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CURVES AND LATTICES OF POINTS IN THE MINKOWSKI PLANE

by

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ABSTRACT. The main goal of this work is to derive an integral formula refering to bounded convex sets (section 3), in order to obtain some results involving lattices of points in the Minkowski plane (seccion 4). To prove such a formula it is necessary to develop some tools of differential geometry in the large. This is done in section 2, where the turning tangents theorem for Euclidean curves is carried over to Minkowski plane.

§1 Preliminares. Let C be a closed convex curve, called *indicatrix*, enclosing the origin 0 of the Euclidean plane $\mathbb{R}^2$ as interior point. There will be assumed throughout that C is "sufficiently" differentiable and has positive finite curvature everywhere.

Let $\mathbf{v}$ be a vector of the plane and let $|\mathbf{v}|$ be the Euclidean length of $\mathbf{v}$. Then the *Minkowski length* of $\mathbf{v}$ is defined to be

$$\|\mathbf{v}\| := \frac{|\mathbf{v}|}{|\mathbf{0A}|}, \tag{1}$$

where $A \in C$ and the vector $\mathbf{0A}$ is parallel to $\mathbf{v}$, as oriented vectors. The plane $\mathbb{R}^2$ equipped with the length (1) shall be referred to as the *Minkowski plane*. If C is a 0-symmetric curve, i.e. $\|\mathbf{v}\| = \|-\mathbf{v}\|$ for every $\mathbf{v} \in \mathbb{R}^2$, then the plane will be called the *symmetric Minkowski plane.*

Let T be the polar reciprocal of C, with respect to the Euclidean unit circle, rotated througth $-\pi/2$. The curve T is a convex closed curve called *isoperimetrix* as it solves the isoperimetric problem [3].

Following Guggenheimer [3], there will be denoted by $\boldsymbol{C}$ a vector from 0 to a point on C and by $\boldsymbol{T}(\boldsymbol{C})$ the vector from 0 to a point on T, such that $[\boldsymbol{T}(\boldsymbol{C}), \boldsymbol{C}] = 1$, where [,] denotes the determinant.

Let $\boldsymbol{x} = \boldsymbol{x}(t)$ be the equation of a curve Γ of class $\mathcal{C}^2$. There will be assumed throughout that all the curves to be considered are regular, i.e. $d\boldsymbol{x}/dt$ is nowhere 0. Then the *Minkowski arc length* of Γ is the new parameter σ defined up to an additive constant by

$$\|d\boldsymbol{x}/d\sigma\| = 1.$$

From now on we shall denote differentiation with respect to σ by a dot.

Let us assume that the curve Γ is parametrized by the Minkowski arc length σ. Denoting the vector $\dot{\boldsymbol{x}}(\sigma)$ by $\boldsymbol{t}(\sigma)$ and the vector $\boldsymbol{T}(\dot{\boldsymbol{x}}(\sigma))$ by $\boldsymbol{n}(\sigma)$, we have the following formulas [3]:

$$\frac{d\,\boldsymbol{t}(\sigma)}{d\sigma} = -h\,\boldsymbol{n}(\sigma), \quad \frac{d\,\boldsymbol{n}(\sigma)}{d\sigma} = k\,\boldsymbol{t}(\sigma). \tag{2}$$

Following Biberstein [1], we shall call h the *curvature* of Γ and k the *anticurvature* of Γ.

Let us denote by α the Minkowski arc length of T and by π the area enclosed by T. Then $0 \le \alpha \le 2\pi$, [3]. We parametrize T with the positive orientation by means of Minkowskian arclength α. Then the vectors $\boldsymbol{C}$ and $\boldsymbol{T}(\boldsymbol{C})$ are uniquely determined by the corresponding value of α, once we have fixed a zero direction, and will be denoted by $\boldsymbol{C}(\alpha)$ and $\boldsymbol{T}(\alpha)$. Two couples of vectors $(\boldsymbol{T}(\alpha), \boldsymbol{C}(\alpha))$ and $(\boldsymbol{T}(\beta), \boldsymbol{C}(\beta))$ are related by [3]:

$$\begin{pmatrix} \boldsymbol{T}(\beta) \\ \boldsymbol{C}(\beta) \end{pmatrix} = \begin{pmatrix} cm(\beta,\alpha) & st(\alpha,\beta) \\ -sm(\alpha,\beta) & cm(\alpha,\beta) \end{pmatrix} \begin{pmatrix} \boldsymbol{T}(\alpha) \\ \boldsymbol{C}(\alpha) \end{pmatrix}. \tag{3}$$

As to the Minkowskian trigonometric function $sm(\alpha,\beta)$, $st(\alpha,\beta)$, $cm(\alpha,\beta)$, introduced by Guggenheimer in [3], we have the following relations:

$$\begin{cases} st(\beta,\alpha) = -st(\alpha,\beta) \\ sm(\beta,\alpha) = -sm(\alpha,\beta) \\ cm(\alpha,\beta)\, cm(\beta,\alpha) + sm(\alpha,\beta)\, st(\alpha,\beta) = 1\,, \end{cases} \tag{4}$$

$$\begin{cases} \dfrac{\mathrm{d}\, cm(\alpha,0)}{\mathrm{d}\alpha} = -sm(0,\alpha) \\ \dfrac{\mathrm{d}\, cm(0,\alpha)}{\mathrm{d}\alpha} = -\chi\, st(0,\alpha) \\ \dfrac{\mathrm{d}\, st(0,\alpha)}{\mathrm{d}\alpha} = cm(0,\alpha) \\ \dfrac{\mathrm{d}\, sm(0,\alpha)}{\mathrm{d}\alpha} = \chi\, cm(\alpha,0)\,, \end{cases} \tag{5}$$

where χ is the unimodular centro-affine curvature of T to the centre 0. Such Minkowskian trigonometric functions can be extended to every real value α and β, so that they become doubly periodic functions in α and β with period 2π.

§2. Guggenheimer has shown in [3] that the isoperimetrix plays a fundamental role in the Minkowski geometry.
In this section we shall use the isoperimetrix, instead of the unit circle, as carrier of a mapping which takes the place of the tangential mapping. Then, by using elementary geometrical arguments, we shall prove a statement which is the natural equivalent of the turning tangents theorem for the Euclidean curves.

Let Γ be an oriented curve of class $\mathcal{C}^2$ with Minkowskian length L, defined by the function $\boldsymbol{x}(\sigma)$ relative to the Minkowski arc length σ. We shall say that Γ is *positively oriented* if it is oriented according to the isoperimetrix T. Let us consider the mapping $\mathfrak{T} : \Gamma \to T$ which carries the point $P \in \Gamma$ to the endpoint of the vector $\boldsymbol{n}(\sigma) = \boldsymbol{T}(\boldsymbol{t}(\sigma))$, where $\boldsymbol{t}(\sigma)$ is the unit tangent vector to Γ at P. Let $\alpha(\sigma)$ be the Minkowskian arc length of the point $\mathfrak{T}(P) \in T$. The map $\mathfrak{T}$ is a continous mapping, whereas the function $\alpha(\sigma)$ is not continous. However the following Lemma shows that there exists a continous function closely related to $\alpha(\sigma)$.

LEMMA 1. *There exists a differentiable function* $\overline{\alpha}(\sigma)$ *such that* $\overline{\alpha}(\sigma) \cong \alpha(\sigma) \bmod 2\Pi$, *for every* $\sigma \in [0,L]$.

We omit the proof which does not differ essentially from that of the Euclidean case [2].

REMARK 1. The difference $\overline{\alpha}(L) - \overline{\alpha}(0)$ is independent of the choice of $\overline{\alpha}$. In fact, if $\widetilde{\alpha}(\sigma)$ is a function satisfying the requirements of Lemma 1, then $\widetilde{\alpha}(\sigma) - \overline{\alpha}(\sigma) = n(\sigma)2\Pi$. Since $n(\sigma)$ is a continous integer valued function, then it must be a constant. Therefore $\widetilde{\alpha}(L) - \widetilde{\alpha}(0) = \overline{\alpha}(L) - \overline{\alpha}(0)$.

PROPOSITION 1. *Let* Γ *be a curve of class* $\mathcal{C}^2$ *with anticurvature* $k(\sigma)$, *relative to the Minkowski arc length* σ. *Then* $k(\sigma) = d\overline{\alpha}(\sigma)/d\sigma$.

Proof. By using the formulas (3), where $\alpha = 0$ and $\beta = \overline{\alpha}(\sigma)$, we get

$$\boldsymbol{t}(\sigma) = \left(- sm(0, \overline{\alpha}(\sigma)),\ cm(0, \overline{\alpha}(\sigma))\right),$$

$$\boldsymbol{n}(\sigma) = \left(cm(\overline{\alpha}(\sigma), 0),\ st(0, \overline{\alpha}(\sigma))\right).$$

Therefore, from (5) we obtain

$$\frac{d\boldsymbol{n}(\sigma)}{d\sigma} = \frac{d\boldsymbol{n}}{d\overline{\alpha}}\frac{d\overline{\alpha}}{d\sigma} = \frac{d\overline{\alpha}}{d\sigma}\left(-sm(0, \overline{\alpha}(\sigma)),\ cm(0, \overline{\alpha}(\sigma))\right) = \frac{d\overline{\alpha}}{d\sigma}\boldsymbol{t}\ .$$

On the other hand, formulae (2) give $dn/d\sigma = k(\sigma)t$, so that $k(\sigma) = d\overline{\alpha}/d\sigma$ as required.

We shall now consider a more general class of curves. Let Γ be a curve of Minkowskian length L, defined by the function $x(\sigma)$ relative to the Minkowski arc length σ. Let us assume that the interval [0,L] is divided into subintervals by points $0 = a_0 < a_1 <..< a_m = L$ such that $x(\sigma)$ is of class $\mathcal{C}^2$ on each interval $[a_i, a_{i+1}]$, with $i = 0,1...,m-1$. The points of Γ corresponding to $\sigma = a_i$ will be referred to as *corners*.

Let us consider the set Δ defined by:

$$\Delta := \{(\sigma_1, \sigma_2) \in \mathbb{R}^2 : 0 \le \sigma_1 \le \sigma_2 \le L\} \setminus \{(\sigma_1, \sigma_2) \in \mathbb{R}^2 : \sigma_1 = \sigma_2 = a_i, \text{ with } i = 1, \ldots, m-1\}.$$

Let $\Phi : \Delta \to T$ be the map which carries the point (σ_1, σ_2) to the endpoint of the vector

$$T([x(\sigma_2) - x(\sigma_1)] / \|x(\sigma_2) - \dot{x}(\sigma_1)\|) \quad \text{or} \quad T(\dot{x}(\sigma_1)),$$

according as $\sigma_1 \neq \sigma_2$ or $\sigma_1 = \sigma_2$. Moreover, let us consider the function $\alpha(\sigma_1, \sigma_2)$ which is defined to be the Minkowski arc length of $\Phi(\sigma_1, \sigma_2) \in T$. It is clear that Φ is a continuous map, whereas $\alpha(\sigma_1, \sigma_2)$ is not a continuouos function. There exists nevertheless a continuous function closely related to $\alpha(\sigma_1, \sigma_2)$, as given by the following Lemma.

LEMMA 2. *There exists a continuous function* $\overline{\alpha}(\sigma_1,\sigma_2)$ *such that* $\overline{\alpha}(\sigma_1, \sigma_2) \cong \alpha(\sigma_1, \sigma_2) \bmod 2\Pi$, *for every* $(\sigma_1,\sigma_2) \in \Delta$.

We omit the proof since it differs slightly from that given by Chern in the Euclidean case [2].

In addition, we now assume that Γ is an oriented simple closed curve. We associate to the i-th corner of Γ a couple of vectors as follows:

$$t_i^+ := \lim_{\sigma \to a_i^+} \dot{x}(\alpha), \quad t_i^- := \lim_{\sigma \to a_i^-} \dot{x}(\alpha)$$

Where $lim_{\sigma \to a_0^-} \dot{x}(\sigma)$ and $lim_{\sigma \to a_m^+} \dot{x}(\sigma)$ are interpreted as $lim_{\sigma \to a_m^-} \dot{x}(\sigma)$ and $lim_{\sigma \to a_0^+} \dot{x}(\sigma)$ respectively.

Then we denote the vector $T(t_i^+)$ by n_i^+ and the vector $T(t_i^-)$ by n_i^-. Let us assume that $n_i^+ \neq n_i^-$. Such a couple of vectors divides the region enclosed by T into two sectors centred at 0. Let Ω_i be the sector relative to the arc described from the endpoint of n_i^- to the endpoint of n_i^+, according to the orientation of Γ. Let us denote twice the area of Ω_i by $|\omega_i|$. Then we may define what we mean by the "exterior angle" of Γ at the i-th corner.

DEFINITION 1. The *exterior angle* of Γ at the i-th corner is defined to be $\omega_i = +|\omega_i|$ or $\omega_i = -|w_i|$ according as Γ is positively or negatively oriented. Moreover, in the case where $n_i^+ = n_i^-$ we shall take $\omega_i = 0$.

We can now define the "rotation number" of a curve Γ.

DEFINITION 2. Let $\bar{\alpha}_i(\sigma)$ be the function defined in the Lemma 1, relative to the interval $[a_i, a_{i+1}]$, with $i = 0,..., m-1$. Then the *rotation number* of Γ is defined to be

$$n_\Gamma := \frac{1}{2\pi}\left\{\sum_{i=0}^{m-1}\left[\bar{\alpha}_i(a_{i+1}) - \bar{\alpha}_i(a_i)\right] + \sum_{i=0}^{m-1}\omega_i\right\}. \quad (6)$$

PROPOSITION 2. *The rotation number of an oriented closed curve Γ, consisting of a finite number of $\mathcal{C}^2$ arcs, is an integer. Moreover*

$$n_\Gamma := \frac{1}{2\pi}\left(\sum_{i=0}^{m-1}\int_{a_i}^{a_{i+1}} k(\sigma)\,d\sigma + \sum_{i=0}^{m-1}\omega_i\right), \quad (7)$$

where $k(\sigma)$ is the anticurvature of Γ.

Proof. The rotation number n_Γ may be rewritten as

$$n_\Gamma := \frac{1}{2\pi}\left\{\sum_{i=0}^{m-1}\left[\overline{\alpha}_{i-1}(a_i) - \overline{\alpha}_i(a_i) + \omega_i\right]\right\},$$

where $\overline{\alpha}_{-1}(\sigma_0)$ is interpreted as $\overline{\alpha}_{m-1}(a_m)$. Since $\overline{\alpha}_{i-1}(a_i) - \overline{\alpha}_i(a_i) \cong \omega_i \mod 2\pi$, there follows that n_Γ is an integer. By remark 1, n_Γ is also independent of the choice of $\overline{\alpha}$. Finally, formula (7) follows from Proposition 1.

Now we shall prove a result analogous to the turning tangents theorem for Euclidean curves.

THEOREM 1. *Let Γ be an oriented simple closed curve, consisting of a finite number of $\mathcal{C}^2$ arcs. Then $n_\Gamma = \pm 1$.*

Proof. Let r be a straight line which cuts Γ and let $P \in r \cap \Gamma$ be a point such that an half-line of r with endpoint P has no other points in common with Γ. Let us denote the unit vector parallel to such an half-line by $\boldsymbol{C}_p$. Since Γ has a finite number of corners we may assume that P is not a corner. Moreover, let us assume that Γ is defined by the function $\boldsymbol{x}(\sigma)$, relative to the Minkowski arc length σ counted from P.

Let $0 = a_0 < a_1 < .. < a_m = L$ be a partition of the interval [0, L], where L is the Minkowski length of Γ, such that $\boldsymbol{x}(\sigma)$ is of class $\mathcal{C}^2$ on each segment.

Let us consider a function $\overline{\alpha}(\sigma_1, \sigma_2)$ satisfying the requirements of Lemma 2. Since $\overline{\alpha}(\sigma_1, \sigma_2)$ is determined up to an integral multiple of 2Π, we can assume that $0 \le \overline{\alpha}(0, L) \le 2\pi$.

i) We will first prove that

$$\overline{\alpha}(L,L) - \overline{\alpha}(0, 0) = \pm 2\pi.$$

Let $\beta(\sigma) = \overline{\alpha}(\sigma, L) - \overline{\alpha}(0, L)$, with $\sigma \in [0, L]$. Since the vector $\boldsymbol{x}(L) - \boldsymbol{x}(\sigma)$ can never be parallel to $-\boldsymbol{C}_p$ and since $\beta(0) = 0$, we have $-2\pi < \beta(\sigma) < 2\Pi$, for every $\sigma \in [0, L]$. Thus, the absolute value $|\beta(L)|$ represents twice the area of the sector centred at 0 and relative to the arc of T described from $\Phi(0, L)$ to $\Phi(L, L)$, according to the orientation of Γ.

Further, let $\gamma(\sigma) - \overline{\alpha}(0,\sigma) - \overline{\alpha}(0,0)$, with $\sigma \in [0,L]$. Since $x(\sigma) - x(0)$ can never be parallel to C_p and since $\gamma(0) = 0$, we have $-2\pi < \gamma(\sigma) < 2\pi$ for every $\sigma \in [0,L]$. Therefore, the absolute value $|\gamma(L)|$ represents twice the area of the sector centred at 0 which is the complement of the sector considered above. Moreover, the sign of $\gamma(L)$ is the same as that of $\beta(L)$. Thus $\overline{\alpha}(L,L) - \overline{\alpha}(0,0) = \beta(L) + \gamma(L) = \pm 2\Pi$.

ii) Let us define, for $i = 1, 2, \ldots, m-1$,

$$\overline{\alpha}_{+}(a_i, a_i) = \lim_{\sigma \to a_i^+} \overline{\alpha}(\sigma, \sigma),$$

$$\overline{\alpha}_{-}(a_i, a_i) = \lim_{\sigma \to a_i^-} \overline{\alpha}(\sigma, \sigma)..$$

We shall prove that

$$\omega_i = \overline{\alpha}_{+}(a_i, a_i) - \overline{\alpha}_{-}(a_i, a_i),$$

where ω_i is the exterior angle of Γ at the i-th corner.

Let us consider the points of Γ corresponding to the values $a_i - \varepsilon, a_i, a_i + \varepsilon$, where we choose $\varepsilon > 0$ so small that $a_{i-1} \notin [a_i - \varepsilon, a_i]$ and $a_{i+1} \notin [a_i, a_i + \varepsilon]$.

For simplicity, we assume that Γ is positively oriented; if not an analogous proof will work. Since $\overline{\alpha}(\sigma_1, \sigma_2)$ is a continous function we may choose $\varepsilon > 0$ so that $\overline{\alpha}(a_i, a_i + \varepsilon) - \overline{\alpha}(\sigma, a_i + \varepsilon) < 2\pi$, for $a_i - \varepsilon < \sigma < a_i$. Thus $\overline{\alpha}(a_i, a_i + \varepsilon) - \overline{\alpha}(a_i - \varepsilon, a_i + \varepsilon)$ represents twice the area of the sector centred at 0 and relative to the arc of T described from $\Phi(a_i - \varepsilon, a_i + \varepsilon)$ to $\Phi(a_i, a_i + \varepsilon)$ in the positive sense. Similary, $\overline{\alpha}(a_i - \varepsilon, a_i + \varepsilon) - \overline{\alpha}(a_i - \varepsilon, a_i)$ represents twice the area of the sector centred at 0 and relative to the arc of T described from $\Phi(a_i - \varepsilon, a_i)$ to $\Phi(a_i - \varepsilon,$

$a_i + \varepsilon$) in the positive sense. Therefore, $\overline{\alpha}(a_i, a_i + \varepsilon) - \overline{\alpha}(a_i - \varepsilon, a_i)$ represents twice the area of the sector centred at 0 and relative to the arc of T described from $\Phi(a_i - \varepsilon, a_i)$ to $\Phi(a_i, a_i + \varepsilon)$ in the positive sense. Moreover, since $\overline{\alpha}(a_i, a_i + \varepsilon) \to \overline{\alpha}_+(a_i, a_i)$ and $\overline{\alpha}(a_i - \varepsilon, a_i) \to \overline{\alpha}_-(a_i, a_i)$ as $\varepsilon \to 0$, by Definition 1 we get the required formula:

$$\omega_i = \overline{\alpha}_+(a_i, a_i) - \overline{\alpha}_-(a_i, a_i).$$

iii) Finally, we shall prove that $n_\Gamma = \pm 1$. By definition 2, noting that $\sigma = 0$ is not a corner of Γ, we have

$$n_\Gamma := \frac{1}{2\pi}\left\{\sum_{i=0}^{m-1}\left[\overline{\alpha}_i(a_{i+1}) - \overline{\alpha}_i(a_i)\right] + \sum_{i=0}^{m-1}\omega_i\right\}.$$

Since $\overline{\alpha}(\sigma, \sigma) \cong \overline{\alpha}_i(\sigma) \bmod 2\pi$, for $\sigma \in (a_i, a_{i+1})$, by Remark 1 there follows

$$n_\Gamma = \frac{1}{2\pi}\left\{\overline{\alpha}(L, L) - \overline{\alpha}(0, 0) + \sum_{i=0}^{m-1}\left[\overline{\alpha}_-(a_i, a_i) - \overline{\alpha}_+(a_i, a_i) + \omega_i\right]\right\}.$$

Thus, by (i) and (ii) we get

$$n_\Gamma = \frac{1}{2\pi}\left[\overline{\alpha}(L, L) - \overline{\alpha}(0, 0)\right] = \pm 1,$$

as required.

Taking proposition 2 into account, this theorem has the immediate.

COROLLARY. *Let be Γ an oriented simple closed curve, consisting of a finite number of $\mathfrak{C}^2$-arcs. Then*

$$\frac{1}{2\pi}\left\{\sum_{i=0}^{m-1}\int_{a_i}^{a_{i+1}} k(\sigma)\,d\sigma + \sum_{i=0}^{m-1}\omega_i\right\} = \pm 1. \tag{8}$$

§3. We shall first recall some notions and results on integral geometry in the Minkowski plane, which are developed in [4].

Let Γ be a curve of class $\mathcal{C}^2$ with anticurvature $k(\sigma)$ relative to the Minkowski arc length σ and let Γ' be a curve of the same class with anticurvature k'. We shall say that Γ' is *congruent* to Γ if and only if $k' = k$ as functions of their restective Minkowskian arc length. This notion may be used to obtain a congruence relation for convex sets. In the following we shall use, except where explicitly mentioned, the expression "convex set" to mean "bounded convex set having interior points and a boundary of class $\mathcal{C}^2$". Then we shall say that a convex set K' is *congruent* to a convex set K if and only if their boundaries are congruent according to the above definition. As in the Euclidean case, a class of congruent convex sets will be also referred to as "moving" convex set.

Let K be a convex set and let ∂K denote the boundary of K. By a Biberstein's theorem [1], the convex set K is uniquely determined into its congruence class by the position of a point $P(x, y)$ fixed on ∂K and by the value α relative to the unit tangent vector $C(\alpha)$ to ∂K at P. Then in order to measure sets of convex sets congruent to K we can introduce the *kinematic density* as

$$dK = dx \wedge dy \wedge d\alpha \tag{9}$$

Let us consider two convex sets K and K_0 having areas S, S_0 and Minkowskian perimeters L, L_0 respectively. Following Guggenheimer [3], we take as area of a convex set its affine area. In [4] we have proved that the measure $\mu(K;\ K \cap K_0 \neq \emptyset)$ of the set of convex sets congruent to K and intersecting K_0 is given by

$$\mu(K;\ K \cap K_0 \neq \emptyset) = 2\pi(S_0 + S) + L_0 L^*\,, \tag{10}$$

where L^* is the Minkowskian perimeter of the set K^* obtained by reflecting K in a point. In particular if K_0 shrinks to a point P, the measure $\mu(K;\ P \in K)$ of the set of convex sets congruent to K and containing P is

$$\mu(K; P \in K) = 2\pi S \tag{11}$$

We shall now generalize this last statement.

THEOREM 2. *Let K be a convex set of area S and Minkowskian perimeter L and let $P_1, \ldots, P_N$ be N fixed points in the plane. Denote by n the number of such points which are covered by the "moving" set K. Then.*

$$\int n \, dK = 2\pi N S, \tag{12}$$

where the integral is taken over all the points P(x,y) in the plane and all the valules of α, with $0 \le \alpha \le 2\pi$.

Proof. Let us consider the curve T_ε^j with $j = 1, \ldots, N$, obtained by transforming the isoperimetrix T by an homothety of ratio ε and by a translation which carries the origin 0 to the point P_j. Let us denote by K_ε^j the convex region enclosed by T_ε^j We choose the value ε so small that $K_\varepsilon^j \cap K_\varepsilon^i = \emptyset$ for $i \neq j$. Moreover, all the curves T_ε^j are assumed to be positively oriented as well as ∂K.

Let Γ_ε^j denote the boundary of $K \cap K_\varepsilon^j$ and let n_ε^j be the rotation number of Γ_ε^j if $\Gamma_\varepsilon^j \neq \emptyset$ or $n_\varepsilon^j = 0$ otherwise. From Theorem 1 there follows that $n_\varepsilon^j = 1$ if $\Gamma_\varepsilon^j \neq \emptyset$. Therefore the value $n_\varepsilon = \sum_{j=1}^{N} n_\varepsilon^j$ gives the number of the curves T_ε^j intersected by K. Moreover $n_\varepsilon \to n$ as $\varepsilon \to 0$. To prove the result it is therefore sufficient to show that

$$\int n_\varepsilon \, dK \to 2\pi N S$$

as $\varepsilon \to 0$.

Since T_ε^j and ∂K are $\mathcal{C}^2$-arcs, the curve Γ_ε^j has only two corners. Let us denote the exterior angle at such corners by ω_1^j and ω_2^j.
Let $k_j(\sigma_j)$, $k(\sigma)$ be the anticurvature of T_ε^j and ∂K respectively, where σ_j and σ_j are the Minkowskian arc lengths. Then by (7) we have

$$\int n\ \varepsilon dK = (1/2\Pi)\sum_{j=1}^{N}\Big(\int_{\{K\cap T_\varepsilon^j\neq\varnothing\}} k_j(\sigma_j)\, d\sigma_j\, dK +$$

$$\int_{\{K_\varepsilon^j\cap\partial K\neq\varnothing\}} k(\sigma) d\sigma dK + \int_{\{\partial K\cap T_\varepsilon^j\neq\varnothing\}} \omega_1^j\, dK + \int_{\{\partial K\cap T_\varepsilon^j\neq\varnothing\}} \omega_2^j\, dK\Big)$$

Let us consider the integral

$$I_1^j = \int_{\{K\cap T_\varepsilon^j\neq\varnothing\}} k_j(\sigma_j)\, d\sigma_j\, dK.$$

Fix a point P of T_ε^j, then by (11) and (8) we get

$$I_1^j = \int_{T_\varepsilon^j} k_j(\sigma_j)\, d\sigma_j \int_{P\in K} dK = (2\Pi)^2 S. \tag{13}$$

Let us now consider the integral

$$I_2^j = \int_{\{K_\varepsilon^j\cap\partial K\neq\varnothing\}} k(\sigma)\, d\sigma\, dK = \int_{\{K_\varepsilon^j\cap K\neq\varnothing\}} dK \int_{Q\in K_\varepsilon^j} k(\sigma)\, d\sigma,$$

where Q is a point of ∂K.
Denoting the maximun of $k(\sigma)$ by $\bar{k}$ we have

$$0 \leq I_2^j \leq \bar{k} \int_{\{K_\varepsilon^j\cap K\neq\varnothing\}} dK \int_{Q\in K_\varepsilon^j} d\sigma.$$

Since $\int_{Q\in K^j_\varepsilon} d\sigma$ represents the length of $\partial K\cap K^j_\varepsilon$, by convexity of K and K^j_ε, recalling that the length of T_ε^j is $2\pi\varepsilon$, we have:

$$0 \leq \int_{Q\in K_\varepsilon^j} d\sigma \leq 2\pi\varepsilon.$$

Therefore, by (10) we get

$$0 \leq I_2^j \leq 4\pi^2\bar{k}(\pi\varepsilon^3 + L^*\varepsilon^2 + S\varepsilon). \tag{14}$$

Finally, let us consider the integral

$$I_3^j = \int_{\{\partial K\cap T_\varepsilon^j\neq\varnothing\}} \omega_1^j\, dK.$$

Let us denote by $C(\beta_j), C(\beta)$ the unit tangent vectors to T^j_ε and ∂K at $T^j_\varepsilon \cap \partial K$, respectively. Then the density (9) may be written as [4]

$$dK = |sm(\beta_j, \beta)|\, d\sigma_j\, d\sigma\, d\alpha\,,$$

so that the integral $I^j{}_3$, becomes

$$I^j_3 = \int_{\{\partial K \cap T^j_\varepsilon \neq \emptyset\}} \omega^j_1\, |sm(\beta_j, \beta)|\, d\sigma_j\, d\sigma\, d\alpha\,.$$

If we fix σ_j and σ, then β_j becomes a constant and β differs from α by a constant. Thus $d\alpha = d\beta$.

Moreover, if we assume that the origin of the arc length α of T coincides with β_j then by definition 1 we have $\omega^j{}_1 = \beta$, so that

$$I^j_3 = \int_0^{2\Pi\varepsilon} d\sigma_j \int_0^L d\sigma \int_0^{2\Pi} \beta |sm(\beta_j, \beta)|\, d\beta\,.$$

Writing $M = max_\beta\, |sm(\beta_j, \beta)|$, we have $0 \leq \beta\, |sm(\beta_j, \beta)| \leq 2\pi M$. So that

$$0 \leq I^j_3 \leq (2\pi)^3\, M\, L\, \varepsilon. \tag{15}$$

Similary, we obtain

$$0 \leq I^j_4 = \int_{\{\partial K \cap T^j_\varepsilon \neq \emptyset\}} \omega^j_2\, dK \leq (2\pi)^3\, M\, L\, \varepsilon\,. \tag{16}$$

Therefore, as $\varepsilon \to 0$ formulae (13), (14), (15) and (16) give the required result.

§4. In this section we shall apply formula (12) to problems involving lattices of points. First we recall the notion of lattice of fundamental regions.

DEFINITION 3. A *lattice of fundamental regions* is a sequence $\{\mathcal{A}_m\}_{m\in\mathbb{N}}$ of regions $\mathcal{A}_m$ such that:

i) every point of the plane belongs exactly to one region $\mathcal{A}_m$;

ii) every region $\mathcal{A}_m$ can be transformed into the region $\mathcal{A}_0$ by a translation $\mathcal{T}_m$ which transforms any $\mathcal{A}_i$ into another $\mathcal{A}_j$, i.e. a translation which leaves the lattice invariant as a whole.

The region $\mathcal{A}_0$ will be referred to as the *fundamental cell* of the lattice.

Let K_0 be a convex set contained in $\mathcal{A}_0$ an let K be a "moving" convex set. Further, let $f(K_0 \cap K)$ be a real-valued function of the intersection $K_0 \cap K$ such that $f(\varnothing) = 0$ and $f(\mathcal{T}(K_0 \cap K)) = f(K_0 \cap K)$ for any translation $\mathcal{T}$ of the plane. In [5] we have proved

$$\int_{\{K_0 \cap K \neq \varnothing\}} f(K_0 \cap K)\, dK = \int_{\mathcal{A}_0} \left[\sum_{m \in \mathbb{N}} f(\mathcal{T}_n K \cap K) \right] dK\,, \tag{17}$$

where the second integral is taken over the positions of K for which the position point P of K belongs to $\mathcal{A}_0$ and $0 \leq \alpha \leq 2\pi$. It is easily seen that formula (17) also holds in the case where K_0 consists of a finite number of points.

DEFINITION 4. Let $\{\mathcal{A}_m\}_{m \in \mathbb{N}}$ be a lattice of fundamental regions. A *lattice of points* is a set $\mathfrak{L}$ of points such that

i) for every m the set $\mathfrak{L} \cap \mathcal{A}_m$ consist of a finite number of points which does not depend on m;

ii) for every m the set $\mathfrak{L} \cap \mathcal{A}_m$ can be transformed into the set $\mathfrak{L} \cap \mathcal{A}_0$ by a translation $\mathcal{T}_m$ which leaves the lattice $\{\mathcal{A}_n\}_{n \in \mathbb{N}}$ invariant as a whole.

If $\mathfrak{L} \cap \mathcal{A}_0$ consists of N points we shall denote such a lattice of points by $\mathfrak{L}(N)$.

We can now prove

THEOREM 3. *Let $\mathfrak{L}(N)$ be a lattice of points and let K be a convex set of area S. Denote by n the number of the points of*

$\mathfrak{L}(N)$ which are covered by the "moving" set K. Then the mean value of n is given by

$$E(n) = \frac{NS}{A_0}, \tag{18}$$

where A_0 is the area of the fundamental cell $\mathcal{A}_0$.

Proof. By formulas (12) and (17), where K_0 is identified with the set $\mathfrak{L}(N) \cap \mathcal{A}_0$, we get

$$\int_{\mathcal{A}_0} n \; dK = 2\pi NS .$$

On the other hand, we have

$$\int_{\mathcal{A}_0} dK = 2\pi A_0 ,$$

so that

$$E(n) = \frac{\int_{\mathcal{A}_0} n \; dK}{\int_{\mathcal{A}_0} dK} = \frac{NS}{A_0}$$

as required.

As a consequence of the previous theorem we have the following.

Blichfeldt's Theorem. *There always exist translates of K which contain $[NS/A_0] + 1$ points of the lattice $\mathfrak{L}(N)$, where $[x]$ denotes the integral part of x.*

This result follows from the same arguments used in the Euclidean case (cfr.[6], pg.137).

THEOREM 4. *Let K be a convex set in the symmetric Minkowski plane. Then it is possible to put n points inside K so that the minimal distance d between two of them is greater than*

$[2S/(A_c n)]^{1/2}$, where S denotes the area of K and A_c the area of the region enclosed by the indicatrix.

Proof. By a Sayrafiezadhe's result [7], to any oval af area A there exists a circumscribed parallelogram having area $\mathfrak{J}$ with $\mathfrak{J} > A > \mathfrak{J}/2$.

Let us assume that such an oval coincides with the curve C_r image of the indicatrix in a homothety of ratio r. Then there exists a parallelogram circumscribed with C_r having area $\mathfrak{J}$ with

$$\mathfrak{J} > A_c r^2 > \frac{\mathfrak{J}}{2} . \qquad (19)$$

Such a parallelogram may be chosen as the fundamental cell of a lattice. Moreover, let us consider the lattice of points consisting of the vertices of the parallelograms of the above lattice. By (18), where in this case $N = 1$, we have

$$E(n) = \frac{S}{\mathfrak{J}} .$$

Denote by d the minimal distance between the points of the lattice. Since $d \geq 2r$, we have

$$E(n) = \frac{S}{\mathfrak{J}} > \frac{S}{2A_c r^2} \geq \frac{2S}{A_c d^2}$$

Therefore, there exists a position of K where it contains n points so that

$$d^2 > \frac{2S}{A_c n}$$

as required.

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