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A COUNTEREXAMPLE IN THE THEORY OF LINEAR SINGULARLY PERTURBED SYSTEMS.

por

Raúl NAULIN

RESUMEN. En esta nota se comparan las soluciones acotadas del sistema lineal singularmente perturbado $\varepsilon x' = A(t)x + f(t)$, con las soluciones del sistema algebráico $A(t)x + f(t) = 0$. Aquí A y f son funciones acotadas de clase C^1, con derivadas acotadas. Suponemos además que los valores propios de $A(t)$ satisfacen la condición $|\Re e\, \lambda(t)| \geq \gamma > 0$. Es sabido que para $f \in C^1$ y ε suficientemente pequeños vale la siguiente estimación: $\|k_\varepsilon(f) + A^{-1}f\| \leq \varepsilon L \|f\|_1$, donde $k_\varepsilon(f)$ denota la única solución acotada de $\varepsilon x' = A(t)x + f(t)$, $\|f\| = sup_{\mathbb{R}} |f(t)|$, $\|f\|_1 := \|f\| + \|f'\|$ y L es una constante que no depende de f ni de ε. Probaremos que esta estimación no puede ser extendida hasta $\|k_\varepsilon(f) + A^{-1}f\| \leq \varepsilon L \|f\|$. Además, si en lugar de exigir que A sea de clase C^1 pedimos que A sea una función de Lipschitz acotada, entonces sigue siendo válida la estimación $\|k_\varepsilon(f) + A^{-1}f\| \leq \varepsilon L \|f\|_1$.

ABSTRACT. In this note we compare the bounded solutions of the linear singularly perturbed system $\varepsilon x' = A(t)x + f(t)$, with the solutions of the algebraic system $A(t)x + f(t) = 0$. Here A and f are bounded C^1 functions with bounded derivatives. We assume that the eigenvalues of $A(t)$ satisfy $|\Re e\, \lambda(t)| \geq \gamma > 0$. It is known that for small ε, the following estimate is valid $\|k_\varepsilon(f) + A^{-1}f\| \leq \varepsilon L \|f\|_1$, where $k_\varepsilon(f)$ denotes the bounded solution of $\varepsilon x' = A(t)x + f(t)$, $\|f\| = sup_{\mathbb{R}} |f(t)|$, $\|f\|_1 := \|f\| + \|f'\|$ and L is a constant. We prove that this estimate cannot be replaced by $\|k_\varepsilon(f) + A^{-1}f\| \leq \varepsilon L \|f\|$. Futhermore, if, instead of the condition that A be C^1, we require that the function be bounded and Lips-

chitz continuous, we show that the same estimate, $\|k_\varepsilon(t) + A^{-1}f\| \leq \varepsilon L \|f\|_1$, can be obtained.

§1. Introduction. In what follows, for a bounded and continuous function $f : \mathbb{R} \to \mathbb{R}^n$, we define $\|f\| := sup \{|f(t)|; t \in \mathbb{R}\}$. If f has a bounded derivative f' we then define $\|f\|_1 := \|f\| + \|f'\|$. We will consider the problem of the existence of bounded solutions on $\mathbb{R}$ of the linear system:

$$\varepsilon x' = A(t)x + f(A), \quad x \in \mathbb{R}^n, \quad (x' := dx/dt) \tag{1}$$

where A is a bounded uniformly continuous function, f is a continuous and bounded function, and ε is a positive small parameter. The following theorem is well known (see [2]).

THEOREM 1. *If the eigenvalues $\lambda(t)$ of $A(t)$ satisfy $|\Re e \lambda(t)| \geq \gamma$, for all, t in $\mathbb{R}$, where $\gamma > 0$ is a constant, then there exist $\varepsilon_0 > 0$ and a positive constant K depending neither on $(0, \varepsilon_0]$ nor f, such that there exists a unique bounded solution on $\mathbb{R}$ of (1), denoted by $k_\varepsilon(f)$, and such that the following estimate holds:*

$$\|k_\varepsilon(f)\| \leq \varepsilon K \|f\|, \quad \varepsilon \in (0, \varepsilon_0] \tag{2}$$

In [2] it is shown that this bounded solution is obtained in the following way: the hypothesis of Theorem 1 assures that for small values of ε, the linear system

$$\varepsilon x' = A(t)x \tag{3}$$

has an exponential dichotomy, confirming the existence of a fundamental matrix $\Phi(t)$ of (3), a constant $H \geq 1$, a constant $\alpha > 0$, and a projection matrix P $(P = PP)$ such that for some small value of ε the following holds:

$$|\Phi(t)P\Phi^{-1}(s)|, \ |\Phi(s)(I-P)\Phi^{-1}(t)| \leq H e^{\alpha(s-t)}, \ t \leq s.$$

Let us construct the function $G(t,s) := \Phi(t)P\Phi^{-1}(s)$ for $t > s$, and $G(t,s) := \Phi(t)(I-P)\Phi^{-1}(s)$ for $t < s$. Then by a direct calculation it is possible to prove that the unique bounded solution $k_\varepsilon(f)$ of (3) is given by

$$k_\varepsilon(f) = \varepsilon^{-1}\int_{\mathbb{R}} G(t,s)\, f(s)\, ds \qquad (4)$$

The purpose of our note is to analyse the following result, whose proof we reproduce. here (see [2]).

THEOREM 2. *Let us assume the hypothesis of Theorem 1. Moreover, suppose that the functions A and f have continuous and bounded derivatives defined on* $\mathbb{R}$. *Then there exist* $\varepsilon_0 > 0$ *and a constant L not depending on* ε *or f, such that for* $\varepsilon \in (0, \varepsilon_0]$ *the following estimate holds:*

$$\|k_\varepsilon(f) + A^{-1}f\| \le \varepsilon L \|f\|_1 . \qquad (5)$$

Proof. From (1) and the definition of $k_\varepsilon(f)$ we have the identity:

$$\varepsilon(k_\varepsilon(f) + A^{-1}f)' = A(t)(k_\varepsilon(f) + A^{-1}f) - \varepsilon(A^{-1}f)'.$$

This identity shows that the function $k_\varepsilon(f) + A^{-1}f$ is a bounded solution of the equation (1) with the nonhomogeneous coefficient $\varepsilon(A^{-1}f)'$. Again the definition of $k_\varepsilon(f)$ leads us to the equality

$$k_\varepsilon(f) + A^{-1}f = k_\varepsilon(-\varepsilon(A^{-1}f)'). \qquad (6)$$

Now (5) follows from (6) and (2).

We will show that estimate (5) cannot be improved to give the following inequality:

$$\|k_\varepsilon(f) + A^{-1}f\| \le \varepsilon L \|f\|.$$

§2. A counterexample. Let us consider a numerical sequence $\alpha_n > 0$, $lim_{n\to\infty}\, \alpha_n = 0$. We define the following sequence of functions:

$$f_n(t) = \begin{cases} n & t < 0 \\ n - nt/\alpha_n & t \in [0, \alpha_n] \\ 0 & t > \alpha_n \end{cases} \qquad (8)$$

If we ask for the unique bounded solution, on $\mathbb{R}$, of the differential equation

$$\varepsilon x' = -x + f_n(t)\ , x \in \mathbb{R}, \tag{9}$$

Then a direct calculation shows that

$$k_\varepsilon(f_n)(t) = \begin{cases} n, & t < 0 \\ n\,(\alpha_n - t + \varepsilon(1 - e^{-t/\varepsilon}))/\,\alpha_n\,, & t \in [0, \alpha_n] \\ k_n\,(e^{-(t-\alpha_n)/\varepsilon} - e^{-t/\varepsilon})\,, & t > \alpha_n \end{cases} \tag{10}$$

Suppose now that (7) is true. Then there should exist a number $\varepsilon_0 > 0$ and a constant $L > 0$ such that for $\varepsilon \in (0,\ \varepsilon_0]$ and for any positive integer n, the following inequality is satisfied: $\|k_\varepsilon(f_n) + A^{-1}f_n\| \le \varepsilon L \|f_n\|$. Because $\|f_n\| = n$ this is equivalent to $|k_\varepsilon(f_n)(t) - f_n(t)| \le L\varepsilon n$, for any t on $\mathbb{R}$. If, we let in this last inequality, for n sufficiently large, $\varepsilon_n = t_n = \alpha_n$, then we will obtain $|k_{\alpha_n}(f_n)(\alpha_n) - f_{\alpha_n}(\alpha_n)| \le L\alpha_n n$. Using (8) and (10) we obtain: $1 - e^{-1} \le L\alpha_n n$, for n large. But $lim_{n\to\infty}\alpha_n = 0$, so that it follows that $1 \le e^{-1}$. This contradiction shows that, in general, (7) is not true.

The estimate (5) cannot be extended to the estimate (7) even if $f(t)$ y $A(t)$ are sufficiently smooth. Using the same equation (6) we can verify this assertion with the aid of the following sequence of functions:

$$f_n(t) = \begin{cases} n, & t < 0 \\ n\,(1 + cos\,(\pi t/\,\alpha_n))/\,2\ , & t \in [0, \alpha_n] \\ 0, & t \ge 0 \end{cases}$$

where α_n is a sequence of positive numbers and $lim_{n\to\infty}\alpha_n = 0$.

§3. Improving Theorem 2. With respect to the inequality (5) we can state another question: is it possible to define a more general class of matrices $A(t)$ for which the estimate (5) is satisfied? In the following Theorem we show that it is possible to weaken the condition that $A(t)$ be differentiable.

THEOREM 3. *Let us suppose that the hypotheses of Theorem 1 are satisfied. Moreover, let us suppose that $A(t)$ is Lipschitz continuous, that is, there exists a constant N such that for any t and s belonging to $\mathbb{R}$ we have $|A(t) - A(s)| \leq N|t - s|$. Let f be a function whose derivative is continous and bounded on $\mathbb{R}$. Then there exist $L>0$, and $\varepsilon_0 > 0$, such that the estimate is satisfied for any $\varepsilon \in (0, \varepsilon_0)$.*

Proof. It is clear that the function $A^{-1}(t)$ exists, and is a bounded Lipschitz continous function. Let us denote the Lipschitz constant by N. Define

$$P(h, t) = h^{-1}\int_t^{t-h} A^{-1}(s)\,ds$$

It follows that $\|P(h, -)\| \leq \|A^{-1}\|$, and that

$$\lim_{h \to \infty} P(h, t) = A^{-1}(t) \tag{12}$$

uniformly with respect to t.

Let ε_0 be the number obtained in the Theorem 1. First we will show that there exists a number $M > 0$, not depending on h, and a function $h_0(\varepsilon)$: $(0, \varepsilon_0] \to (0, \infty)$, such that

$$\|k_\varepsilon(f) + P(h, -)f\| \leq \varepsilon M \|f\|_1, \quad h \in (0, h_0(\varepsilon)). \tag{13}$$

We note that the differentiability of function $P(h,t)$ with respect to t and the definition of the operator k_ε give the following:

$$\varepsilon(k_\varepsilon(f) + P(h, \cdot)f)'(t) = A(t)(k_\varepsilon(f) + P(h, \cdot)f)(t) +$$

$$f(t) - AP(h, t)f(t) + \varepsilon(P(h, \cdot)f)'(t).$$

Again, from the definition of k_ε, the above identity shows that

$$k_\varepsilon(f) + P(h, t)f = k_\varepsilon(f - AP(h, \cdot)f + \varepsilon(P(h, \cdot)f)') \tag{14}$$

The inequality (2), fullfilled for any $\varepsilon \in (0, \varepsilon_0]$, implies:

$$\|k_\varepsilon(f) + P(h, \cdot)f\| \leq K\|f - AP(h, \cdot)f + \varepsilon(P(h, \cdot)f)'\|. \tag{15}$$

Now, from (12), for each $\varepsilon \in (0, \varepsilon_0]$, we obtain a number $h_0(\varepsilon) > 0$, such that

$$\|I - AP(h, \cdot)\| \leq 1, \text{ for } 0 < h < h_0(\varepsilon). \tag{16}$$

From (15) and (16) we obtain, for $\varepsilon \in (0, \varepsilon_0]$ and $h \in (0, h_0(\varepsilon))$:

$$\|k_\varepsilon(f) + P(h, \cdot)f\| \leq \varepsilon K(\|f\| + \|(P(h, \cdot)f)'\|). \tag{17}$$

An explicit expression for the function $(P(h, \cdot)f)'(t)$ is given by:

$$(P(h,\cdot)f)'(t) = h^{-1}(A^{-1}(t+h) - A^{-1}(t))f(t) + P(h,t)f'(t)$$

and, in virtue of the Lipschitz condition over $A^{-1}(t)$ we have:

$$\|P(h, \cdot)f)'\| \leq N\|f\| + \|P(h,\cdot)\|\,\|f'\|. \tag{18}$$

Introducing (18) into (17) we can write for $h \in (0,h_0)$:

$$\|k_\varepsilon(f) + P(h, \cdot)f\| \leq \varepsilon K(\|f\| + N\|f\| + \|P(h,t)\|\,\|f'\|)$$

Defining $L := K(1+N + \|A^{-1}\|)$ we obtain for $\varepsilon \in (0, \varepsilon_0]$, $h \in (0, h_0(\varepsilon))$

$$\|k_\varepsilon(f) + P(h, \cdot)f\| \leq \varepsilon L\|f\|_1. \tag{19}$$

From (11), letting $h \to 0+$ in (19), we obtain

$$\|k_\varepsilon(f) + A^{-1}(\cdot)f\| \leq \varepsilon L\|f\|_1.$$

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REFERENCES

[1] Coppel, W.A., *Dichotomies in Stability Theory*. Lecture Notes in Mathematics 629, Springer Verlag, Berlin, 1978

[2] Mitropolskii, Y.A., Likova, O.B, *Integral Manifolds in Non Linear Mechanic*, Nauka, Moscú, 1973. (Russian)

[3] Hale, J., K., *Ordinary Differential Equations*, Wiley Interscience, New York, 1969.

Universidad de Oriente
Departamento de Matemáticas
Cumaná, Venezuela.

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