UNA NOTA SOBRE UNAS PROPIEDADES DE LA FUNCION GAMA $\Gamma(x)$

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1.

La función gama $\Gamma(x)$ se define, para todo valor $x > 0$, por medio de la integral

$$(1) \qquad \Gamma(x) = \int_0^\infty e^{-t}\, t^{x-1}\, dt,$$

ó por el límite

$$(2) \qquad \Gamma(x) = \lim_{n\to\infty} \frac{n!\, n^x}{x(x+1)(x+2)\cdots(x+n)}$$

ó también por el producto infinito

$$(3) \qquad \frac{1}{\Gamma(x)} = x\, e^{\gamma x} \prod_{n=1}^{\infty} \left[\left(1+\frac{x}{n}\right) e^{-\frac{x}{n}} \right]$$

en donde

$$\gamma = \lim_{m\to\infty} \left(\sum_{n=1}^{\infty} \frac{1}{n} - \log m \right) = 0.57721 \ldots .$$

Integrando por partes, se obtiene de (1), para todo $x > 0$

$$\Gamma(x+1) = x\,\Gamma(x).$$

Cuando x es un número natural, aplicando reiteradamente la fórmula anterior, y teniendo en cuenta que $\Gamma(1) = 1$, se obtiene que

$$\Gamma(n+1) = n!\,.$$

Sean x_i , y_j , z_k números reales, tales que

$$x_i > 0 \ (1 \leqq i \leqq \alpha), \quad y_j > 0 \ (1 \leqq j \leqq \beta) \quad \text{y} \quad z_k > 0 \ (1 \leqq k \leqq \delta),$$

y sean s_i , t_j , u_k, n, n', n'' números enteros tales que

$$s_i \geqq 0 \ (1 \leqq i \leqq \alpha), \quad t_j \geqq 0 \ (1 \leqq j \leqq \beta) \quad \text{y} \quad u_k \geqq 0 \ (1 \leqq k \leqq \delta),$$

y

$$n \geqq 0, \quad n' \geqq 0, \quad n'' \geqq 0.$$

entonces :

TEOREMA I (α, β, δ)

$$\sum_{\substack{n',n''\\ n'+n''=n}} \; \sum_{\substack{s_i,\, t_j\\ s_1+\cdots+s_\alpha+t_1+\cdots+t_\beta=n'}} (-1)^{t_1+\cdots+t_\beta} \prod_{i=1}^{\alpha} \frac{1}{s_i!\,(x_i-s_i)!} \prod_{j=1}^{\beta} \frac{\Gamma(y_j+t_j)}{t_j!} (-1)^{n''} \times$$

$$\times \sum_{\substack{u_k\\ u_1+\cdots+u_\delta=n''}} \prod_{k=1}^{\delta} \frac{\Gamma(z_k+u_k)}{u_k!} =$$

$$= \frac{\prod_{j=1}^{\beta}\Gamma(y_j)\prod_{k=1}^{\delta}\Gamma(z_k)}{\prod_{i=1}^{\alpha}\Gamma(x_i)} \; \frac{\Gamma(x_1+\cdots+x_\alpha-y_1-\cdots-y_\beta-z_1-\cdots-z_\delta-\alpha+1)}{\Gamma(x_1+\cdots+x_\alpha-y_1-\cdots-y_\beta-z_1-\cdots-z_\delta-\alpha+1-n)\; n!},$$

si $\quad x_1+\cdots+x_\alpha-y_1-\cdots-y_\beta-z_1-\cdots-z_\delta-\alpha+1-n>0$

TEOREMA I' (α, β, δ)

$$\sum_{\substack{n',n''\\ n'+n''=n}} \; \sum_{\substack{s_i,\, t_j\\ s_1+\cdots+s_\alpha+t_1+\cdots+t_\beta=n'}} (-1)^{t_1+\cdots+t_\beta} \prod_{i=1}^{\alpha} \frac{1}{s_i!\,(t-s_i)!} \prod_{j=1}^{\beta} \frac{\Gamma(y_j+t_j)}{t_j!} (-1)^{n''} \sum_{\substack{u_k\\ u_1+\cdots+u_\delta=n''}} \prod_{k=1}^{\delta} \frac{\Gamma(z_k+u_k)}{u_k!} =$$

$$= (-1)^n \frac{\prod_{j=1}^{\beta}\Gamma(y_j)\prod_{k=1}^{\delta}\Gamma(z_k)}{\prod_{i=1}^{\alpha}\Gamma(x_i)} \; \frac{\Gamma(y_1+\cdots+y_\beta+z_1+\cdots+z_\delta-x_1-\cdots-x_\alpha+\alpha-n)}{\Gamma(y_1+\cdots+y_\beta+z_1+\cdots+z_\delta-x_1-\cdots-x_\alpha+\alpha)!\; n!}$$

si $\quad y_1+\cdots+y_\beta+z_1+\cdots+z_\delta-x_1-\cdots-x_\alpha+\alpha-n>0$

TEOREMA II $(\delta=0)$

$$\sum_{\substack{s_i,\, t_j\\ s_1+\cdots+s_\alpha+t_1+\cdots+t_\beta=n}} (-1)^{t_1+\cdots+t_\beta} \prod_{i=1}^{\alpha} \frac{1}{s_i!\,\Gamma(x_i-s_i)!} \prod_{j=1}^{\beta} \frac{\Gamma(y_j+t_j)}{t_j!} =$$

$$= \frac{\prod_{j=1}^{\beta} \Gamma(y_j)}{\prod_{i=1}^{\alpha} \Gamma(x_i)} \frac{\Gamma(x_1+\cdots+x_\alpha-y_1-\cdots-y_\beta-\alpha+1)}{\Gamma(x_1+\cdots+x_\alpha-y_1-\cdots-y_\beta-\alpha+1-n)\, n!},$$

$$\text{si} \quad x_1+\cdots+x_\alpha-y_1-\cdots-y_\beta-\alpha+1-n>0$$

TEOREMA II' $(\delta=0)$.

$$\sum_{\substack{s_i,\,t_j \\ s_1+\cdots+s_\alpha+t_1+\cdots+t_\beta=n}} (-1)^{t_1+\cdots+t_\beta} \prod_{i=1}^{\alpha} \frac{1}{s_i!\,\Gamma(x_i-s_i)} \prod_{j=1}^{\beta} \frac{\Gamma(y_j+t_j)}{t_j!} =$$

$$= \frac{\prod_{j=1}^{\beta} \Gamma(y_j)}{\prod_{i=1}^{\alpha} \Gamma(x_i)} \frac{\Gamma(y_1+\cdots+y_\beta-x_1-\cdots-x_\alpha+\alpha-n)}{\Gamma(y_1+\cdots+y_\beta-x_1-\cdots-x_\alpha+\alpha)\, n!},$$

$$\text{si} \quad y_1+\cdots+y_\beta-x_1-\cdots-x_\alpha+\alpha-n>0$$

TEOREMA III $(\alpha=\beta=0)$

$$\sum_{\substack{s_i \\ s_1+\cdots+s_\alpha=n}} \prod_{i=1}^{\alpha} \frac{1}{s_i!\,\Gamma(x_i-s_i)} = \frac{1}{\prod_{i=1}^{\alpha} \Gamma(x_i)} \cdot \frac{\Gamma(x_1+\cdots+x_\alpha-\alpha+1)}{\Gamma(x_1+\cdots+x_\alpha-\alpha+1-n)\, n!}$$

$$\text{si} \quad x_1+\cdots+x_\alpha-\alpha+1-n>0$$

TEOREMA IV $(\alpha=\delta=0)$, [2]:

$$\sum_{\substack{t_j \\ t_1+\cdots+t_\beta=n}} \prod_{j=1}^{\beta} \frac{\Gamma(y_j+t_j)}{t_j!} = \prod_{j=1}^{\beta} \Gamma(y_j) \frac{\Gamma(y_1+\cdots+y_\beta+n)}{\Gamma(y_1+\cdots+y_\beta)\; n!}$$

$$\text{si} \quad y_1+\cdots+y_\beta \geq 1.$$

TEOREMA V $(\beta=0)$:

$$\sum_{\substack{n',n'' \\ n'+n''=n}} \sum_{\substack{s_i \\ s_1+\cdots+s_\alpha=n'}} \prod_{i=1}^{\alpha} \frac{1}{s_i!\,\Gamma(x_i-s_i)} (-1)^{n''} \sum_{\substack{u_\kappa \\ u_1+\cdots+u_\delta=n''}} \prod_{\kappa=1}^{\delta} \frac{\Gamma(z_\kappa+u_\kappa)}{u_\kappa!} =$$

$$= \frac{\prod_{\kappa=1}^{\delta}\Gamma(z_\kappa)}{\prod_{i=1}^{\alpha}\Gamma(x_i)} \cdot \frac{\Gamma(x_1+\cdots+x_\alpha-z_1-\cdots-z_\delta-\alpha+1)}{\Gamma(x_1+\cdots+x_\alpha-z_1-\cdots-z_\delta-\alpha+1-n)\,n!}$$

$$\text{si} \quad x_1+\cdots+x_\alpha-z_1-\cdots-z_\delta-\alpha+1-n>0$$

TEOREMA V' $(\beta=0)$:

$$\sum_{\substack{n',n'' \\ n'+n''=n}} \sum_{\substack{s_i \\ s_1+\cdots+s_\alpha=n'}} \prod_{i=1}^{\alpha} \frac{1}{s_i!\,\Gamma(x_i-s_i)} (-1)^{n''} \sum_{\substack{u_\kappa \\ u_1+\cdots+u_\delta=n''}} \prod_{\kappa=1}^{\delta} \frac{\Gamma(z_\kappa+u_\kappa)}{u_\kappa!} =$$

$$= (-1)^n \frac{\Gamma(z_1+\cdots+z_\delta-x_1-\cdots-x_\alpha+\alpha-n)}{\Gamma(z_1+\cdots+z_\delta-x_1-\cdots-x_\alpha+\alpha)\,n!},$$

$$\text{si} \quad z_1+\cdots+z_\delta-x_1-\cdots-x_\alpha+\alpha-n>0$$

2.

Los Teoremas II $\sim$ V son casos especiales del Teorema I, y el Teorema I es también un caso especial del Teorema II. Porque, si en el Teorema II, se hace

$$y_j = y'_j, \; t_j = t'_j \quad (1 \leq j \leq \beta'),$$

$$y'_j = z'_\kappa, \; t_j = u'_\kappa \quad (\beta'+1 \leq j \leq \beta), \; (1 \leq k \leq \beta-\beta'),$$

$$u'_1 + \ldots\ldots + u'_{\delta'} = n'',$$

entonces, en este caso se obtiene el Teorema I (α, β', δ'), y por lo tanto, basta demostrar el Teorema II .

Se demuestra primeramente el Teorema III ($\alpha = 2$). por inducción completa. La afirmación es cierta para n $=$ 0, 1. En el caso n $=$ 1, se obtiene :

$$(4) \qquad \frac{1}{x_1+x_2-2} \sum_{s_1+s_2=1} \frac{1}{s_1!\, s_2!\, \Gamma(x_1-s_1)\Gamma(x_2-s_2)} = \frac{1}{\Gamma(x_1)\Gamma(x_2)}$$

Suponiendo que para n = n el Teorema III (α = 2 n = n) es cierto, entonces se tendrá :

$$\sum_{\substack{s_1, s_2 \\ s_1+s_2=n+1}} \frac{1}{s_1!\, s_2!\, \Gamma(x_1-s_1)\Gamma(x_2-s_2)} = \sum_{\substack{s_1, s_2 \\ s_1+s_2=n}} \frac{1}{\Gamma(s_1+1)\Gamma(s_2+1)\Gamma(x_1-s_1)\Gamma(x_2-s_2)} =$$

y, según el Teorema III (α = 2, n = 1) (4), se tiene:

$$= \frac{1}{n+1}\left(\sum_{\substack{s_1, s_2 \\ s_1+s_2=n+1}} \frac{1}{\Gamma(s_1+1)\Gamma(s_2)\Gamma(x_1-s_1)\Gamma(x_2-s_2)} + \sum_{\substack{s_1, s_2 \\ s_1+s_2=n}} \frac{1}{\Gamma(s_1)\Gamma(s_2+1)\Gamma(x_1-s_1)\Gamma(x_2-s_2)} \right) =$$

y, por la hipótesis de III (α = 2, n = n) :

$$= \frac{1}{n+1}\left(\frac{\Gamma(x_1+x_2-2)}{\Gamma(x_1+x_2-2-n)\Gamma(x_1)\Gamma(x_2-1)\, n!} + \frac{\Gamma(x_1+x_2-2)}{\Gamma(x_1+x_2-2-n)\Gamma(x_1-1)\Gamma(x_2)\, n!} \right) =$$

$$= \frac{1}{n+1} \cdot \frac{\Gamma(x_1+x_2-2)}{\Gamma(x_1+x_2-2-n)} \left(\frac{1}{\Gamma(x_1)\Gamma(x_2-1)} + \frac{1}{\Gamma(x_1-1)\Gamma(x_2)} \right)$$

$$= \frac{1}{\Gamma(x_1)\Gamma(x_2)} \cdot \frac{\Gamma(x_1+x_2-2)}{\Gamma(x_1+x_2-2-n)} \cdot \frac{1}{(n+1)!},$$

lo que demuestra completamente el Teorema III (α = 2).

Se hace ahora la demostración del Teorema III por inducción completa sobre el número α . La afirmación es cierta para α = 1 (trivial) y α = 2 (III (α = 2)) . Entonces

$$\sum_{\substack{s_i \\ s_1+\cdots+s_\alpha+s_{\alpha+1}=n}} \prod_{i=1}^{\alpha+1} \frac{1}{s_i!\, \Gamma(x_i-s_i)}$$

$$= \sum_{\substack{s_{\alpha+1},\, n' \\ s_{\alpha+1}+n'=n}} \frac{1}{s_{\alpha+1}!\, \Gamma(x_{\alpha+1}-s_{\alpha+1})} \sum_{\substack{s_i \\ s_1+\cdots+s_\alpha=n'=n-s_{\alpha+1}}} \prod_{i=1}^{\alpha} \frac{1}{s_i\, \Gamma(x_i-s_i)} =$$

y por la hipótesis del Teorema III ($\alpha = \alpha$) :

$$= \frac{\Gamma(x_1+\cdots+x_\alpha-\alpha+1)}{\prod_{i=1}^{\alpha}\Gamma(x_i)} \sum_{s_{\alpha+1}+n'=n}' \frac{1}{s_{\alpha+1}!\,\Gamma(x_{\alpha+1}-s_{\alpha+1})} \frac{1}{\Gamma(x_1+\cdots+x_\alpha-\alpha+1-n')\,n'!} =$$

Según el Teorema III ($\alpha = 2$), se obtiene :

$$= \frac{\Gamma(x_1+\cdots+x_\alpha-\alpha+1)}{\prod_{i=1}^{\alpha}\Gamma(x_i)} \frac{\Gamma(x_1+\cdots+x_\alpha+x_{\alpha+1}-\alpha-1)}{\Gamma(x_1+\cdots+x_{\alpha+1}-\alpha-n)\,\Gamma(x_1+\cdots+x_\alpha-\alpha+1)\,n!} =$$

$$= \frac{\Gamma(x_1+\cdots+x_{\alpha+1}-\alpha)}{\prod_{i=1}^{\alpha+1}\Gamma(x_i)\,\Gamma(x_1+\cdots+x_{\alpha+1}-\alpha-n)\,n!},$$

lo que demuestra el Teorema III .

El Teorema II ($\alpha = \beta = 1$) se demuestra por inducción completa sobre el número n. La afirmación es cierta para n = 0.

Suponiendo que para n = n se tiene el Teorema II ($\alpha = \beta = 1$ n = n) es decir :

$$\sum_{\substack{s_1,t_1\\ s_1+t_1=n}} (-1)^{t_1} \frac{1}{s_1!\,\Gamma(x_1-s_1)} \frac{\Gamma(y_1+t_1)}{t_1!} = \frac{\Gamma(y_1)}{\Gamma(x_1)} \frac{\Gamma(x_1-y_1)}{\Gamma(x_1-y_1-n)\,n!};$$

entonces se tendrá [véase: la demostración del III ($\alpha = 2$)] :

$$\sum_{\substack{s_1,t_1\\ s_1+t_1=n}} (-1)^{t_1} \frac{1}{s_1!\,\Gamma(x_1-s_1)} \frac{\Gamma(y_1+t_1)}{t_1!} =$$

$$= \frac{1}{n+1}\left(\sum_{s_1+t_1=n} (-1)^{t_1} \frac{1}{\Gamma(s_1+t_1)\,\Gamma(t_1)} \frac{\Gamma(y_1+t_1)}{\Gamma(x_1-s_1)} + \sum_{s_1+t_1=n+1} (-1)^{t_1} \frac{\Gamma(y_1+t_1)}{\Gamma(s_1)\Gamma(t_1+1)\Gamma(x_1-s_1)} \right) =$$

$$= \frac{1}{n+1} \frac{\Gamma(x_1-y_1-1)}{\Gamma(x_1-y_1-(n+1))} \left(-\frac{\Gamma(y_1+1)}{\Gamma(x_1)} + \frac{\Gamma(y_1)}{\Gamma(x_1-1)} \right) =$$

$$= \frac{\Gamma(x_1-y_1)\,\Gamma(y_1)}{\Gamma(x_1-y_1-(n+1))\,\Gamma(x_1)} \cdot \frac{1}{(n+1)!}$$

Se hace ahora la demostración del Teorema II por inducción completa sobre los números α, β. La afirmación es cierta para ($\alpha, \beta = 0$).

(*TEOREMA III*).

Suponiendo que para $\alpha = \beta$ se tiene el Teorema II (α, β), entonces

$$\sum_{\substack{s_i, t_j \\ s_1+\cdots+s_\alpha+t_1+\cdots+t_{\beta+1}=n}} (-1)^{t_1+\cdots+t_\beta+t_{\beta+1}} \prod_{i=1}^{\alpha} \frac{1}{s_i!\,\Gamma(x_i-s_i)} \prod_{j=1}^{\beta} \frac{\Gamma(y_j+t_j)}{t_j!} =$$

$$= \sum_{t_{\beta+1}+n'=n} (-1)^{t_{\beta+1}} \frac{\Gamma(y_{\beta+1}+t_{\beta+1})}{t_{\beta+1}!} \sum_{\substack{s_i, t_j \\ s_1+\cdots+s_\alpha+t_1+\cdots+t_\beta=n'=n-t_{\beta+1}}} (-1)^{t_1+\cdots+t_\beta} \prod_{i=1}^{\alpha} \frac{1}{s_i!\,\Gamma(x_i-s_i)} \prod_{j=1}^{\beta} \frac{\Gamma(y_j+t_j)}{t_j!} =$$

y según el Teorema II (α, β):

$$= \sum_{t_{\beta+1}+n'=n} (-1)^{t_{\beta+1}} \frac{\Gamma(y_{\beta+1}+t_{\beta+1})}{t_{\beta+1}!} \frac{\prod_{j=1}^{\beta}\Gamma(y_j)}{\prod_{i=1}^{\alpha}\Gamma(x_i)} \cdot \frac{\Gamma(x_1+\cdots+x_\alpha-y_1-\cdots-y_\beta-\alpha+1)}{\Gamma(x_1+\cdots+x_\alpha-y_1-\cdots-y_\beta-n')\; n'!} =$$

$$= \frac{\prod_{j=1}^{\beta}\Gamma(y_j)}{\prod_{i=1}^{\alpha}\Gamma(x_i)} \Gamma(x_1+\cdots+x_\alpha-y_1-\cdots-y_\beta-\alpha+1) \sum_{t_{\beta+1}+n'=n} (-1)^{t_{\beta+1}} \frac{\Gamma(y_{\beta+1}+t_{\beta+1})}{t_{\beta+1}!\,\Gamma(x_1+\cdots+x_\alpha-y_1-\cdots-y_\beta-\alpha-n')\; n'!} =$$

y según el Teorema II

$$= \frac{\prod_{j=1}^{\beta+1}\Gamma(y_j)}{\prod_{i=1}^{\alpha}\Gamma(x_i)} \cdot \frac{\Gamma(x_1+\cdots+x_\alpha-y_1-\cdots-y_\beta-(\alpha+1)+1)}{\Gamma(x_1+\cdots+x_\alpha-y_1-\cdots-y_\beta-(\alpha+1)+1-n)\; n!},$$

c. q. d.

lo que demuestra el Teorema II (α, $\beta+1$), y de la misma manera se puede demostrar el Teorema II ($\alpha+1$, β).

De la misma manera se pueden demostrar los Teoremas I, II', V'.

Nota. Muchas igualdades y muchas fórmulas relativas a los coeficientes binomiales están contenidas en los Teoremas I $\sim$ V. [3]

Referencias :

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(3) E. Netto: Lehrbuch der Combinatorik, Leipizg, 1901 .