the isolated poles of* $X(x)$, *which are all simple, and* $x \in D_\mu$ *if and only if*

$$\mu(\{x\}) = \sigma(x)-\sigma(x-0) = \mathrm{Res}(X,x) \neq 0. \tag{2.25}$$

Points in P_μ are called *mass points* of μ; those in D_μ, *isolated mass points*. The points in $P_\mu \cap$ interior (S_μ) are the *embedded mass points*. If S_μ happens to be a union of intervals, the end points of the infinite intervals in this class that are in P_μ, but not in the interior of S_μ, are the *end-points masses* of μ.

The measure μ can be written

$$\mu = \mu_c + \mu_s + \mu_j \tag{2.26}$$

where μ_c is *abolutely continuous*, μ_s *is singular continuous* and μ_j is a *jump measure*. The measure μ_c is concentrated on C_μ and μ_s on a subset of Lebesgue measure 0 of this set; μ_j is concentrated on P_μ. The measure μ_c can be written $g(x)dx$, where $g(x)$ is an integrable function (locally integrable if Suppμ is not assumed to be bounded).

REMARK 2.1. Let $X_\varepsilon(x) = X(x-i\varepsilon) - X(x+i\varepsilon)$ and assume that

$$\lim_{\varepsilon\to 0} \frac{1}{2\pi i} X_\varepsilon(x) \tag{2.27}$$

exists a.e. in an open interval (a,b). Further assume that for $a < c < d < b$ there are $C = C(c,d) > 0$ and $\varepsilon_0 = \varepsilon(c,d) > 0$ such that $|X_\varepsilon(x)| \leqslant C$ for almost all $x \in [c,d]$ and all $\varepsilon \leqslant \varepsilon_0$. Then $g(x) = \lim_{\varepsilon\to 0}(1/2\pi i)\cdot X_\varepsilon(x)$, a.e. on (a,b), and μ is free of masses in (a,b). If $g(x) \neq 0$ a.e. in (a,b) then $(a,b) \subseteq C_\mu$.

Proofs of Markov's theorem can be found in [1], [26]. See also [6]. The relevant facts about continued fractions and its relation to O.P.S. are in [3], [9], [26]. For the Cauchy-Sieltjes transform and the Stieltjes inversion formula see [1], [5], [6], [19]. A proof of relation (2.24) is in [1], [24], and one of (2.25), in [1], [6].

§3. The Associated Ultraspherical Polynomials.

The notation

$$(\alpha)_n = \begin{cases} 1, & n = 0 \\ \alpha(\alpha+1)\ldots(\alpha+n-1), & n \geqslant 1, \end{cases} \tag{3.1}$$

will be used in the sequel. Clearly

$$(\alpha)_n = \frac{\Gamma(\alpha+n)}{\Gamma(\alpha)}, \quad \alpha \neq 0, \tag{3.2}$$

where $\Gamma(x)$ denotes the Gamma function. The asymptotic formula

$$\frac{\Gamma(a+n)}{\Gamma(b+n)} \approx n^{a-b}, \quad n \to \infty \tag{3.3}$$

follows from Stirling's formula ([20],[22]).

We also recall the binomial formula ([20]),[22])

$$(1-x)^{-\alpha} = \sum_{n=0}^{\infty} \frac{(\alpha)_n}{n!} \cdot x^n, \quad |x| < 1. \tag{3.4}$$

The associated ultraspherical polynomials $\{C_n^{\tau}(x;i)\}$ are given by

$$2(n+\tau+i)xC_n^{\tau}(x;i)$$
$$= (n+i+1)C_{n+1}^{\tau}(x;i) + (n+2\tau+i-1)C_{n-1}^{\tau}(x;i), \quad n \geqslant 0, \tag{3.5}$$

and the initial conditions

$$C_{-1}^{\tau}(x;i) = 0, \quad C_0^{\tau}(x;i) = 1. \tag{3.6}$$

We assume $i = 0,1,2,3,\ldots$, and write $C_n^{\tau}(x) = C_n^{\tau}(x;0)$. The generating functions

$$\sum_{n=0}^{\infty} C_n^{\tau}(x)t^n = (t^2-2tx+1)^{-\tau} \tag{3.7}$$

and

$$\sum_{n=0}^{\infty} C_n^{\tau}(x;i)t^{n+1} = i(t^2-2tx+1)^{-\tau}\int_0^t u^{i-1}(u^2-2ux+1)^{\tau-1}du, \quad i \geqslant 1, \tag{3.8}$$

are easily obtained from (3.5) and (3.6). Observe that

$$C_n^{\tau}(x;i) = \sum_{k=0}^{n} \frac{i}{k+i} C_k^{1-\tau}(x)C_{n-k}^{\tau}(x), n \geqslant 0, i \geqslant 1, \tag{3.9}$$

follows from (3.8). Hence

$$C_n^{\tau}(1) = \frac{(2\tau)_n}{n!}, \quad n \geqslant 0, \tag{3.10}$$

and also

$$C^{\tau}_{n-1}(1;1) = \frac{1}{2\tau-1}\left[\frac{(2\tau)_n}{n!}-1\right], \quad \tau \neq \tfrac{1}{2} \tag{3.11}$$
$$C^{\frac{1}{2}}_{n}(1;1) = \sum_{k=0}^{n} \frac{1}{k+1}, \quad n \geqslant 0.$$

The continued fraction of $\{C^{\tau}_{n}(x)\}$ is

$$X_0(x) = 2\tau z\int_0^1 \left(1-\frac{z}{Z}\cdot u\right)^{\tau-1}(1-u)^{\tau-1}du \tag{3.12}$$

where $z = z(x) = x-\sqrt{x^2-1}$, $Z = Z(x) = x+\sqrt{x^2-1}$ and $\sqrt{x^2-1}$ is the branch of the square root of x^2-1 which is analytic in $\mathbb{C}-[-1,1]$ and behaves like x when $x\to\infty$. (It follows that $|z(x)| \leqslant |Z(x)|$ with $|z(x)| = |Z(x)|$ if and only if $x\in[-1,1]$, $z=1/z$, $z+Z = 2x, Z-z = 2\sqrt{x^2-1}$). Formula (3.12) can be obtained from (3.7) and (3.8) by means of (3.4) and Darboux's asymptotic method ([21], Sec.8.9). Stieltjes inversion formula and Remark 2.1 then yield for the orthogonality measure μ_0 of $\{C^{\tau}_{n}(x)\}$ the formula

$$d\mu_0(x) = -\frac{1}{\pi}\Phi(x)\,\mathrm{Im}X_0(x)\,dx \tag{3.13}$$
$$= \frac{2^{2\tau-1}}{\pi}\,\frac{\Gamma(\tau)^2}{\Gamma(2\tau)}(1-x^2)^{\tau-\frac{1}{2}}\Phi(x)\,dx$$

where $\Phi(x)$ is the characteristic function of $(-1,1)$. Hence, μ_0 is absolutely continuous.

The continued fraction of $\{C^{\tau}_{n}(x;1)\}$ is

$$X_1(x) = 2(1+\tau)z\frac{\displaystyle\int_0^1 u\left(1-\frac{z}{Z}\cdot u\right)^{\tau-1}(1-u)^{\tau-1}du}{\displaystyle\int_0^1 \left(1-\frac{z}{Z}\cdot u\right)^{\tau-1}(1-u)^{\tau-1}du} \tag{3.14}$$

which is analytic for $x\notin[-1,1]$. Since the denominator $\mathcal{D}_1(x)$ in (3.14) does not vanish in $(-1,1)$(it is $X_0(x)$, which does not vanish in $(-1,1)$), then

$$y_1(x) = \frac{1}{2\pi i}\lim_{\varepsilon\to 0+}(X_1(x-i\varepsilon)-X_1(x+i\varepsilon)) \tag{3.15}$$

is continuous on $\mathbb{R}$ except, perhaps, at ± 1. A simple but rather long calculation gives for $x \in (-1,1)$ that

$$y_1(x) = -\frac{1}{\pi}\cdot \mathrm{Im}\, X_1(x) \tag{3.16}$$

$$= 2^{2\tau-1}\cdot\frac{\Gamma(\tau)^2(\tau+1)}{\pi\Gamma(2\tau)\tau}\cdot(1-x^2)^{\tau-\frac{1}{2}}$$

$$\frac{z[(z/\tau+1)F(1-\tau,2,\tau+2:z^2)-xF(1-\tau,1,\tau+1:z^2)]}{|F(1-\tau,1,\tau+1:z^2)|^2}$$

where

$$F(a,b,c:x) = \sum_{n=0}^{\infty}\frac{(a)_n(b)_n}{n!(c)_n}\cdot x^n. \tag{3.17}$$

Other expressions for $y_1(x)$ can be found in [4], [9], [22], [25]. We will write $y_1(x) = (1-x^2)^{\tau-\frac{1}{2}}\tilde{y}(x)$, where $\tilde{y}(x) = y_1(x)(1-x^2)^{\frac{1}{2}-\tau}$ for $x \in (-1,1)$ and $\tilde{y}(x) = 0$ for $x \notin [-1,1]$. Then $\tilde{y}(x)$ is continuous for $x \neq \pm 1$ and $\tilde{y}(x) \neq 0$ a.e. in $(-1,1)$. The support Supp ν of *the orthogonality measure* ν of $\{C_n^{\tau}(x;1)\}$ is then $[-1,1]$.

More details about the associated ultraspherical polynomials can be found in the references above.

§4. Basic relationships. Let $\{P_n(x)\}$ be given by (1.1), (1.2) and (1.3). Eliminating $P_{2n+1}(x)$ and $P_{2n-1}(x)$ form (1.1) gives

$$(x^2-\tfrac{1}{2})P_{2n}(x) = P_{2(n+1)}(x)+\frac{1}{4^2}\cdot\frac{n}{n+\tau}\cdot\frac{n+2\tau-1}{n+\tau-1}P_{2(n-1)}(x) \tag{4.1}$$

which hols for $n \geq 2$. Let

$$W = W(x) = 2x^2-1 \tag{4.2}$$

and

$$Q_n(x) = \frac{4^n(\tau)_n}{n!}\cdot P_{2n}(x), \quad n \geq 0. \tag{4.3}$$

Then

$$2(n+\tau)WQ_n(x) = (n+1)Q_{n+1}(x)+(n+2\tau-1)Q_{n-1}(x), \quad n \geq 2, \tag{4.4}$$

and from (2.9), (2.10) it follows that

$$Q_n(x) = \alpha C_n^\tau(W) + 4\tau(1-\alpha)x^2 C_{n-1}^\tau(W;1), \quad n \geq 1. \tag{4.5}$$

Hence,

$$P_{2n}(x) = \frac{n!}{4^n(\tau)_n}[\alpha C_n^\tau(W) + 4\tau(1-\alpha)x^2 C_{n-1}^\tau(W;1)], \quad n \geq 1. \tag{4.6}$$

In particular (3.10) and (3.11) yield

$$P_{2n}(0) = (-1)^n \frac{(2\tau)_n}{4^n(\tau)_n}\cdot\alpha \tag{4.7}$$

$$P_{2n}(\pm 1) = \frac{(2\tau)_n}{4^n(\tau)_n}\left[\frac{4\tau-(2\tau+1)\alpha}{2\tau-1}\right] - \frac{4\tau(1-\alpha)n!}{(2\tau-1)4^n(\tau)_n}, \quad n \geq 1, \tau \neq \tfrac{1}{2}, \tag{4.8}$$

and

$$P_{2n}(\pm 1) = \frac{n!}{4^n(\frac{1}{2})_n}\left[\alpha + 2(1-\alpha)\sum_{k=0}^{n-1}\frac{1}{k+1}\right], \quad n \geq 1, \tau = \tfrac{1}{2}. \tag{4.9}$$

Let μ denote the orthogonality measure of $\{P_n(x)\}$. Then

$$\operatorname{Supp}\mu \subseteq [-(2+\sqrt{\alpha})/\sqrt{2}, (2+\sqrt{\alpha})/\sqrt{2}] \tag{4.10}$$

and

$$\int_{-\infty}^{+\infty} P_n(x)P_m(x)\,d\mu = k_n\delta_{mn}; m, \quad m, n \geq 0, \tag{4.11}$$

with

$$k_0 = 1, k_1 = \tfrac{1}{2}$$

$$k_{2n} = \frac{\tau\alpha(2\tau)_n\cdot n!}{(n+\tau)4^{2n}\cdot(\tau)_n^2}, k_{2n+1} = (\tau\alpha)\frac{(2\tau)_{n+1}\cdot n!}{(\tau)_{n+1}^2\cdot 4^{2n+1}}, \quad n \geq 1. \tag{4.12}$$

A calculation entirely similar to the one above gives for $\{P_n^{(1)}(x)\}$ that

$$P_{2n+1}^{(1)}(x) = \frac{(n+1)!}{4^n(\tau+1)_n} x C_n^\tau(W;1), \quad n \geq 0. \tag{4.13}$$

The continued fraction $X(x)$ of $\{P_n(x)\}$ then is

$$X(x) = \lim_{n\to\infty}\frac{P_{2n+1}^{(1)}(x)}{P_{2n+2}(x)} = \frac{2xX_0(W)}{\alpha+2(1-\alpha)x^2X_0(W)} \tag{4.14}$$

$$x \notin [-(2+\sqrt{\alpha})/\sqrt{2}, (2+\sqrt{\alpha})/\sqrt{2}],$$

where $X_o(W)$, given by (3.12), is the continued fraction of $\{C_n^\tau(W)\}$. Hence $X(x)$ is an analytic function of x for $x \notin [-1,1]$, except, perhaps, for simple poles located in $[-(2+\sqrt{\alpha})/\sqrt{2},-1) \cup (1,(2+\sqrt{\alpha})/\sqrt{2}]$. Let

$$g(x) = \lim_{\varepsilon \to o} \frac{1}{2\pi i}\cdot\{X(x-i\varepsilon)-X(x+i\varepsilon)\} \tag{4.15}$$

Then $g(x) = 0$ for $x \notin [-(2+\sqrt{\alpha})/\sqrt{2},(2+\sqrt{\alpha})/\sqrt{2}]$ and

$$g(x) = -\frac{2\alpha|x|\cdot \operatorname{Im}X_o(W)}{\pi|\alpha+2(1-\alpha)x^2X_o(W)|^2}, x \in (-1,0)\cup(0,1). \tag{4.16}$$

Let $\mathcal{D}(x) = \alpha+2(1-\alpha)x^2X_o(W)$. Since

$$\operatorname{Im}X_o(W) = -2^{2\tau}\tau\frac{\Gamma(\tau)^2}{\Gamma(2\tau)}(1-W^2)^{\tau-\frac{1}{2}}, \quad x \in (-1,0)\cup(0,1), \tag{4.17}$$

as follows from (3.13), and

$$\operatorname{Im}\mathcal{D}(x) = 2(1-\alpha)x^2\cdot\operatorname{Im}X_o(W), \tag{4.18}$$

$\mathcal{D}(x)$ does not vanish on $(-1,0)\cup(0,1)$. An argument based on Remark 2.1 then shows that

$$\int_{-\infty}^{+\infty} f\,d\mu = \int_{-\infty}^{+\infty} f(x)\,g(x)\,dx \tag{4.19}$$

for any continuous function with compact support contained in $(-1;0)\cup(0,1)$, that g is integrable on this set, and that this set is free of masses of μ. Since $g(x) > 0$ on $(-1,0)\cup(0,1)$, this set is part of C_μ. Now, $\mathcal{D}_\mu$ is the subset of $[-(2+\sqrt{\alpha})/\sqrt{2},-1)\cup(1,(2+\sqrt{\alpha})/\sqrt{2}]$ where $\mathcal{D}(x)$ vanishes, and $P_\mu \subseteq \mathcal{D}_\mu\cup\{0,\pm1\}$. In the next section we will show that $0 \notin P_\mu$, so that $(-1,1) \subseteq C_\mu$. We will also prove that $\pm1 \in P_\mu$ if and only if $\tau > 3/2$ and $\alpha = 4\tau/(2\tau+1)$. Hence, if $\alpha = 1, C_\mu = [-1,1]$ and $P_\mu = \phi$, which agrees with results in [2].

Now we turn to $\{P_n^{(1)}(x)\}$. Since

$$xP_{2n}^{(1)}(x) = P_{2n+1}^{(1)}(x) + \frac{n+2\tau}{4(n+\tau)}P_{2n-1}^{(1)}(x), \quad n \geq 0, \tag{4.20}$$

(4.13) yields

$$P_{2n}^{(1)}(x) = \frac{(n+1)!}{(\tau+1)_n 4^n}\{C_n^\tau(W;1) + \frac{n+2\tau}{n+1}C_{n-1}^\tau(W;1)\}, \quad n \geq 0. \qquad (4.21)$$

From (2.10) and (2.11) we then obtain that

$$P_{2n}^{(1)}(0) = (-1)^n \frac{n!}{(\tau+1)_n 4^n}, \quad n \geq 0, \qquad (4.22)$$

$$P_{2n}^{(1)}(\pm 1) = \frac{2(2\tau+n)n!}{(2\tau-1)(\tau+1)_n 4^n}\{\frac{(2\tau)_n}{n!} - \frac{2n+2\tau+1}{4\tau+2n}\}, \quad n \geq 0, \tau \neq \tfrac{1}{2} \qquad (4.23)$$

and

$$P_{2n}^{(1)}(\pm 1) = \frac{n!}{4^n(\tau+1)_n}\{2\sum_{k=0}^{n-1} \frac{n+1}{k+1} + 1\}, \quad n \geq 0, \tau = \tfrac{1}{2}. \qquad (4.24)$$

The orthogonality relation for $\{P_n^{(1)}(x)\}$ is

$$\int_{-\infty}^{+\infty} P_n^{(1)}(x) \cdot P_m^{(1)}(x)\, d\nu = k_n' \delta_{mn}; \; m, n \geq 0, \qquad (4.25)$$

where

$$k_{2n}' = \frac{(2\tau+1)_n n!}{(\tau+1)_n^2 4^{2n}}, \quad k_{2n+1}' = \frac{(2\tau+1)_n (n+1)!}{(\tau+n+1)(\tau+1)_n^2 4^{2n+1}}, \quad n \geq 0. \qquad (4.26)$$

For $P_{2n}^{(2)}(x)$ we obtain, as before, that

$$P_{2n}^{(2)}(x) = \frac{(n+1)!}{4^n(\tau+1)_n}[C_n^\tau(W;1) + \tfrac{1}{2}C_{n-1}^\tau(W;2)], \quad n \geq 0, \qquad (4.27)$$

and the continued fraction for $\{P_n^{(1)}(x)\}$ is

$$X^*(x) = \lim_{n\to\infty} \frac{P_{2n}^{(2)}(x)}{P_{2n+1}^{(1)}(x)} = \frac{1}{x}[1 + \tfrac{1}{2}X_1(W)], \qquad (4.28)$$

where $X_1(W)$, given by (4.13), is the continued fraction of $\{C_n^\tau(W;1)\}$. Hence $X^*(x)$ is an analytic function of x for $x \notin [-1,1]$. We have that

$$\begin{aligned} \mathcal{Y}(x) &= \lim_{\varepsilon\to 0+} \frac{1}{2\pi i}\{X^*(x-i\varepsilon) - X^*(x+i\varepsilon)\} \\ &= \frac{2^{2\tau-2}}{\pi}|x|^{2\tau-2}(1-x^2)^{\tau-\frac{1}{2}}\tilde{\mathcal{Y}}(2x^2-1) \end{aligned} \qquad (4.29)$$

where $\tilde{\mathcal{Y}}$ is given at the end of Section 3. As before we conclude that $\mathcal{Y}(x)$ is integrable on $(-1,0)\cup(0,1)$, and it is continuous in $(-1,0)\cup(0,1)$ if $\tau > \frac{1}{2}$. The absolutely continuous part of ν is $\mathcal{Y}(x)dx, (-1,0)\cup(0,1) \subseteq C_\nu$, and $P_\nu \subseteq \{0,\pm 1\}$. In the next section we will prove that $P_\nu = \emptyset$ if $\tau \leqslant \frac{1}{2}$ and $P_\nu = \{0\}$ if $\tau > \frac{1}{2}$, so that $C_\nu = [-1,1]$ or $C_\nu = [-1,0)\cup(0,1]$, accordingly.

§5. Main results.

THEOREM 5.1. *The orthogonality measure* μ *of* $\{P_n(x)\}$ *bears masses at* $x = \pm 1$ *if and only if* $\tau > 3/2$ *and*

$$\alpha = \frac{4\tau}{2\tau+1}. \tag{5.1}$$

In such case, ± 1 *are end points masses of* μ.

Proof. From (4.8) and (4.12) we obtain, when $\tau \neq \frac{1}{2}$, that

$$\frac{P_{2n}(\pm 1)}{\sqrt{k_{2n}}} = \left[\frac{4\tau-(2\tau+1)\alpha}{2\tau-1}\right]\sqrt{\left[\frac{(n+\tau)(2\tau)_n}{n!\tau\alpha}\right]}+\left[\frac{4\tau(1-\alpha)}{2\tau-1}\right]\sqrt{\left[\frac{n!(n+\tau)}{(2\tau)_n\tau\alpha}\right]}, \quad n \geqslant 1,$$

and we have, from (3.3), that

$$\sqrt{\left[\frac{(n+\tau)(2\tau)_n}{n!\tau\alpha}\right]} \approx \frac{n^\tau}{\sqrt{[\tau\alpha\Gamma(2\tau)]}}$$

and

$$\sqrt{\left[\frac{n!(n+\tau)}{(2\tau)_n\tau\alpha}\right]} \approx \sqrt{\left[\frac{\Gamma(2\tau)}{\tau\alpha}\right]}n^{1-\tau}.$$

Hence

$$\frac{P_{2n}(\pm 1)}{\sqrt{k_{2n}}} \approx \frac{n^\tau}{\sqrt{[\tau\alpha\Gamma(2\tau)]}}$$

if $\alpha = 1$ (even if $\tau = \frac{1}{2}$), and also

$$\frac{P_{2n}(\pm 1)}{\sqrt{k_{2n}}} \approx \frac{2\tau-(2\tau+1)\alpha}{2\tau-1}\,\frac{n^\tau}{\sqrt{[\tau\alpha\Gamma(2\tau)]}}$$

if $\alpha \neq 1, \alpha \neq 4\tau/2\tau+1$ and $\tau > \frac{1}{2}$. In both cases $\sum_{n=0}^{\infty} \frac{P_{2n}^2(\pm 1)}{k_{2n}}$ diverges and μ has no mass points at ± 1. Now assume $\alpha = 4\tau/2\tau+1$.

Then
$$\frac{P_{2n}(\pm 1)}{k_{2n}} \approx \sqrt{\left[\frac{2\alpha\Gamma(2\tau+2)}{\tau}\right]_n} 1-\tau,$$

even if $\tau = \frac{1}{2}$. If $\tau \leqslant 3/2$, $\sum_{n=0}^{\infty} P_{2n}^2(\pm 1)/k_{2n}$ is still divergent. But if $\tau > 3/2$, it is convergent, and μ may have masses at ± 1. To see that such is the case we have to show that $\sum_{n=0}^{\infty} \frac{P_{2n+1}^2(\pm 1)}{k_{2n+1}}$ is convergent, but this follows at once from

$$\pm \frac{P_{2n+1}(\pm 1)}{k_{2n+1}} = \sqrt{\left[\frac{k_{2n+2}}{k_{2n+1}}\right]} \cdot \frac{P_{2n+2}(\pm 1)}{\sqrt{k_{2n+2}}} + \frac{n+2\tau}{4(n+\tau)} \sqrt{\left[\frac{k_{2n}}{k_{2n-1}}\right]} \frac{P_{2n}(\pm 1)}{\sqrt{k_{2n}}}$$
$$= \tfrac{1}{2}\sqrt{\left[\frac{n+1}{n+\tau+1}\right]} \cdot \frac{P_{2n+2}(\pm 1)}{\sqrt{k_{2n+2}}} + \tfrac{1}{2}\sqrt{\left[\frac{n+2\tau}{n+\tau}\right]} \cdot \frac{P_{2n}(\pm 1)}{\sqrt{k_{2n}}}.$$

All that is left to prove is that when $\tau = \frac{1}{2}$ and $\alpha = 4\tau/2\tau+1$, $\sum_{n=0}^{\infty} \frac{P_{2n}^2(\pm 1)}{k_{2n}}$ diverges. But, in such case,

$$\frac{P_{2n}(\pm 1)}{\sqrt{k_{2n}}} = \sqrt{\left[\frac{2n+1}{\alpha}\right]} \{\alpha + 2(1-\alpha) \sum_{k=0}^{n-1} \frac{1}{k+1}\} \sim \sqrt{[2/\alpha]} \cdot 2(1-\alpha) n^{\frac{1}{2}} \log n$$

as follows from (4.9) and from

$$\sum_{k=0}^{n-1} \frac{1}{k+1} \sim \log n, \quad n \to \infty, \tag{5.2}$$

which is a consequence of the formula

$$\lim_{n \to \infty} \{\sum_{k=0}^{n} \frac{1}{k+1} - \log n\} = c \tag{5.3}$$

for the Euler constant c (See [22], p.8.). Then $\sum_{n=0}^{\infty} \frac{P_{2n}^2(\pm 1)}{k_{2n}}$ is divergent, and the proof is complete. ▲

THEOREM 5.2. *The orthogonality measure* μ *of* $\{P_n(x)\}$ *bears no mass at* $x = 0$.

Proof. From (4.7) and (4.12)

$$\frac{P_{2n}(0)}{\sqrt{k_{2n}}} = \sqrt{\left[\frac{(2\tau)_n(n+\tau)\alpha}{\tau n!}\right]} \approx \sqrt{\left[\frac{\alpha}{\tau\Gamma(2\tau)}\right]}n^{\tau}.$$

Since $\tau > 0$, $\sum_{n=0}^{\infty}\frac{P_{2n}^2(0)}{k_{2n}}$ diverges. ▲

THEOREM 5.3. *If $\tau > \frac{1}{2}$, and only in such case, the orthogonality measure ν of $\{P_n^{(1)}(x)\}$ carries a mass at $x = 0$, which is then an embedded mass point of ν.*

Proof. In fact from (4.22) and (4.26),

$$\frac{P_{2n}^{(1)}(0)}{\sqrt{k'_{2n}}} = (-1)^n\sqrt{\left[\frac{n!}{(2\tau+1)_n}\right]} \approx (-1)^n\sqrt{[\Gamma(2\tau)]}n^{-\tau} .$$

Hence, $\sum_{n=0}^{\infty}\left[\frac{P_{2n}^{(1)}(0)}{\sqrt{k'_{2n}}}\right]^2$ converges if and only if $\tau > \frac{1}{2}$. Since $P_{2n+1}^{(1)}(0) = 0$, this also holds for $\sum_{n=0}^{\infty}\left[\frac{P_n^{(1)}(0)}{\sqrt{k'_n}}\right]^2$. Since $(-1,0)\cup(0,1) \subseteq C_\nu$, 0 is in such case *an embedded mass point of* ν. ▲

THEOREM 5.4. *The measure ν bears no masses at $x = \pm 1$.*

Proof. From (4.23) and (4.26) it follows that, for $\tau \neq \frac{1}{2}$,

$$\frac{P_{2n}^{(1)}(\pm 1)}{\sqrt{k'_{2n}}} = \frac{2\tau}{2\tau-1}\sqrt{\left[\frac{(2\tau+1)_n}{n!}\right]} - \frac{2n+2\tau-1}{2\tau-1}\sqrt{\left[\frac{n!}{(2\tau+1)_n}\right]}.$$

But

$$\sqrt{\left[\frac{(2\tau+1)_n}{n!}\right]} \approx \frac{n^{\tau}}{\sqrt{\Gamma(2\tau)}}$$

and

$$\frac{2n+2\tau-1}{2\tau-1}\sqrt{\left[\frac{n!}{(2\delta-1)_n}\right]} \approx \frac{2}{2\tau-1}\sqrt{[\Gamma(2\tau)]}\ n^{1-\tau}.$$

Hence

$$\frac{P_{2n}^{(1)}(\pm 1)}{\sqrt{k'_{2n}}} \approx \frac{2}{2\tau-1}\cdot\frac{n^{\tau}}{\sqrt{\Gamma(2\tau)}}, \qquad \tau > \tfrac{1}{2};$$

$$\frac{P_{2n}^{(1)}(\pm 1)}{\sqrt{k'_{2n}}} \approx \frac{2\sqrt{\Gamma(2\tau)}}{2\tau-1}\cdot n^{1-\tau}, \qquad \tau < \tfrac{1}{2}.$$

The measure ν

In both cases $\sum_{n=0}^{\infty}\left|\frac{P_{2n}^{(1)}(\pm 1)}{\sqrt{k'_{2n}}}\right|^2$ is divergent. For $\tau = \frac{1}{2}$ we get, from (4.24) and (5.2), that

$$\frac{P_{2n}^{(1)}(\pm 1)}{\sqrt{k'_{2n}}} \approx n^{\frac{1}{2}} \log n,$$

and $\sum_{n=0}^{\infty}\left|\frac{P_{2n}^{(1)}(\pm 1)}{\sqrt{k'_{2n}}}\right|^2$ also diverges. ▲

It follows, using (2.24), that

THEOREM 5.5. *For $\tau > \frac{1}{2}$ the measure ν is*

$$d\nu(x) = Y(x)\,dx + a\delta(x)\,dx \tag{5.4}$$

where $Y(x)$, given by (4.29), is integrable and $\delta(x)dx$ is the Dirac measure at $x = 0$. Furthermore

$$a = \nu(\{0\}) = \frac{2\tau-1}{2\tau}. \tag{5.5}$$

REMARK 5.1. Observe that the measure ν tends to concentrate on $x = 0$ when $\tau \to +\infty$.

REMARK 5.2. When $\tau = 1$, (5.4) takes the particulary simple form

$$d\nu(x) = \frac{1}{\pi}\sqrt{[1-x^2]}\,\Phi(x)\,dx + \tfrac{1}{2}\delta(x)\,dx. \tag{5.6}$$

where $\Phi(x)$ is the characteristic function of $[-1,1]$.

For ν, the discrete support $\mathcal{D}_\nu$ is empty. We now describe $\mathcal{D}\mu$.

THEOREM 5.6. *The discrete support $\mathcal{D}\mu$ of the orthogonality measure μ of $\{P_n(x)\}$ is a follows:*

(1) If $\alpha \leqslant 1, \mathcal{D}\mu = \phi$.

(2) If $\alpha > 1$ *and* $\tau \leqslant \frac{1}{2}, \mathcal{D}\mu = \{-c, c\}$, *where* $1 < c \leqslant (\sqrt{\alpha}+2)/\sqrt{2}$ *is such that* $\mathcal{D}(c) = 0$.

(3) If $\tau > \frac{1}{2}$ *and* $1 < \alpha \leqslant 4\tau/(2\tau+1)$, $\mathcal{D}\mu = \phi$.

(4) *If* $\tau > \frac{1}{2}$ *and* $\alpha > 4\tau/2\tau+1$, $\mathcal{D}\mu = \{-c,c\}$, *where, as before,* $1 < c \leqslant (\sqrt{\alpha}+2)/\sqrt{2}$ *is such that* $\mathcal{D}(c) = 0$.

Proof. Let $\mathcal{D}(x) = \alpha+2(1-\alpha)x^2X_0(W)$. Clearly $\mathcal{D}(x) > 0$ if $\alpha \leqslant 1$. Let $\mathcal{D}(+\infty) = \lim_{x\to+\infty} \mathcal{D}(x)$, $\mathcal{D}(1^+) = \lim_{x\to 1+} \mathcal{D}(x)$. Then $\mathcal{D}(+\infty) = 1$, and

$$\mathcal{D}(1+) = -\infty, \tau \leqslant \tfrac{1}{2}; \mathcal{D}(1^+) = \frac{4\tau-(2\tau+1)\alpha}{2\tau-1}, \quad \tau > \tfrac{1}{2}$$

Hence $\mathcal{D}\mu = \phi$ if $\alpha \leqslant 1$. If $\alpha > 1$, $\mathcal{D}(1+)$ and $\mathcal{D}(+\infty)$ have opposite signs in cases (2) and (4), and there must be $1 < c < +\infty$ such that $\mathcal{D}(c) = 0$. Since the numerator $N(x)$ in (4.14) is positive for $x > 0$, μ has a mass at $x = c$ whose value is

$$\mu(\{c\}) = \mathrm{Res}(X,c) = \frac{N(c)}{\mathcal{D}'(c)}. \tag{5.7}$$

Now, the set $\mathcal{D}\mu$ is discrete. If there were $c' > c$ sich that $(c,c') \cap \mathcal{D}\mu = \phi$ and $\mathcal{D}(c') = 0$ then $\mathcal{D}'(c)$ and $\mathcal{D}'(c')$ would carry opposite signs, which is absurd, since $\mu(\{c'\}) > 0$. Hence $\mathcal{D}\mu = \{-c,c\}$ in cases (2) and (4). Since it is impossible to have $N(x) = \mathcal{D}(x) = 0$ for $x > 0$, it follows that $\mathcal{D}\mu = \phi$ in case (3). ▲

COROLLARY 5.1. *If* $\tau > 3/2$ *and* $\alpha = 4\tau/2\tau+1$, *the orthogonality measure* μ *of* $\{P_n(x)\}$ *is*

$$d\mu(x) = Y(x)\,dx + a[\delta(x-1) + \delta(x+1)]\,dx \tag{5.8}$$

where $Y(x)$, *given by* (4.16) *for* $x \in (-1,0)\cup(0,1)$ *and* $Y(x) = 0$ *elsewhere, is integrable,* $\delta(x\pm 1)$ *is the Dirac measure at* ± 1, *and*

$$a = \frac{2\tau-3}{4\tau-2}. \tag{5.9}$$

Proof. Let $\mathcal{D}(x) = \alpha+2(1-\alpha)x^2X_0(W)$, $x \in [-1,1]$, $W = 2x^2-1$, and extend $\mathcal{D}(x)$ to $(-1,1)$ by taking $X_0(W(x)) = \lim_{\varepsilon\to 0+} X_0(W(x+i\varepsilon))$ for $x \in (-1,1)$. Then $\mathcal{D}(x)$ is anlytic in $\mathbb{C}-[-1,1]$ and its restriction to $(-1,1)$ is also continuous. It is readily seen that for $\tau > 3/2$ and $\alpha = 4\tau/2\tau+1$ we have

$$\mathcal{D}(x) = (1-x^2)F(x), \quad x \in [-1,1],$$

where $F(x)$ is continuous and non vanishing in $(-1,1)$, and

$$F(\pm 1) = -4\tau/2\tau-3,$$

so that

$$\mathcal{V}(x) = \begin{cases} 4^{2\tau+1} \cdot \frac{\Gamma(\tau)^2}{\Gamma(2\tau)} \cdot \frac{|x|^{2\tau}(1+x)^{\tau-5/2}(1-x)^{\tau-5/2}}{|F(x)|^2}, & x \in (-1,1) \\ 0, & x \notin [-1,1], \end{cases}$$

which obviously is integrable.

To show that a is given by (5.9), let $\varepsilon, \delta > 0$ and integrate over the positively oriented rectangular path $\beta_{\varepsilon,\delta}$ with center at 1, basis 2δ and height 2ε. We have

$$\int_{1-\delta}^{1+\delta} [X(x-i\varepsilon)-X(x+i\varepsilon)]\,dx + i\int_{-\varepsilon}^{\varepsilon} [X(1+\delta+it)-X(1-\delta-it)]\,dt = \int_{\beta_{\varepsilon,\delta}} X(x)\,dx$$

Now, the right hand side integral can be replaced by the integral on β_δ, the positively oriented circle with center at 1 and radius δ (because $X(x)$ is analytic on $\operatorname{Im} x > 0$ and $\operatorname{Im} x < 0$ and both $\lim_{\varepsilon \to 0} X(x \pm i\varepsilon)$ exist and are finite for $x \in [1-\delta, 1) \cup (1, 1+\delta]$). Moreover, the second integral on the left vanishes when $\varepsilon \to 0$. Hence

$$\sigma(1+\delta)-\sigma(1-\delta) = \lim_{\varepsilon\to 0} \frac{1}{2\pi i} \int_{1-\delta}^{1+\delta} (X(x-i\varepsilon)-X(x+i\varepsilon))\,dx = \frac{1}{2\pi i}\int_{\beta_\delta} X(x)\,dx,$$

and

$$a = \mu(\{1\}) = \sigma(1)-\sigma(1-0) = \lim_{\delta\to 0} \frac{1}{2\pi i} \int_{\beta_\delta} X(x)\,dx = -\frac{4\tau/2\tau-1}{2F(1)} = \frac{2\tau-3}{2(2\tau-1)}.$$

The proof is complete. ▲

REMARK 5.3. Observe that μ tends to concentrate on ± 1 when $\tau \to +\infty$.

§6. Final Remarks.

Let $\{p_n(x)\}$ be an O.N.P.S. given by

$$xp_n(x) = b_n p_{n-1}(x) + a_n p_n(x) + b_{n+1} p_{n+1}(x), \quad n \geq 0, \tag{6.1}$$
$$p_{-1}(x) = 0, \quad p_o(x) = 1.$$

Assume

$$|a_n| \leq M, 0 < b_{n+1} \leq M, \quad n \geq 0. \tag{6.2}$$

(b_o is arbitrary). The tridiagonal infinite matrix

$$J = \begin{bmatrix} a_o & b_1 & 0 & 0 & \cdot & \cdot & \cdot \\ b_1 & a_1 & b_2 & 0 & \cdot & \cdot & \cdot \\ 0 & b_2 & a_2 & b_3 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{bmatrix} \tag{6.3}$$

is called the *Jacobi Matrix* of $\{p_n(x)\}$. It defines on the Hilbert space $\ell_2(\mathbb{C})$ of square summable, complex sequences $\{x_n/n \geq 0\}$ a bounded self-adjoint opperator L by

$$L(e_n) = b_n e_{n-1} + a_n e_n + b_{n+1} e_{n+1}, \quad n \geq 0, \tag{6.4}$$

($e_{-1} = 0$) and continuous linear extension. Here $\{e_n/n \geq 0\}$ is the canonical basis of $\ell_2(\mathbb{C})$ ($e_n = (\delta_{on}, \delta_{1n}, \ldots)$, $n \geq 0$). It can be shown that $\|L\| \leq 3M$. Observe that $x = (x_o, x_1, \ldots)$ is an eigenvector of L for the eigenvalue β if and only if $\sum_{k=o}^{\infty} |x_k|^2 < \infty$ and

$$\beta x_n = b_n x_{n-1} + a_n x_n + b_{n+1} x_{n+1}, \quad n \geq 0, \tag{6.5}$$

i.e, if and only if $x \in \ell_2(\mathbb{C})$ and $x_n = x_o p_n(\beta)$, $n > 0$.

Let μ be the orthogonality measure of $\{p_n(x)\}$ and for each $\tau \in \mathbf{R}$ let $E_\tau : \ell_2 \to \ell_2$ be defined by

$$E_\tau e_n = \sum_{k=o}^{\infty} \{\int_{-\infty}^{\tau} p_n(t) p_k(t) d\mu(t)\} e_k, \tag{6.6}$$

and continuous linear extension. Then $\{E_\tau | \tau \in \mathbb{R}\}$ is a resolution of the identity for L and

$$\int_{-\infty}^{+\infty} f(x)\,d\mu = \int_{-\infty}^{+\infty} f(t)\,d(E_\tau e_o; e_o) \tag{6.7}$$

for any continuous function on $\mathbb{R}((x;y) = \sum_{n=o}^{\infty} x_n \bar{y}_n$ is the inner product in $\ell_2(\mathbb{C})$). From (6.7) it easily follows that the spectrum $\sigma(L)$ of L coincides with Supp μ; $\sigma_c(L)$, the continuous spectrum, is C_μ; and $\sigma_p(L)$, the point spectrum, is P_μ. Details of above can be found in [1], [6].

It follows that the operator L of the Jacobi matrix

$$a_n = 0, \quad n \geq 0$$
$$b_1 = \sqrt{[\alpha/2]},\ b_{2n} = \tfrac{1}{2}\sqrt{\left[\frac{n}{n+\tau}\right]},\ b_{2n+1} = \tfrac{1}{2}\sqrt{\left[\frac{n+2\tau}{n+\tau}\right]}, \quad n \geq 1 \tag{6.8}$$

has, for $\tau > 3/2$ and $\alpha = 4\tau/2\tau+1$, eigenvalues at the end points ± 1 of the spectrum $\sigma(L) = [-1,1]$. These are the only eigenvalues of L. As for the operator L of the Jacobi matrix

$$a_n = 0, \quad n \geq 0$$
$$b_{2n} = \tfrac{1}{2}\sqrt{\left[\frac{n+2\tau}{n+\tau}\right]},\ b_{2n+1} = \tfrac{1}{2}\sqrt{\left[\frac{n+1}{n+\tau+1}\right]}, \quad n \geq 0, \tag{6.9}$$

it has, for $\tau > \tfrac{1}{2}$, an embedded eigenvalue, 0, in the spectrum $\sigma(L) = [-1,1]$. This is its only eigenvalue.

Now consider the recurrence relation

$$xp_{kn+j}(x) = p_{kn+j+1}(x) + a_{j,n}p_{nk+j-1}(x), n \geq 0, \quad j = 0,\ldots,k-1,$$
$$p_{-1}(x) = 0, \quad p_o(x) = 1, \tag{6.10}$$

where $k \geq 2$ is an integer, $a_{o,n}$ and $a_{1,n}$, $n \geq 0$, are given by (1.3), and

$$a_{j,n} = 1/4, j = 2,3,\ldots,k-1, \quad n > 0. \tag{6.11}$$

Recurrence relation (6.10) reduces to that of the sieved ultraspherical polynomials of the first kind in [2] when $\alpha = 1$. As it is to be expected, the orthogonality measure μ of $\{p_n(x)\}$ bears masses at ± 1 when $k \geq 2, \tau > 3/2$ and $\alpha = 4\tau/2\tau+1$. When $\tau > \frac{1}{2}$, the orthogonality measure of $\{p_n^{(1)}(x)\}$ carries embedded masses at the points x of $(-1,1)$ where $T_k(x) = \pm 1$ ($T_k(x)$ is the k^{th} Chebichev polynomial of the first kind ([22], [25])). The technical details of the proofs will appear elsewhere.

Acknowledgements. The authors are deeply indebted to M.E.H. Ismail for suggestions, criticism and advice which greatly contributed to many improvements. The first author thanks the University of South Florida in Tampa for granting him a visiting position during the spring semester of 1989, and to Colciencias, for support to attend the first U.S.-U.R.S.S. Conference on Approximation Theory, Tampa, where these ideas were presented.

REFERENCES

[1] Ahkiezer, N.I., *The classical Moment Problem*, Hafner, New York, 1965.

[2] Al-Salam, W., Allaway, W. and Askey, R., *Sieved Ultraspherical polynomials*, Transactions Amer. Math. Soc, 284 (1984), 39-55.

[3] Askey, R. and Ismail, M.E.H., *Recurrense relations, continued fractions and orthogonal polynomials*, Memoirs Amer. Math. Soc. 300 (1984), 1-108.

[4] Bustoz, J. and Ismail, M.E.H., *The associated ultraspherical polynomials and their q-analogues*, Canadian Journal of Mathematics, 34 (1982), 718-736.

[5] Bremermann, H., *Distributions, Complex Variables and Fourier Transforms*, Addison Wesley, Reading, Mass., 1965.

[6] Charris, J.A. and Gomez, L.A., *Functional analysis, orthogonal polynomials and a theorem of Markov*, Rev. Col. de Mat., 22 (1988), 77-128.

[7] Charris, J.A. and Ismail, M.E.H., *On sieved orthogonal polynomials II: Randon walk polynomials*, Canadian Journal of Mathematicas, 38 (1986), 397-415.

[8] Charris, J.A. and Ismail, M.E.H., *On sieved orthogonal polynomials V: Sieved Pollaczek polynomials*, SIAM J. of Math. Analysis, 18 (1987), 1177-1218.

[9] Chihara, T.S., *An Introduction to Orthogonal Polynomials*, Gordon and Breach, New York, 1978.

[10] Chihara, T.S., *Orthogonal polynomials and measures with end point masses*, Rocky Mountain J. Math., 15 (1985), 705-719.
[11] Geronimo, J.S. and Van Assche, W., *Orthogonal polynomials on several intervals via a polynomial mapping*, Transactions Amer. Math. Soc., 308 (1988), 559-581.
[12] Ismail, M.E.H., *On sieved orthogonal polynomials I: symmetric Pollaczek analogues*, SIAM J. Math. Analysis, 16 (1985), 1093-1113.
[13] Ismail, M.E.H., *On sieved orthogonal polynomials, III: Orthogonality on several intervals*, Transactions Amer. Math. Soc., 294 (1986), 89-111.
[14] Isamil, M.E.H., *On sieved orthogonal polynomials, IV: Generating functions*, J. Approx. Theory, 46(1986), 284-296.
[15] Koekoek, R., *Generalizations of Laguerre (type) polynomials*, Report 87-58, Delft University of Technology, Delft, the Netherlands, 1987.
[16] Koekoek, R., *Koornwinder's generalized Laguerre polynomials and its q-analogues*, Report 88-87, Delft University of Technology, Delft, the Netherlands, 1988.
[17] Koekoek, R., *Generalizations of Laguerre polynomials*, Report 82-25, Delft University of Technology, Delft, the Netherlands, 1989.
[18] Koornwinder, T.H., *Orthogonal polynomials with weight function* $(1-x)^{\alpha}(1+x)^{\beta}+M\delta(x+1)+N\delta(x-1)$, Can. Math. Bulletin, 27 (1984), 205-241.
[19] Lang, S., *Real Analysis*, Addison Wesley, Reading, Mass., 1984.
[20] Lebedev, N.N., *Special Functions and their Applications*, Prentice Hall, Englewood Clifs, N.J., 1965.
[21] Olver, F.W.J., *Asymptotics and Special Functions*, Academic Press, New York, 1974.
[22] Rainville, E.D., *Special Functions*, MacMillan, New York, 1960.
[23] Slim, H.A., *On co-recursive orghogonal polynomials and their applications to potential scattering*, J. Math. Anal. and Applic., 136 (1988), 1-19.
[24] Shohat, J. and Tamarkin, J.D., *The Problem of Moments*, Math. Surveys, Vol.1, Amer. Math. Soc, Providence, R.I., 1963.
[25] Szegö, G., *Orthogonal Polynomials*, 4th Edition, Colloquium Publications, Vol. 23, Amer. Math Soc., Providence, R.I., 1975.
[26] Wall, H.S., *The Analytic Theory of Continued Fractions*, Van Nostrand, New York, 1948.
[27] Wheeler, J.C., *Modified moments and continued fraction coefficients for the diatomic linear chain*, J. Chem. Phys.. 80 (1984), 472-476.

Departamento de Matemáticas y Estadística
Universidad Nacional de Colombia
Bogotá, Colombia.

Departamento de Matemáticas
Universidad Antonio Nariño
y Universidad de los Andes
Bogotá, Colombia.

(Recibido en diciembre de 1989, la versión revisada en febrero de 1990).