STUDY OF SOME PROPERTIES OF ANALYTIC FUNCTIONS (*)

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1. - DEFINITION OF ANALYTIC FUNCTIONS

Reminder. - The field C is an algebra over the field R. The elements 1,i form a basis of C over R.

An analytic function f, may be regarded as a map : $(x,y) \to u(x,y) + iv(x,y) = f(x,y)$, from R^2 into the algebra C, such that,

$$iD_x f - D_y f = (iD_x - D_y)f = 0$$

This formula remains meaningful when f is a distribution.

We are going to denote by K a field equal to R orto C; by A a commutative algebra over K (K-algebra) with unit element, so that K may be identified with a subalgebra of A; by D, the space of the k-valued C^∞ funtions with compact support defined in R^n, equipped with the usual topology (Ref. 1, pag 67); and by D' the topological dual space of D, called the space of distributions.

Let us fix a base $(a_\lambda)_{\lambda \in L}$ of A. Let us denote pr_μ the map $\sum_\lambda z_\lambda a_\lambda \longrightarrow z_\mu$ from A onto K.

Let L^* be the space of all K-linear maps T of D into A, such thar $pr_\lambda \circ T \in D$ for every $\lambda \in L$.

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The map

$$D' \times A \longrightarrow L^*$$

$$(S, a) \longrightarrow \{g \longrightarrow \langle S, g\rangle a\}$$

is bilinear, hence it defines a linear map h

$$h : D' \otimes A \longrightarrow L^*$$

$$S \otimes a \longrightarrow \{g \longrightarrow \langle S, g\rangle a\}$$

(Ref. 2, pag 7, scholie). Let us denote $D' \otimes A$ by $D'(A)$.

Proposition 1,1. a) h is injective. b) if $\dim_K A$ is infinite h is not surjective.

Proof. a) Every element $u \in D'(A)$ may be written in the form $\sum_\lambda T_\lambda \otimes a_\lambda$ (Loc. cit. Pag. 10, cor. 1), where $T_\lambda = 0$ except for a finite number of indices. Then, if we have $\sum \langle T_\lambda, g\rangle a_\lambda = 0$ for every $g \in D$, this implies $\langle T_\lambda, g\rangle = 0$, for every g and for every λ, which implies $T_\lambda = 0$ for every λ.

b) For every positive integer $\nu > 0$, let B_ν be the ball of radius ν and center 0 of R^n, and let $(a_{\lambda_\nu})_{\nu \in N}$ be a sequence of different elements of the basis $(a_\lambda)_{\lambda \in L}$. Let T_ν be a sequence of elements of D' such that supp. $T_\nu \subset \complement \bar{B}_\nu \cap \mathring{B}_\nu$.

The map : $g \longrightarrow \sum_\nu \langle T_\nu, g\rangle a_{\lambda_\nu}$

of D into A belongs to L^* but is not the image by h of any element $\sum T_\lambda \otimes a_\lambda$ since one of these elements contains only a finite number of a_λ. ∎

Remarks. 1) The proposition 1,1 allows us to identify $D'(A)$ with a subspace of L^*.

2) Henceforth, we will write Sa instead of $S \otimes a$, for $S \in D'$ and $a \in A$.

If E is a ring let, $E[X_1 \dots X_n]$ be the ring of polynomials in the letters $X_1 \dots X_n$ with coefficients in E. Then we can consider $K[X_1 \dots X_n]$ as a subring of $A[X_1 \dots X_n]$.

By putting $X_j Ta = D_{x_j} Ta$, $(T \in D', a \in A)$, we define on $D'(A)$ a structure of module over the ring $A[X_1 \dots X_n]$.

The elements Ta of $D'(A)$ will be called distributions with values in A or simply distributions. $X_i Ta$ will be called the i-th partial derivate of Ta.

Let $\lambda = (\lambda_1, \dots, \lambda_n)$ be a sequence of n distinct indices and $\Lambda \subset (1,n) \times (1,n)$.

Definition 1. 1. We will say that an element $T \in D'(A)$ is analytic for the couple (λ, Λ), or simply analytic, if the annihilator of T in the $A[X_1 \dots X_n]$-module $D'(A)$ includes the ideal $\mathfrak{a}$ of $A[X_1 \dots X_n]$ generated by the polynomials $a_{\lambda_j} X_i = a_{\lambda_i} X_j$, $(i,j) \in \Lambda$.

Definition 1.2. Let T be analytic for the couple (λ, Λ). If the a_{λ_i} $(1 \leq i \leq n)$ are invertible in A, and if from the relation

$$\frac{X_j T}{a_{\lambda_j}} = \frac{X_i T}{a_{\lambda_i}}$$

for $(i,j) \in \Lambda$, we can conclude that the same relation holds for each $(i,j) \in (1,n) \times (1,n)$, then we define the derivative of T as

$$\frac{dT}{dz} = \frac{X_i T}{a_{\lambda_i}}$$

z being defined as $z = x_1 a_{\lambda_1} + \dots + x_n a_{\lambda_n}$.

Example 1. 1. We can take in definition 1, 1: $K = R$; $A = C$; $n = 2$; $L = \{1, 2\}$; $(a_\lambda)_{\lambda \in L} = \{a_1 = 1,\ a_2 = i\}$; $\lambda = (1,2)$; $\Lambda = (1,2)^2$. How if we take an analytic element of the couple (λ, Λ) in the sense of the definition 1. 1 we have got an analytic function in the usual sense.

Example 1. 2. Let A be an algebra over R, and let $g_1 \dots g_n$ be n linearly independent elements of A. Let $f : R^n \longrightarrow A$ be defined by $f(x_1 \dots x_n) = x_1 g_1 + \dots + x_n g_n$. Then,

$$(X_i g_j = X_j g_i)\ f = g_i g_j - g_i g_j = 0$$

So, f is analytic for the couple λ, Λ where $\lambda = (1, 2, \ldots n)$, $\Lambda = \lambda + \lambda$
Furthermore, if the g_i $(1 \leq i \leq n)$ are invertible in A, the derivate of f is,

$$\frac{df}{dz} = 1 \text{ (u unit element of A).}$$

Example 1. 3. We can take the example 1. 2. $A = R^{R}$,

$$g_j(x) = \begin{cases} 1 \text{ if } x \leq j \\ 2 \text{ if } x > j \end{cases}$$

g_j has an inverse :

$$g_j^{-1}(x) = \begin{cases} 1 \text{ if } x \leq j \\ \frac{1}{2} \text{ if } x > j \end{cases}$$

The unit element of A is the function equal to 1.

2 - ANALYTIC DISTRIBUTIONS WITH VALUES IN A QUOTIENT ALGEBRA.

Reminder. It is possible to define the R-algebra C in the following way : consider the homogeneus polynomial

$$P(X_1, X_2) = X_1^2 + X_2^2 \in R[X_1, X_2]$$

and the ring $R[Z_2]$, Then $A = C$ is isomorphic to $R[Z_2] / (P(1, Z_2))$.

Let $\quad \phi : R[Z_2] \longrightarrow R[Z_2] / (P(1, Z_2))$

be the canonical epimorphism, let $\phi(Z_2) = g_2$ which is usually denoted by i. We have $P(1, g_2) = g_2^2 + 1 = 0$, in A.

In this case

$$X_1^2 + X_2^2 = -(X_1 g_2 - X_2)(X_1 g_2 + X_2)$$

or in other words, $X_1^2 + X_2^2$, which may be regarded as an element of $A[X_1, X_2]$, belongs to the ideal $(X_1 g_2 - X_2) \subset A$.

From this fact we can prove that if $f(x, y) = u(x, y) + iv(x, y)$

is analytic, then

$$(D_x^2 + D_y^2)\, u = (D_x^2 + D_y^2)\, v = 0$$

Let

(1) $P(X_1, \ldots X_n) = \sum_{p=0}^{m} \alpha_{1p} X_1^{m-p} X_2^{p} + \ldots + \sum \alpha_{n,p} X_1^{m-p} X_n^{p}$, with $\alpha_{j,p} \in K$ $(2 \leq j \leq n,\ 0 \leq p \leq m)$. Suppose $\alpha_{j,m} = 1$ for $2 \leq j \leq n$. Let us write

(2) $P_j(X_1, X_j) = \sum_{p=0}^{m} \alpha_{j,p} X_1^{m-p} X_j^{p}$ $(2 \leq j \leq n)$,

so that

(3) $P = \sum_{j=2}^{m} P_j$.

Let

$A = K[Z_2, \ldots, Z_n] / (P(1, Z_2, \ldots, Z_n))$, Φ the canonical epimorphism from $K[Z_2, \ldots, Z_n]$ onto A, and $\Phi(Z_j) = g_j$, $(2 \leq j \leq n)$. We have

(4) $P(1, g_2, \ldots, g_n) = 0$

Lemma 2.1. If we put

(5) $P_j^*(X_1, X_j) = P(X_1, g_2 X_1, \ldots, g_{j-1} X_1, X_j, g_{j+1} X_1, \ldots, g_n X_1)$

then we have

(6) $P = \sum_{j=2}^{n} P_j^*$

Proof. $P_j^*(X_1, X_j) = P_j(X_1, X_j) + \sum_{k \geq 1} P_k(X_1, g_k X_1)$. Then

$$\sum_{j=2}^{n} P_j^*(X_1, X_j) = \sum_{j=2}^{n} P_j(X_1, X_j) + \sum_{j=2}^{n} \Big(\sum_{\substack{k \geq 2 \\ k \neq j}} P_k(X_1, g_k X_1) \Big) =$$

$$= P(X_1, \ldots, X_n) + (n-2)\, P(1, g_2, \ldots, g_n)\, X_1^m = P(X_1, \ldots, X_n)$$

since from (4), $(n-2)\, P(1, g_2, \ldots, g_n)\, X_1^m = 0$. ∎

Lemma 2. 2. The element $P_j^*(X_1, X_j) \in A[X_1, X_j]$ is divisible by $X_1 g_j - X_j$.

Proof. $P_j^*(X_1, X_j)$ may be considered as an element of the ring of polynomials in X_j with coefficients in the ring $A[X_1]$. If we replace in P_j^*, X_j by $X_1 g_j$, we get

$$P_j^*(X_1, X_1 g_j) = P(1, g_2, \ldots, g_n)\, X_1^m = 0$$

from (4). Then $(X_1 g_j - X_j)$ divides P_j^*. (Ref. 3, pag. 23, pro. 5).∎

Let $\mathfrak{a}$ be the ideal of A generated by the elements $X_1 g_j - X_j$ $(2 \le j \le n)$.

Proposition 2. 1. $P(X_1, \ldots, X_n)$ considered as an element of $A[X_1, \ldots, X_n]$ belongs to $\mathfrak{a}$.

Proof. Follows inmediately from lemmas 1 and 2.∎

Now, choose in A a basis $(a_\lambda)_{\lambda \in N}$, such that

$a_1 = 1$, $a_2 = g_2, \ldots, g_n = a_n$.

Let $\lambda = (1, \ldots, n)$, $\Lambda = \{1\} \times \lambda$. We then have

Corollary 2. 1. If $T = \sum T_\lambda a_\lambda$ is analytic for (λ, Λ) then

$$P(X_1, \ldots X_n) T_\lambda = 0, \text{ for each } \lambda \in N.$$

Proof. $0 = \sum_{j=2}^{n} P_j^*(X_1, X_j)\, T = P(X_1, \ldots, X_n)\, (\sum T_\lambda a_\lambda) =$

$= \sum_\lambda P(X_1, \ldots, X_n)\, T_\lambda a_\lambda$.

The conclusion follows by remarking that the a_λ are linearly independent.∎

Example 2. 1. Let us consider the polynomial

$P(X_1, \ldots, X_n) = X_1^m + \ldots + X_n^m$. Then

$A = K[Z_2, \ldots, Z_n] / (1 + Z_2^m + \ldots + Z_n^m)$

$1 + g_2^m + \ldots + g_n^m = 0$

$$P_j^*(X_1, X_j) = X_j^m + X_1^m (1 + \sum_{k \neq j} g_k^m) = X_j^m - g_j^m X_1^m =$$

$$= (X_j - g_j X_1)\, (\sum_{j=2}^{n} X_j^{m-1-k} X_1^k g_j^k) \qquad (2 \le j \le n)$$

$$P(X_1, \ldots, X_n) = \sum_{j=2}^{n} P_j^* = \sum_{j=2}^{n} (X_j - g_j X_1)\, (\sum_{k=0}^{m-1} X_j^{m-1-k} X_1^k g_j^k).$$

Example 2. 2. Let us consider the polynomial

$P(X_1, \ldots, X_n) = X_1^m - X_2^m - \ldots - X_n^m$. Then

$$A = K\left[Z_2, \dots, Z_m\right] / (1 - Z_2^m - \dots - Z_n^m)$$

$$1 - g_2^m - \dots - g_n^m = 0$$

$$P_j^*(X_1, X_j) = -X_j^m + X_1^m (1 - \sum_{\substack{k \neq j \\ k \geq 2}} g_k^m (= (X_1 g_j - X_j) \left(\sum_{k=0}^{m-1} X_1^k g_j^k \quad X_j^{m-1-k}\right)$$

$$P(X_1, \dots, X_n) = \sum_{j=2}^{n} P_j^* = \sum_{j=2}^{n} (X_1 g_j - X_j)\left(\sum_{k=0}^{m-1} X_1^k g_j^k X_j^{m-1-k}\right)$$

Remarks. Let us consider again the general case considered in definition 1, 1, where A, λ, Λ, were arbitrary. We get : if $P(X_1, \dots, X_n) \in K[X_1, \dots, X_n]$ belongs to the ideal $\mathfrak{a} \subset A[X_1, \dots, X_n]$ generated by the polynomials $a_{\lambda_i} X_j - a_{\lambda_j} X_i$, $(i, j) \in \Lambda$ then the elements $a_{\lambda_1}, \dots, a_{\lambda_n}$ are algebraic over K, since

$$P(a_{\lambda_1}, \dots, a_{\lambda_n}) = 0 .$$

Let $D^{(1)}$ be the subspace of D' of the continuously differentiable function.. $D^{(1)}$ is an algebra over K. $D^{(1)} \otimes A$ A may be identified with a linear subspace of $D' \otimes A$ (Ref. 2, pag. 11, cor. 3). $D^{(1)} \otimes A$ will be called the space of continuously differentiable functions with values in A. If $f \in D^{(1)} \otimes A$ is analytic, we will say that f is an analytic function.

By writing $(f \otimes a)(g \otimes b) = fg \otimes ab$ for all f and $g \in D^{(1)}$, a and $b \in A$, we define on $D^{(1)} \otimes A$ an structure of K-algebra (Ref.2 pag. 30).

The formula

$$(a_{\lambda_j} X_i - a_{\lambda_i} X_j)((f_k a_{\lambda_k})(f_\ell a_{\lambda_\ell})) = ((a_{\lambda_j} X_i - a_{\lambda_i} X_j) f_k a_{\lambda_k}) f_\ell a_{\lambda_\ell} + ((a_{\lambda_j} X_i - a_{\lambda_i} Z_j) f_\ell a_{\lambda_\ell}) f_k a_{\lambda_k}$$

for $f_k a_{\lambda_k}$ and $f_\ell a_{\lambda_\ell}$ belonging to $D^{(1)} \otimes A$ proves that the product of two analytic functions is an analytic function. Since the sum of two analytic functions is an analytic fuction we conclude that the set of all analytic functions forms a subalgebra $\mathcal{A}$ of $D^{(1)} \otimes A$.

Now, if with the notations of proposition 2, 1, we put

$$z(x_1, \dots, x_n) = x_1 + x_2 g_2 + \dots + x_n g_n$$

then the set of elements of the form $P(z)$, where $P(X) \in K[X]$ form a subalgebra of $\mathcal{A}$, which we will denote by $\mathcal{P}$.

3 - CAUCHY'S THEOREM

Notations. Notation and terminology used in this chapter are those of Chern (Ref. 4.). We consider differentiable chains in the sense of reference 5, page 27.

Remider. If f is an analytic function in the disk $|z| \leq r$, $r > 1$, then

$$\int_\gamma f(z)\,dz = 0$$

where γ is the circle $|z| = 1$.

The n functions

$$R^n \longrightarrow R$$

$$(x_j)_{1 \leq j \leq n} \longrightarrow x_i \qquad (1 \leq i \leq n)$$

define an structure of differentiable manifold on R^n. Let $x \in R^n$. Let $V^*(x)$ be the space of covectors at the poin x, $\Lambda^*(x)$ the exterior algebra of $V^*(x)$, $\Lambda(R^n) = \bigcup_{x \in R^n} \Lambda^*(x)$ the bundle of exterior forms.

By extension of the ring of scalars from R to A (Ref. 2, Pag. 20) we define the A-module $V^*_A(x) = A \otimes V^*(x)$. By the same procedure we can define from the R-algebra $\Lambda^*(x)$ the A-algebra $A \otimes \Lambda^*(x)$. (Re. 2, Pag. 36), and the bundle $\Lambda_A(R^n) = \bigcup_{x \in R^n} (A \otimes \Lambda^*(x))$.

Let us take a look at $A \otimes V^*(x)$ and $A \otimes \Lambda^*(x)$. The differentials dx_j $(1 \leq j \leq n)$ form a basis of $V^*(x)$. Let $(a_\lambda)_{\lambda \in L}$ be a basis of A. Then the elements

$$a_\lambda \otimes dx_i \quad \lambda \in L, \quad 1 \le i \le n$$

form a basis of the R-linear space $A \otimes V^*(x)$.

If $\overset{r}{\Lambda} V^*(x)$ is the space of (exterior) r-covectors, we know that $\Lambda^*(x)$ is direct sum of the $\overset{r}{\Lambda} V^*(x)$, $0 \le r \le n$.

The exterior r-covectors

$$dx_{i_1} \wedge \ldots \wedge dx_{i_r} \qquad i_1 < \ldots < i_r$$

form a basis of $\overset{r}{\Lambda} V^*(x)$. The elements

$$a_\lambda \otimes (dx_{i_1} \wedge \ldots \wedge dx_{i_r}), \quad \lambda \in L, \quad i_1 < \ldots < i_r \le n$$

form a basis of the R-linear space $A \otimes (\overset{r}{\Lambda} V^*(x))$. $A \otimes \Lambda^*(x)$ is direct sum of the $A \otimes \overset{r}{\Lambda} V^*(x)$, $0 \le r \le n$.

In the A-modules $A \otimes V^*(x)$, and $A \otimes \overset{r}{\Lambda} V^*(x)$ the multiplication by an scalar is given by

$$a(a_\lambda \otimes dx_i) = a\, a_\lambda \otimes dx_i, \qquad a \in A, \lambda \in L, \qquad 1 \le i \le n.$$

$$a(a_\lambda \otimes (dx_{i_1} \wedge \ldots \wedge dx_{i_r})) = a\, a_\lambda \otimes (dx_{i_1} \wedge \ldots \wedge dx_{i_r})$$

$$a \in A, \lambda \in L, \quad i_1 < \ldots < i_r.$$

Now we are in position to extend the definition of exterior differential form, exterior differentiation, and integral of an exterior differential form.

Definition 3.1 An exterior differential form of degree r and class 1 with coefficients in A (or simply an exterior differential form) is a map

$$\alpha_A : R^n \longrightarrow \Lambda_A(R^n)$$

$$x \longrightarrow \sum_{i_1 < \ldots < i_r} \alpha_{i_1, \ldots, i_r}(x_1, \ldots, x_n)\, dx_{i_1} \wedge \ldots \wedge dx_{i_r}$$

where the "coefficients" $\alpha_{i_1 \ldots i_r}(x_1, \ldots, x_n)$ belong to $D^{(1)} \otimes A$.

With the help of the basis $(a_\lambda)_{\lambda \in L}$ we can write

$$(1)\qquad \alpha_A = \sum_{\lambda, i_1 < \cdots < i_r} \alpha^{(\lambda)}_{i_1 \cdots i_r}(x_1 \ldots x_n) a_\lambda dx_{i_1} \wedge \ldots \wedge dx_{i_r} =$$

$$= \sum_\lambda a_\lambda \Big(\sum_{i_1 < \cdots < i_r} \alpha^{(\lambda)}_{i_1 \cdots i_r}(x_1 \ldots x_n) \; dx_{i_1} \wedge \ldots \wedge dx_{i_r} \Big)$$

where $\alpha^{(\lambda)}_{i_1 \cdots i_r}(x_1, \ldots, x_n) \in D^{(1)}$.

When α is a usual exterior differential form, we know the meaning of its exterior differential $d\alpha$, and integral $\int_\sigma \alpha$ (σ being a chain).

Definition 3. 2. The exterior differential of the exterior differential form (1) is defined by

$$(2)\qquad d\alpha_A = \sum_\lambda a_\lambda d\Big(\sum_{i_1 < \cdots < i_r} \alpha^{(\lambda)}_{i_1 \cdots i_r} \; dx_{i_1} \wedge \ldots \wedge dx_{i_r} \Big).$$

The integral over the chain σ is defined by

$$(3)\qquad \int_\sigma \alpha_A = \sum_\lambda a_\lambda \Big(\int_\sigma \sum \alpha^{(\lambda)}_{i_1 \cdots i_r} dx_{i_1} \wedge \ldots \wedge dx_{i_r} \Big).$$

Let us return to de notations of proposition 2.1, and consider the exterior differential form :

$$dz = (g_2 + \ldots + g_n) d\hat{x}_1 + d\hat{x}_2 + \ldots + d\hat{x}_n . \quad \text{where}$$

$$d\hat{x}_j = (-1)^j \; dx_1 \wedge \ldots \wedge dx_{i-1} \wedge dx_{i+1} \wedge \ldots \wedge dx_n \quad (2 \leq j \leq n)$$

$$d\hat{x}_1 = dx_2 \wedge dx_3 \wedge \ldots \wedge dx_n$$

Proposition 3.1. If $\phi \in D' \otimes A$, and σ is a chain, we have

$$(4)\qquad \int_{\partial\sigma} \phi \; dz \; v = \int_\sigma ((g_2 X_1 - X_2) + \ldots + (g_n X_1 - X_n)) \phi \, dx_1 \wedge \ldots \wedge dx_n$$

Proof. We have

$$d(\phi\, dz) = \frac{\partial \phi}{\partial x_1} dx_1 \wedge \ldots \wedge dx_n (g_2 + \ldots + g_n) +$$

$$+ \frac{\partial \phi}{\partial x_2} dx_2 \wedge dx_1 \wedge dx_3 \wedge \ldots \wedge dx_n + \ldots$$

$$+ \frac{\partial \phi}{\partial x_n} (-1)^n dx_n \wedge dx_1 \wedge dx_2 \wedge \ldots \wedge dx_{n-1} =$$

$$= ((g_2 X_1 - X_2) + \ldots + (g_n X_1 - X_n))\, \phi \;\; dx_1 \wedge \ldots \wedge dx_n$$

The conclusion follows from Stokes' formula

(4')
$$\int_{\partial\sigma} \alpha_A = \int_{\sigma} d\alpha_A$$

applied to

$$\alpha_A = \phi\, dz. \blacksquare$$

Corollary 3.1. If $F(x_1, \ldots, x_n) \in D^{(1)} \otimes A$ is analytic for the couple (λ, Λ) then

(5)
$$\int_{\partial\sigma} F dz \;\; v \;\; 0$$

for every chain σ .

4 - APPENDIX

The article "A Generalization of the Cauchy-Riemann Equations" by J. Horváth studies an analytic function by starting from an homogeneous polynomial

$$P(X_1, X_2) = \sum_{m=0}^{K} \alpha_m X_1^m X_2^{K-m}.$$

I was interested in doing the same work starting from a general homogeneous polynomial

$$P(X_1, X_2, \ldots, X_n) = \sum_{m_1+\cdots+m_n=m} \alpha_{m_1\cdots m_n} X_1^{m_1} \cdots X_n^{m_n}.$$

I found several problems which I could not solve.

Here are stated three of them, pointing out some difficulties to overcome.

Problem A.1. To prove proposition 2,1 we took a particular homogeneous polynomial

(1) $$P(X_1, \ldots, X_n) = P_2(X_1, X_2) + \ldots + P_n(X_1, X_n).$$

The question is whether proposition 2,1 remains true when we take a general homogeneous polynomial

(2) $$P(X_1, \ldots, X_n) = \sum_{m_1+\cdots+m_n=m} \alpha_{m_1\cdots m_n} X_1^{m_1} \cdots X_n^{m_n}.$$

Problem A.2. Suppose $n=2$ in (2). Then if E is a fundamental solution of the polynomial (2), and if we put

(3) $$P(X_1, X_2) = (g_2 X_1 - X_2) P^{**}(X_1, X_2)$$

then

(4) $$P^{**}(X_1, X_2) E$$

is a fundamental solution of the operator $g_2 X_1 - X_2$.

The question is how to generalize this fact to the case $n > 2$.

Problem A,3. For the case $n=2$ the following theorem is proved (Loc. Cit.) : Let f be analytic. Then we have

$$f(x_0, y_0) = - \int_{\gamma} E(x_0 - x, y_0 - y) f(x, y) dz$$

where γ is a simple closed piecewise differentiable curve containing the point (x_0, y_0) in its interior.

The proof of this theorem makes use of the fact that there exist a fundamental solution of the operator $g_2 X_1 - X_2$.

The problem is how to generalize this theorem to the case $n > 2$.

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