Revista Colombiana de Matemáticas
Vol. XXV (1991) pgs. 103 - 122

THE OPTIMUM SHAPE OF AN HYDROFOIL WITH NO CAVITATION

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ABSTRACT. We consider a two-dimensional hydrofoil at rest in the (xy)-plane embedded in a steam with a uniform flow at infinity and we pose the problem of finding the optimum shape of the hydrofoil of a given length and prescribed mean curvature for which the lift is a maximum. Using the lifting line theory and standard variational calculus techniques we show that the slope of the mean chord of the hydrofoil has to satisfy a differential equation of the second order. The Rayleigh-Ritz method is used to solve the second order differential equation which gives the optimal values.

I. Introduction. The purpose of this paper is to evaluate the optimum shape of a two-dimensional hydrofoil of given length and prescribed mean curvature which produces maximum lift. The hydrofoil as in the accompanying diagram (Fig.1) is placed in a uniform flow of an incompressible non-viscous liquid filling an infinite space. The liquid flow is taken to be two-dimensional irrotational, steady, and a linearized theory is assumed.

A two-dimensional vortex distribution over the hydrofoil is used to simulate the two-dimensional zero cavity flow past the hydrofoil. This method leads to a system of integral equations and these are solved exactly using Carleman-Muskhelishvili technique. This method is similar to that used by T.V. Davies, [1], [2].

We use variational calculus techniques to obtain the optimum shape of the hydrofoil in order to maximize the lift coefficient subject problem is that extremizing a functional depending on γ (the vortex, strength), and z (the hydrofoil slope) when these two functions are related by a singular integral equation. The analytical solution for the unknown shape z and the unknown singularity distributions has branch type singularities at the two ends of the hydrofoil. The external solutions $\gamma(x;\ \lambda_1,\ \lambda_2)$ and $z(x;\ \lambda_1,\ \lambda_2)$ will involve two Lagrange multiplier constants $\lambda_1,\ \lambda_2$ which can be determined by substituting the external solutions $\gamma(x;\ \lambda_1,\ \lambda_2)$ and $z(x;\ \lambda_1,\ \lambda_2)$ in the constraints. Analytical solution by a singular integral equations and Rayleight-Ritz method are discussed.

In a previous paper A.H. Essawy, [9] studied this problem and the resulting equations were solved numerically using the NAG-library routine $D_{02}ADF$ which solves a two points boundary value problem for a system of two ordinary differential equations

$$\frac{dw_i}{dx} = f_i\,(x, w_1, w_2), \qquad i = 1, 2$$

In this paper analytical solutions by a singular integral equations, variation of parameters, and the Rayleight-Ritz methods are given. A sufficient condition for the extremum to be a maximum is derived by considering the variation.

II-A. Expression of the problem in integral equations.

$\widehat{OA}$ in the figure 1 represents a hydrofoil of arbitrary shape. The problem will be solved on the basis of linearized theory and for this purpose we distribute: vortices of strength $\gamma(x)$ per unit length in $0<x<a$ ($\gamma>0$ clockwise) along the x-axis to replace the above physical configuration, and $\gamma(x)$ being an unknown distribution.

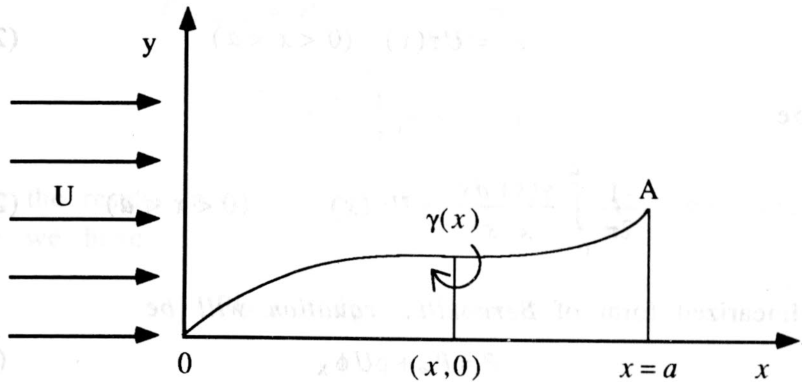

Fig. 1.

The velocity potential due to the distribution of vortices in $0 < x < a$ is given by

$$\phi(x,y) = \frac{-1}{2\pi}\int_0^a \gamma(s)\tan^{-1}\left(\frac{y}{x-s}\right)ds \qquad (0<x<a) \qquad (2.a.1)$$

and the corresponding velocity in y-direction will be

$$v = \frac{-\partial\phi}{\partial y} = \frac{1}{2\pi}\int_0^a \frac{\gamma(s)(x-s)\,ds}{(x-s)^2+y^2} \qquad (2.a.2)$$

As $y \to 0\pm$ we have, for all x

$$\lim_{y\to 0\pm} v = \frac{1}{2\pi}\int_0^a \frac{\gamma(s)\,ds}{(x-s)} \qquad (0<x<a). \qquad (2.a.3)$$

The boundary condition on the hydrofoil is

$$z(x) = \frac{v}{U+u}, \quad z(x) = y'(x) \quad 0<x<a \qquad (2.a.4)$$

where u,v are the components of liquid velocity along x,y axis respectively, U is a uniform stream at infinity, parallel to x-axis and $y'(x)$ is the gradient of the hydrofoil at position x. The equation (2.a.4) is approximated in the usual way to

$$v = Uz(x) \quad (0 < x < a) \tag{2.a.5}$$

hence

$$\frac{-1}{2\pi}\int_0^a \frac{\gamma(s)\,ds}{x-s} = Uz(x) \qquad (0 < x < a) \tag{2.a.6}$$

The linearized form of *Bernoulli's equation will be*

$$P = P_\infty + \rho U \phi_x \tag{2.a.7}$$

where P is the pressure, P_∞ the pressure at infinity and ρ is the constant density of the liquid.

From (2.a.1) we can write

$$\frac{\partial\phi}{\partial x} = \frac{1}{2\pi}\int_0^a \frac{\gamma(s)\,y\,ds}{(x-s)^2 + y^2} \tag{2.a.8}$$

the limiting value of $\frac{\partial\phi}{\partial x}$ as $y \to 0\pm$ is

$$\lim_{y\to 0\pm}\left(\frac{\partial\phi}{\partial x}\right) = \pm\frac{1}{2}\gamma(x) \quad (0 < x < a)\cdot \tag{2.a.9}$$

II-B Determinating the general formula for the lift and drag. Let the x and y components of the hydrodynamic force acting on the hydrofoil be denoted by drag D and lift L, then the complex forces acting on a hydrofoil calculated within the linearized theory are given by

$$D + iL = \int_0^a \left\{ P|_{y=0^-} - P|_{y=0^+} \right\} i\,dZ \tag{2.b.1}$$

Using the results in (2.a.7) and (2.a.9) as $y \to 0+$ through positive value we obtain

$$P|_{y=0+} = P_\infty + \rho U \lim_{y\to 0+} \phi_x$$

$$= P_\infty + \frac{1}{2}\rho U \gamma(x), \quad (0<x<a). \tag{2.b.2}$$

Using the results in (2.a.7) and (2.a.9) as $y \to 0-$ through negative value we have

$$P|_{y=0-} = P_\infty + \rho U \lim_{y\to 0-} \phi_x$$

$$= P_\infty - \frac{1}{2}\rho U \gamma(x) \quad (0<x<a) \tag{2.b.3}$$

It follows that we can write from (2.b.1), (2.b.2) and (2.b.3) the hydrodynamic forces acting on the hydrofoil

$$L = \int_0^a \left\{ P|_{y=0-} - P|_{y=0+} \right\} dx$$

$$= -\rho U \int_0^a \gamma(x)\, dx \quad (0<x<a) \tag{2.b.4}$$

and

$$D = -\int_0^a \left\{ P|_{y=0-} - P|_{y=0+} \right\} dy$$

$$= \rho U \int_0^a \gamma(x)\, y'(x)\, dx \quad (0<x<a) \tag{2.b.5}$$

II-C. The optimum shape of a hydrofoil using variational calculus techniques, so that the lift is a maximum. We pose the problem of maximizing the lift coefficient

$$L^* = \frac{L}{\rho U^2}$$

$$= \frac{-1}{U}\int_0^a \gamma(X)\,dx\ , \qquad (2.c.1)$$

subject to a constraint on the curvature of the form

$$k = \int_0^a z'^2(x)\,dx\ , \qquad (2.c.2)$$

where k is prescribed, together with a constraint on the length of the hydrofoil of the form

$$C = \int_0^a \sqrt{1+z^2(x)}\,dx \qquad (2.c.3.)$$

where C is prescribed and $z(x) = y'(x)$ is the gradient of the hydrofoil at position x.

Statement of the problem. The general optimum problem considered here may be stated as follows: To find the real, extremal function $\gamma(x)$ of a real variable, required to be *Holder* continous (see, e.g., Tricomi, [3]) in the region $(0 < x < a)$ together with

$$z(x) = -\frac{1}{2\pi U}\int_0^a \frac{\gamma(s)\,ds}{s-x} \qquad (0 < x < a) \qquad (2.c.4)$$

so that $\gamma(x)$ and $z(x)$ minimize the functional

$$I\left[\gamma(x), z(x), z'(x), x\right] = -L^* + \lambda_1 C + \lambda_2 k$$

$$= \int_0^a F\left[\gamma(x), z(x), z'(x), x; \lambda_1, \lambda_2\right] dx \qquad (2.c.5.)$$

where

$$F\left[\gamma(x), z(x), z'(x), x; \lambda_1, \lambda_2\right] = \lambda_1\sqrt{1+z^2(x)} + \lambda_2 z'^2(x) + \frac{1}{U}\gamma(x), \quad (2.c.6)$$

and $\gamma(x)$, $z(x)$ are related by (2.c.4) and λ_1,λ_2 are *Lagrange* multipliers. We define an admissible function as any function $\gamma(x)$ which satisfies the *Hölder* condition μ ($\mu < 1$) and the constraints (2.c.2) and (2.c.3), and we assume that the optimal function is an admissible function which minimizes the function $I[\gamma,z,z',x]$.

The solution of (2.c.4) satisfying the *Kutta* condition (the liquid leaves the hydrofoil smoothly along the tangent at the trailing edge i.e. the velocity must vanish at $x = a$).

$$\gamma(x) = 0 \quad (2.c.7)$$

is well known and is given by

$$\gamma(x) = \frac{2U}{\pi}\sqrt{\frac{a-x}{x}}\int_0^a \sqrt{\frac{s}{a-s}}\,\frac{z(s)ds}{s-x} \qquad (0 < x < a) \quad (2.c.8)$$

The necessary condition of optimalty. Let $\gamma(x)$, $z(x)$ denote the required optimal vortex distribution function and optimal hydrofoil slope respectively, we write

$$\gamma_1(x) = \gamma(x) + \varepsilon\xi(x),$$

$$z_1(x) = z(x) + \varepsilon\eta(x), \quad (2.c.9)$$

and we can use (2.c.8) to obtain the following relation between $\xi(x)$ and $\eta(x)$

$$\xi(x) = \frac{2U}{\pi}\sqrt{\frac{a-x}{x}}\int_0^a \sqrt{\frac{s}{a-s}}\,\frac{\eta(s)ds}{s-x} \quad (2.c.10)$$

If $\xi(x)$, is an admissible variation, then $I[\gamma(x)+\varepsilon\xi(x), z(x)+\varepsilon\eta(x), z'(x)+\varepsilon\eta'(x), x]$ in (2.c.5) is a function of ε which has an extreme value when $\varepsilon = 0$.

For sufficiently small ε, the expansion of (2.c.5) in a *Taylor* series yields

$$\Delta I = \varepsilon\delta I + \frac{\varepsilon^2}{2!}\delta^2 I + \ldots \tag{2.c.11}$$

we have

$$\Delta I = \int_0^a F[\gamma+\varepsilon\xi, z+\varepsilon\eta, z'+\varepsilon\eta', x]dx - \int_0^a F[\gamma, z, z', x]dx, \tag{2.c.12}$$

where

$$\delta I = \int_0^a [\xi(x)F_\lambda(\gamma, z.z', x)+\eta(x)F_z(\gamma, z, z, z', x)+\eta'(x)F_{z'}(\gamma, z, z', x)]dx \tag{2.c.13}$$

in which the sub-indices denote partial derivatives; it may be noted that ξ and η are related by (2.c.10), the variations δI, $\delta^2 I$, ... depend on $\xi(x)$ as well as $\gamma(x)$.

$$\delta I = \Big[\eta(x)F_{z'}(\gamma, z, z', x)\Big]_0^a + \int_0^a \Big(\xi(x)F_\gamma(\gamma, z, z', x)+\eta(x)[F_z(\gamma, z, z', x) - \frac{d}{dx}F_{z'}(\gamma, z, z', x)]\Big)dx \tag{2.c.14}$$

substituting (2.c.10) into (2.c.14) we obtain

$$\delta I = \Big[\eta(x)F_{z'}(\gamma, z, z', x)\Big]_0^a + \int_0^a \Big([F_z(\gamma, z, z', x) - \frac{d}{dx}F_{z'}(\gamma, z, z', x)]\eta(x) + \frac{2U}{\pi}\sqrt{\frac{a-x}{x}}F_\gamma(\gamma, z, z', x)\int_0^a \sqrt{\frac{s}{a-s}}\frac{\eta(s)ds}{s-x}\Big) \tag{2.c.15}$$

it is permissible to interchange the order of the double integral on the right-hand side of (2.c.15) [see, e.g., Muskhelishvili [4]] and interchange the variables x,t and when we do so we obtain

$$\delta I = \Big[\eta(x) F_{z'}(\gamma,z,z',x)\Big]_0^a + \int_0^a \Big\{ \Big[F_z(\gamma,z,z',x) - \frac{d}{dx} F_{z'}(\gamma,z,z',x)\Big]$$

$$- \frac{2U}{\pi}\sqrt{\frac{a-x}{x}} \int_0^a \sqrt{\frac{a-s}{s}}\ \frac{F_\gamma(\gamma,z,z',x)}{s-x} \Big\}\eta(x)\,dx \qquad (2.c.16)$$

we have from (2.c.6)

$$F_\gamma(\gamma,z,z',x) = \frac{1}{U}$$

$$F_z(\gamma,z,z',x) = \frac{\lambda_1 z(x)}{\sqrt{1+z^2(x)}} \qquad (2.c.17)$$

$$F_{z'}(\gamma,z,z',x) = 2\lambda_2 z'(x)$$

substituting (2.c.17) in (2.c.16) we obtain

$$\delta I = \Big[2\lambda_2 \eta(x) z'(x)\Big]_0^a + \int_0^a \Big(\frac{\lambda_1 z(x)}{\sqrt{1+z^2(x)}} - 2\lambda_2 z''(x)$$

$$- \frac{2}{\pi}\sqrt{\frac{x}{a-x}} \int_0^a \sqrt{\frac{a-s}{s}}\ \frac{ds}{(s-x)}\Big)\eta(x)\,dx \qquad (2.c.18)$$

For $I[z]$ to be a minimum we must have for all admissible function $\eta(x)$

$$\delta I[z,\eta] = 0, \qquad (2.c.19)$$

and this implies that the coefficients of $\eta(x)$ in (2.c.18) should vanish that is

$$2\lambda_2 z''(x) - \lambda_1 \frac{z(x)}{\sqrt{1+z^2(x)}} = -\frac{2}{\pi}\sqrt{\frac{x}{a-x}} \int_0^a \sqrt{\frac{a-s}{s}}\ \frac{ds}{(s-x)} = 2\sqrt{\frac{x}{a-x}} \qquad (2.c.20)$$

while at the end points it is necessary that

$$\eta(x)z'(0) = 0, \qquad \eta(a)z'(a) = 0 \tag{2.c.21}$$

be satisfied. In all the examples considered subsequently $z(x)$ is postulated at $x = 0$ and $x = a$ and this implies that

$$\eta(0) = \eta(a) = 0 \tag{2.c.22}$$

Equation (2.c.20) is a nonlinear differential equation for $z(x)$. We consider the solution of (2.c.20) for the slope $z(x)$ only in the case of small slope, and we approximate to (2.c.20) as follows:

$$z''(x) - n\,z(x) = E\sqrt{\frac{x}{a-x}},\ n = \frac{\lambda_1}{2\lambda_2},\ E = \frac{1}{\lambda_2},\ \lambda_2 \neq 0,\ (0 < x < a) \tag{2,c,23}$$

It is assumed at this stage that $\lambda_1/\lambda_2 < 0$ and we show later that $\lambda_1 < 0$, $\lambda_2 > 0$ are sufficient conditions for a true maximization of L. We write the differential equation in the form

$$z''(x) + m^2 z(x) = E\sqrt{\frac{x}{a-x}},\ \left(m^2 = -n = -\frac{\lambda_1}{2\lambda_2}\right),\ (0 < x < a) \tag{2.c.24}$$

To derive the solution of the nonhomogeneous equation, (2.c.24) we apply the usual method of variation of parameters then we can write $z(x)$ in the form

$$z(x) = \frac{E}{m}\int_0^x \sqrt{\frac{\xi}{a-\xi}}\ \mathrm{Sin}\, m(x-\xi)d\xi + A\ \mathrm{Sin}\, mx + B\ \mathrm{Cos}\, mx,\ (0 < x < a) \tag{2.c.25}$$

where A and B are arbitrary constants. Using the boundary conditions

$$z(0) = 0,\ z(a) = \beta, \tag{2.c.26}$$

we obtain

$$A = \frac{-E}{m\ \mathrm{Sin}\, ma}\int_0^a \sqrt{\frac{\xi}{a-\xi}}\ \mathrm{Sin}\, m(a-\xi)d\xi + \beta\,\mathrm{Cosec}\, m a,\quad B = 0 \tag{2.c.27}$$

substituting (2.c.27) into (2.c.25) we obtain

$$z(x)=y'(x)=\frac{E}{m}\int_0^x\sqrt{\frac{\xi}{a-\xi}}\,\mathrm{Sin}\,m(x-\xi)d\xi+\beta\frac{\mathrm{Sin}\,mx}{\mathrm{Sin}\,ma}$$

$$-\frac{E\,\mathrm{Sin}\,mx}{m\,\mathrm{Sin}\,ma}\int_0^a\sqrt{\frac{\xi}{a-\xi}}\,\mathrm{Sin}\,m(a-\xi)d\xi,\ (0<x<a) \tag{2.c.28}$$

we integrate (2.c.28) with respect to x, and use the boundary condition $y(0)=0$ to obtain.

$$y(x)=\frac{E}{m}\int_0^x d\sigma\int_0^\sigma\sqrt{\frac{\xi}{a-\xi}}\,\mathrm{Sin}\,m(x-\xi)d\xi-\beta\frac{(\mathrm{Cos}\,mx-1)}{m\,\mathrm{Sin}\,ma}$$

$$+\frac{E}{m}\frac{(\mathrm{Cos}\,mx-1)}{\mathrm{Sin}\,ma}\int_0^a\sqrt{\frac{\xi}{a-\xi}}\,\mathrm{Sin}\,m(a-\xi)d\xi,(0<x<a) \tag{2.c.29}$$

Equation (2.c.29) can be written as follows when the order of integration of the double integral is inverted:

$$y(x)=\frac{E}{m}\int_0^x d\xi\int_\xi^x\sqrt{\frac{\xi}{a-\xi}}\,\mathrm{Sin}\,m(\sigma-\xi)d\sigma-\beta\frac{(\mathrm{Cos}\,mx-1)}{m\,\mathrm{Sin}\,ma}$$

$$+\frac{E}{m^2}\frac{(\mathrm{Cos}\,mx-1)}{\mathrm{Sin}\,ma}\int_0^a\sqrt{\frac{\xi}{a-\xi}}\,\mathrm{Sin}\,m(a-\xi)d\xi,\ (0<x<a)$$

$$=-\frac{E}{m^2}\int_0^x\sqrt{\frac{\xi}{a-\xi}}\,[\mathrm{Cos}\,m(x-\xi)-1]d\xi-\beta\frac{(\mathrm{Cos}\,mx-1)}{m\,\mathrm{Sin}\,ma}$$

$$+\frac{E}{m^2}\frac{(\mathrm{Cos}\,mx-1)}{\mathrm{Sin}\,ma}\int_0^a\sqrt{\frac{\xi}{a-\xi}}\,\mathrm{Sin}\,m(a-\xi)d\xi,\ (0<x<a) \tag{2.c.30}$$

When we substitute for $z(x)$ an $z'(x)$, using (2.c.28) into the constraints (2.c.2) and (2.c.3) we obtain two equations, in the two unknowns E, m, which have to be evaluated numerically.

We do not complete the solution of this problem using this method since there is an alternative methods of solving the problem, the *Rayleigh-Ritz* method, discussed in detail in section II-E.

II-D. The sufficient condition for the extremum to be a minimum. A sufficient condition for the extremum of I to be a minimum is derived from consideration of the second variation of I. Since

$$\delta I\,[\gamma(x), z(x), z'(x), x] = 0 \qquad (2.d.1)$$

the condition for I to be a minimum requires that

$$\delta^2 I\,[\gamma(x), z(x), z'(x), x] > 0\ , \ (0 < x < a) \qquad (2.d.2)$$

for all admissible variations $\xi(x)$ and $\eta(x)$ consistent with

$$\xi(x) = \frac{2U}{\pi}\sqrt{\frac{a-x}{x}}\int_0^a \sqrt{\frac{s}{a-s}}\,\frac{\eta(s)\,ds}{s-x}, \qquad (0 < x < a) \quad (2.d.3)$$

where $\eta(x)$ satisfies the boundary conditions

$$\eta(0) = 0\ , \quad \eta(a) = 0 \qquad (2.d.4)$$

Using *Taylor's theorem* we can write the increment of the functional $I\,(\gamma, z, z', x)$ in the form

$I\,[\gamma + \varepsilon\xi, z + \varepsilon\eta, z' + \varepsilon\eta', x] - I(\gamma, z, z', x) =$

$$\varepsilon\int_0^a \Big(\xi(x)F_\gamma(\gamma,z,z',x) + \eta(x)[F_z(\gamma,z,z',x) - \frac{d}{dx}F_{z'}(\gamma,z,z',x)]\Big)dx$$

$$+\frac{1}{2}\varepsilon^2\int_0^a \Big\{\xi^2(x)F_{\gamma\gamma}(\gamma,z,z',x) + \eta^2(x)F_{zz}(\gamma,z.z',x) + \eta'^2(x)F_{z'z'}(\gamma,z,z',x)$$

$$+ 2\xi(x)\eta(x)F_{\gamma z}(\gamma,z,z',x) + 2\xi(x)\eta'(x)F_{\gamma z'}(\gamma,z,z',x)$$

$$+ 2\eta(x)\eta'(x)F_{zz'}(\gamma,z,z',x)\Big\}dx + O(\varepsilon^3), \qquad (0 < x < a) \qquad (2.d.5)$$

Denoting that coefficient ε by δI and that of ε^2 by $\delta^2 I$ at a stationary value fo I, we have from (2.d.1), (2.d.3) and (2.d.5)

$$F_z(\gamma,z,z',x) - \frac{d}{dx}F_{z'}(\gamma,z,z',x)$$

$$= \frac{2U}{\pi}\sqrt{\frac{x}{a-x}}\int_0^a \sqrt{\frac{a-s}{s}}\;\frac{F_\gamma(\gamma,z,z',x)}{s-x}\,ds \qquad (2.d.6)$$

and

$$\delta^2 I = \int_0^a \{\xi^2 F_{\gamma\gamma} + \eta^2 F_{zz} + \eta'^2 F_{z'z'} + 2\xi\eta F_{\gamma z} + 2\xi\eta' F_{\gamma z'} + 2\eta\eta' F_{zz'}\}dx \qquad (2.d.7)$$

where by (2,c.6) we have

$$F_{\gamma\gamma}[\gamma,z,z',x] = 0$$

$$F_{zz}[\gamma,z,z',x] = \frac{\lambda_1}{[1+z^2(x)]^{3/2}}$$

$$F_{z'z'}[\gamma,z,z',x] = 2\lambda_2 \qquad (2.d.18)$$

$$F_{\gamma z'}[\gamma,z,z',x] = 0$$

$$F_{zz'}[\gamma,z,z',x] = 0$$

substituting (2.d.8) in (2.d.7) we obtain

$$\delta^2 I = \int_0^a \left(\frac{\lambda_1}{1+z^2(x)]^{3/2}}\eta^2(x) + 2\lambda_2\eta'^2(x)\right)dx\,. \qquad (2.d.9)$$

Using *Friedrich's* inequality (see, [5], p.192, (18-28))

$$\int_c^d u^2(x)\mathrm{d}x < \frac{(d-c)^2}{\pi^2}\int_c^d u'^2(x)dx\ ,\ u(c)=0\ ,\ u(d)=0, \qquad (2.d.10)$$

we can write

$$\delta^2 I > \left(2\lambda_2 + \frac{a^2}{\pi^2}\lambda_1\right)\int_0^a \eta'^2\,(x)dx. \qquad (2.d.11)$$

The sufficient condition for $\delta^2 I$ to be positive is

$$\lambda_1 + \frac{2\pi^2}{a}\lambda_2 > 0. \qquad (2.d.12)$$

II-E. Analytical solution by the Rayleigh-Ritz method. We use the *Rayliegh-Ritz* method (see, e.g. [6] and [7]) to solve equation [2.c.23], namely:

$$z''(x) - n\ z(x) = E\sqrt{\frac{x}{a-x}}\ ,\quad n = \frac{\lambda_1}{2\lambda_2}\ ,\quad E = \frac{1}{\lambda_2} \qquad (2.e.1)$$

where λ_1, λ_2 are *Lagrange* multipliers, and $z(x)$ is subject to the boundary conditions (2.c.26). Equation (2.e.1) is the necessary condition for the integral

$$J = \int_0^a \left\{\frac{1}{2}z'^2(x) + \frac{1}{2}n\ z^2(x) + E\sqrt{\frac{x}{a-x}}\ .\ z(x)\right\}dx \qquad (2.e.2)$$

to be minimized.

The *Rayliegh-Ritz* method can be applied to this problem in the following way. We select a basic set of a linearly independent polynomial functions and we assume an expression for $z(x)$ of the form

$$z(x) = \frac{\beta}{a^2}x^2 + a_1 x(a-x) + a_2 x^2(a-x) \qquad (0 \le x \le a) \quad (2.e.3)$$

which satisfies the end conditions (2.c.26), a_1 and a_2 being arbitrary constants. The values of $z(x)$ and $z'(x)$ are obtained from equation (2.e.3) and are substituted in (2.e.2); the result is a quadratic from in a_1 and a_2, namely

$$J = \frac{2}{3}\frac{\beta^2}{a} + \frac{1}{6}a^3a_1^2 + \frac{1}{15}a^5a_2^2 + \frac{1}{6}a_1a_2a^4 - \frac{1}{3}\beta a_1 a$$
$$- \frac{1}{6}\beta a_2 a + \frac{1}{10}n\,\beta^2 + \frac{1}{60}n\,a_1^2a^5 + \frac{1}{210}\,n\,a_2^2\,a^7$$
$$+\frac{1}{20}n\,a_1 B\,a^3 + \frac{1}{30}n\,a_2 B\,a^4 + \frac{1}{60}a_1a_2 n\,a^6$$
$$+ \frac{5\pi\beta a}{16}E + \frac{5\pi a^3 a_1}{16}E + \frac{5\pi a^4 a_2}{128}E = 0 \qquad (2.e.4)$$

The necessary conditions for minimizing J, with respect to a_1 and a_2 are

$$\frac{\partial J}{\partial a_1} = \frac{a_1 a^3}{30}[10+n\,a^2] + \frac{a_2 a^4}{60}[10+n\,a^2] - \frac{1}{60}\beta a[20-3n\,a^2] + \frac{\pi a^3}{16}E = 0 \qquad (2.e.5)$$

and

$$\frac{\partial J}{\partial a_2} = \frac{a_1 a^4}{60}[10+n\,a^2] + \frac{a_2 a^5}{150}[14+n\,a^2] - \frac{1}{30}\beta a^2[5-n\,a^2] + \frac{5\pi a^4}{16}E = 0. \qquad (2.e.6)$$

Using (2.e.5) and (2.e.6) the quantities n and E can now be expressed in terms of a_1 and a_2, but for convenience we introduce

$$\xi = a_1 a^2$$
$$\eta = a_2 a^3 \qquad (2.e.7)$$

and we then have

$$n = W/V \qquad (2.e.8)$$

with

$$W = \left(\frac{5}{24} - \frac{1}{60}\right)\xi + \left(\frac{5}{48} - \frac{2}{15}\right)\eta - \left(\frac{5}{24} - \frac{1}{6}\right)\beta$$

$$V = \left[\left(\frac{1}{48} - \frac{1}{60}\right)\xi + \left(\frac{1}{96} - \frac{1}{105}\right)\eta + \left(\frac{1}{32} - \frac{1}{30}\right)\beta\right] a^2 \tag{2.e.9}$$

and

$$E = \frac{-16}{\pi a^2}\left[\frac{\xi}{30}(10+n\,a^2) + \frac{\eta}{60}(10+n\,a^2) - \frac{1}{60}\beta(20 - 3n\,a^2)\right] \tag{2.e.10}$$

From (2.d.12) it follows that the sufficient condition for the lift to be a maximum can be expressed in the form:

$$\frac{[n + \frac{\pi^2}{16}]}{E} > 0 \quad , \quad E = \frac{1}{\lambda_2} \; , \; n = \frac{\lambda_1}{2\lambda_2} \tag{2.e.11}$$

Substituting from (2.e.3) in the constraints, (2.c.2) and (2.c.3) we obtain

$$\ell = \int_0^a \left[1 + \frac{1}{2} z^2(x)\right] dx = a + \frac{1}{10}\beta^2 a + \frac{1}{60} a_1^2 a^5$$

$$+ \frac{1}{210} a^7 a_2^2 + \frac{1}{20}\beta a^3 a_1 + \frac{1}{30}\beta a^4 a_2 + \frac{1}{60} a_1 a_2 a^6 \tag{2.e.12}$$

and

$$K = \int_0^a z'^2(x)\, dx = \frac{4}{3}\frac{\beta^2}{a} + \frac{1}{3} a^3 a_1^2 + \frac{2}{15} a^5 a_2^2$$

$$- \frac{2}{3}\beta a_1 a - \frac{1}{3}\beta a_2 a^2 + \frac{1}{3} a_1 a_2 a^4 \tag{2.e.13}$$

Equation (2.e.12) and (2.e.13) can be written as follows:

$$S_1 = A_1\xi^2 + 2H_1\xi\eta + B_1\eta^2 + 2P_1\xi + 2Q_1\eta + C_1 = 0$$
$$S_2 = A_2\xi^2 + 2H_2\xi\eta + B_2\eta^2 + 2P_2\xi + 2Q_2\eta + C_2 = 0 \qquad (2.e.14)$$

where

$$\begin{aligned} A_1 &= 1/60, & A_2 &= 1/3 \\ H_1 &= 1/120, & H_2 &= 1/6 \\ B_1 &= 1/210, & B_2 &= 2/15 \\ P_1 &= (1/40)\,\beta, & P_2 &= (-1/3)\,\beta \\ Q_1 &= (1/60)\,\beta, & Q_2 &= (-1/6)\,\beta \\ C_1 &= (\ell - a)/a + (1/10)\,\beta^2, & C_2 &= -Ka + (4/3)\,\beta^2 \end{aligned} \qquad (2.e.15)$$

we shall consider the special case

$$\begin{aligned} \ell &= 4.02 \text{ ft} \\ a &= 4.00 \text{ ft} \\ K &= 0.0148 \text{ ft} \\ b &= -\tan 12 = -0.21256\,. \end{aligned} \qquad (2.e.16)$$

Regarding $S_1 = 0$ and S_2 as two conics the condition upon λ for the quadratic

$$S_1 + \lambda S_2 = 0 \qquad (2.e.17)$$

to represent a pair of straight lines is

$$233661\,\lambda^3 - 27702.7\,\lambda^2 + 1712.08\,\lambda + 34.663 = 0 \qquad (2.e.18)$$

(see, e.g. [8], which can be solved to give the following roots

$$\lambda \equiv -0.01572\,, \quad 0.067139 \pm 0.070215\,i \qquad (2.e.19)$$

using the real value of λ we can write equation (2.e.15) in the form:

$$1.1427\xi^2 + 1.1427\xi\eta + 0.2666\eta^2 - 1.2855\xi - 0.8199\eta - 0.04983 = 0 \qquad (2.e.20)$$

By factorizing equation (2.e.20), we obtain

$$\xi + 0.3708\ \eta - 1.1625 = 0 \qquad (2.e.21)$$

$$\xi + 0.6292\ \eta + 0.0375 = 0 \qquad (2.e.22)$$

The straight line (2.e.21) when combined with $S_1 = 0$ produces

$$\xi = 1.5576 \pm 1.7002\ i,\ \eta = -1.06656 \pm 3.15517\ i \qquad (2.e.23)$$

in other words there is no real intersection of this straight line with the conic; the points of intersection between the straight line (2.e.22) and S_1 are real and are as follows:

$$\begin{array}{llll} \text{(i)} & \xi = -0.30834, & \eta = 0.43045 & \\ \text{(ii)} & \xi = 0.062619, & \eta = -0.15915. & \end{array} \qquad (2.e.24)$$

Using (2.e.16) and (2.e.24) we can write the values of η and E, (2.e.8) and (2.e.10) in the forms

$$\begin{array}{ll} \text{(i)} & \eta = -2.2599,\ E = -0.17072 \\ \text{(ii)} & \eta = -1.7925,\ E = -0.12293. \end{array} \qquad (2.e.25)$$

We find that both values of n and E in (2.e.25) satisfy the sufficient condition (2.e.11), but the values

$$\eta = -2.2599, \qquad E = -0.17072 \qquad (2.e.26)$$

actually provide the maximum values of lift, namely

$$L = 121260 \text{ Lbs} \qquad (2.e.27)$$

Thus the appropriate values of ξ and η are

$$\xi = -0.30834, \qquad \eta = 0.43045 \qquad (2.e.28)$$

using (2.e.28) and (2.e.7) we obtain

$$a_1 = -0.01927,\ a_2 = 0.006726 \qquad (2.e.29)$$

Now we can write the solution $z(x)$, (2.e.3) of the differential equation (2.e.1), using (2.e.9) and (2.e.16) as follows:

$$z(x) = -0.07708\ x + 0.05946\ x^2 - 0.006726\ x^3 \quad (0 \le x \le a) \qquad (2.e.30)$$

We integrate (2.e.30) with respect to x and we obtain

$$y(x) = -\ 0.03854\ x^2 + 0.01982 x^3 - 0.001682 x^4 \quad (0 \le x \le a) \qquad (2.e.31)$$

there being no arbitrary constant since

$$y(0) = 0 \tag{2.e.32}$$

The graph of y(x) is shown in Fig.2.

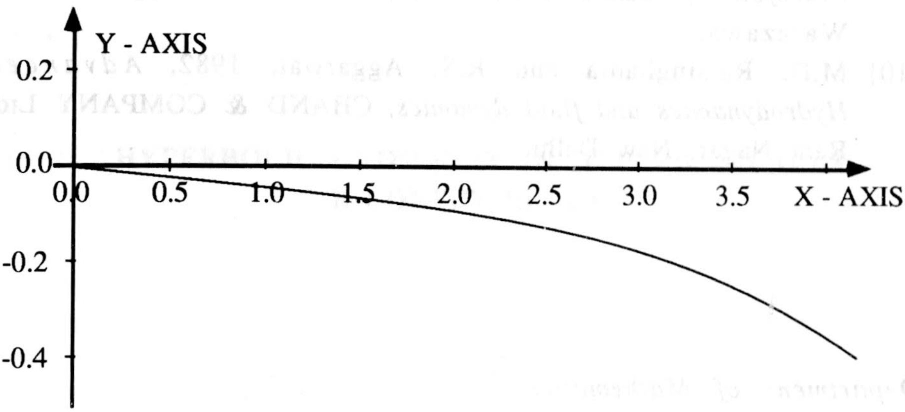

Fig. 2

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(Recibido en marzo de 1990, la versión revisada en abril de 1991)