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HYPERBOLIC GEOMETRY IN HYPERBOLICALLY K-CONVEX REGIONS

by

Diego MEJIA and David MINDA[1]

§I. Introduction. This paper is the third and final part of a trilogy dealing with the concept of k-convexity in various geometries. Our first paper [MM_1] dealt with k-convexity in euclidean geometry and the second [MM_2] with k-convexity in spherical geometry. In this paper we treat the concept of k-convexity relative to hyperbolic geometry on the unit disk $\mathbb{D} = \{z : |z| < 1\}$.

We assume that the reader is familiar with both [MM_1] and [MM_2]; we frequently omit details of proofs when they are similar to proofs of analogous results in either one of these papers.

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A Jordan region Ω in the unit disk $\mathbb{D}$ is called hyperbolically k-convex if the hyperbolic curvature of the boundary is at least k at each point of $\partial\Omega$. This assumes that the boundary of Ω is smooth. A definition of hyperbolic k-convexity that applies to arbitrary regions is given in section III. Our goal is to obtain sharp hyperbolic geometric estimates for various quantities in hyperbolically k-convex regions. These estimates lead to sharp distortion and covering theorems (including the Bloch-Landau constant for $k \geq 2$ and the Koebe set) for the family $K_h(k, \alpha)$ of normalized ($f(0) = 0, f'(0) = \alpha$) conformal mappings of the unit disk $\mathbb{D}$ onto hyperbolically k- convex regions.

§II Hyperbolic Geometry. We begin by recalling some basic facts about hyperbolic geometry on the unit disk $\mathbb{D}$. The hyperbolic metric is $\lambda_{\mathbb{D}}(z)\,|dz| = |dz|/(1 - |z|^2)$; it has Gaussian curvature -4. The group of conformal automorphisms of $\mathbb{D}$ is

$$Aut(\mathbb{D}) = \{T(z) = \frac{e^{i\theta}(z - a)}{1 - \bar{a}z} : \theta \in \mathbb{R},\ \ a \in \mathbb{D}\}.$$

The hyperbolic metric is invariant under the group $Aut(\mathbb{D})$; that is, $T^*(\lambda_{\mathbb{D}}(z)\,|dz|) = \lambda_{\mathbb{D}}(z)\,|dz|$, or equivalently, $|T'|/(1 - |T|^2) = 1/(1 - |z|^2)$ for all $T \in \mathrm{Aut}(\mathbb{D})$. The hyperbolic distance between $a,b \in \mathbb{D}$ is defined by

$$d_{\mathbb{D}}(a,b) = \inf \int_{\gamma} \lambda_{\mathbb{D}}(z)\,|dz|,$$

where the infimum is taken over all paths γ in $\mathbb{D}$ which join a and b. Moreover, this infimum is actually a minimum and is uniquely attained for the arc δ of the circle trough a and b which is orthogonal to $\partial\mathbb{D}$. The path δ between a and b such that

$$d_{\mathbb{D}}(a,b) = \int_{\delta} \lambda_{\mathbb{D}}(z)\,|dz|$$

is called the hyperbolic geodesic joining a and b. The explicit formula for the hyperbolic distance is $d_{\mathbb{D}}(a,b) = \operatorname{artanh}|(b-a)/(1-\bar{a}b)|$. The hyperbolic distance is invariant under the group

Aut($\mathbb{D}$). Let $\mathbb{D}_h(a, r)$ denote the hyperbolic disk with center a and radius r. Sometimes it is more convenient to employ a related quantity in place of the hyperbolic distance. Set $E_{\mathbb{D}}(a,b)$ = tanh $d_{\mathbb{D}}(a,b)$ and note that this quantity is also invariant under the group Aut($\mathbb{D}$).

We shall also make use of the notion of hyperbolic curvature. We briefly recall this concept; for more details the reader should consult ([FO], [M_3]). Suppose γ is a C^2 curve on $\mathbb{D}$ with nonvanishing tangent at the point $a \in \mathbb{D}$. The hyperbolic curvature of γ at a is

$$k_h(a,\gamma) = \frac{k(a,\gamma) - (\partial/\partial n)\log \lambda_{\mathbb{D}}(a)}{\lambda_{\mathbb{D}}(a)}$$

where $k(a, \gamma)$ is the euclidean curvature of γ at a and $n = n(a)$ is the unit normal at a which makes an angle $+\pi/2$ with the tangent vector to γ at a. Note that hyperbolic and euclidean curvature coincide at the origin. In euclidean (spherical) geometry each path of constant positive euclidean (spherical) curvature is a subarc of the boundary of a euclidean (spherical) disk of the appropriate radius. This is no longer true in hyperbolic geometry when $0 < k \leq 2$ and helps to explain some of the intriguing differences between this paper and our two earlier papers ([MM_1], [MM_2]) on k-convexity in euclidean and spherical geometry. Some of the geometric techniques from the first two papers do not extended to the hyperbolic setting.

EXAMPLE 1. We begin by listing three types of paths in $\mathbb{D}$ with constant hyperbolic curvature. First, suppose γ is the hyperbolic circle $\{z \in \mathbb{D} : d_{\mathbb{D}}(a,z) = r\})$, oriented so that the center a lies on the left-hand side of γ. Then $k_h(z, \gamma) = 2\coth(2r) > 2$. Next, suppose γ is a horocycle, that is, a euclidean circle which is internally tangent to the unit circle. Then $k_h(z, \gamma) = 2$. The final case is when γ is a subarc of a euclidean circle which intersects the unit circle at two distinct points; such an arc is sometimes called a hypercycle. If φ is the acute angle between γ and the unit circle, then $K_h(z, \gamma) = 2 \cos \varphi < 2$. For more information see [C, p.25]

In fact, every arc of constant hyperbolic curvature in $\mathbb{D}$ is of one of these three types. This can be established as follows. From [S, Thm. 2.13] it follows that if γ is a path in $\mathbb{D}$ with constant hyperbolic curvature, then γ also has constant euclidean curvature. In particular, γ is a euclidean circular arc and so is one of the three types considered.

If f is a locally schlicht function mapping $\mathbb{D}$ into itself an γ is a path in $\mathbb{D}$, then for $z \in \gamma$.

$$k_h(f(z), f \circ \gamma) f^h(z) = k_h(z, \gamma) \lambda_{\mathbb{D}}(z) + \mathrm{Im}\left(\left[\frac{2\bar{z}}{1-|z|^2} + \frac{f''(z)}{f'(z)} - \frac{2\overline{f(z)} f'(z)}{1-|f(z)|^2}\right] t(z)\right),$$

where $t(z)$ is the unit tangent to γ at z. Here $f^h(z) = |f'(z)|/(1-|f(z)|^2)$ denotes the hyperbolic derivative of holomorphic function mapping $\mathbb{D}$ into itself.

Now we turn to an issue involving hyperbolic geometry on subregions of the unit disk. For a Hyperbolic region Ω in the unit disk $\mathbb{D}$ it is natural to consider the "hyperbolic density" of the hyperbolic metric in place of the (euclidean) density of the hyperbolic metric. For a hyperbolic region $\Omega \subset \mathbb{D}$, the hyperbolic density is defined by

$$\nu_\Omega(z) = \lambda_\Omega(z)\,|dz| \,/\, \lambda_{\mathbb{D}}(z)\,|dz| = (1-|z|^2)\,\lambda_\Omega(z)$$

The hyperbolic density is a continous function on Ω which is invariant under $\mathrm{Aut}(\mathbb{D})$.

We shall frequently make use of the hyperbolic distance (relative to $\mathbb{D}$) to the boundary of a region $\Omega \subset \mathbb{D}$. Let $\gamma_\Omega(z) = \min\{d_{\mathbb{D}}(z, c) : c \in \partial w\}$.. This is the hyperbolic distance from z to $\partial\Omega$ and is clearly invariant under $\mathrm{Aut}(\mathbb{D})$. In some applications our formulas become much simpler if we employ a related quantity in place $\gamma_\Omega(z)$. Define $\Gamma_\Omega(z) = \tanh \gamma_\Omega(z)$. This quantity is also invariant under Aut($\mathbb{D}$).

We reformulate several basic results for the hyperbolic metric in terms of the hyperbolic density. These results are well-known for the euclidean density of the hyperbolic metric.

Principle of hyperbolic metric. Suppose Ω is a simply connected subregion of $\mathbb{D}$ and f is holomorphic on the unit disk $\mathbb{D}$ with $f(\mathbb{D}) \subset \Omega$. Then $\nu_\Omega(f(z))\, f^h(z) \le \lambda_{\mathbb{D}}(z)$ for $z \in \mathbb{D}$ with equality if and only if f is a conformal mapping of $\mathbb{D}$ onto Ω.

Monotoniciy property. Suppose Ω and Δ are subregions of $\mathbb{D}$ and $\Omega \subset \Delta$. Then $\nu_\Delta(z) \le \nu_\Omega(z)$ for $z \in \Omega$ with equality if and only if $\Omega = \Delta$.

§III. Geometric properties of hyperbolically *k*-convex regions. In this section we introduce the concept of a hyperbolically k-convex subregion of $\mathbb{D}$ and study some of the basic properties of such regions.

Suppose that $k > 0$, $a, b \in \mathbb{D}$ and $d_{\mathbb{D}}(a,b) <$ artanh $(2/k)$. We first assume $k > 2$. Then there are two distinct closed hyperbolic disks $\overline{D}_1$ and $\overline{D}_2$ each having hyperbolic radius $(1/2)$ artanh $(2/k)$, such that $a, b \in \partial\overline{D}_j$ (j = 1, 2); note that $\partial\overline{D}_j$ has constant hyperbolic curvature k. Let $H_k[a,b] = \overline{D}_1 \cap \overline{D}_2$. The boundary of $H_k[a,b]$ consists of two closed circular arcs Γ_1 and Γ_2, each with constant hyperbolic curvature k. For $d_{\mathbb{D}}(a,b) =$ artanh $(2/k)$, we let $H_k[a,b]$ be the closed hyperbolic disk with center at the midpoint of the hyperbolic geodesic joining a and b and radius $(1/2)$ artanh $(2/k)$. The case $k = 2$ is similar except that $\overline{D}_1$ and $\overline{D}_2$ are replaced by horodisks. When $0 < k < 2$, there are two unique arcs of constant hyperbolic curvature k joining a and b. In this situation $H_k[a,b]$ is the closed Jordan region determined by the union of these two arcs. This latter definition also applies when $2 \leq k$ except that we need to specify the shorter arc of constant curvature k joining a and b when $k > 2$. We define $H_0[a,b]$ to be the hyperbolic geodesic between a and b. Note that for $0 \leq k' < k \leq 2 / \tanh(d_{\mathbb{D}}(a,b))$, it follows that $H_{k'}[a,b] \subset H_k[a,b]$.

DEFINITION. Let $k \in [0, \infty)$. A region $\Omega \subset \mathbb{D}$ is called *hyperbolically k-convex* if for any pair of points $a, b \in \Omega$, $d_{\mathbb{D}}(a,b) <$ ar-tanh(2/k) and $H_k[a,b] \subset \Omega$.

Clearly, hyperbolically 0-convex is equivalent to hyperbolic convexity, so we shall employ the phrase "hyperbolic k convexity" only when $k > 0$ and use "hyperbolic convexity" instead of "hyperbolic 0-convexity". Note that if Ω is hyperbolically k-convex, then it is also hyperbolically k'-convex for $0 \leq k' \leq k$. In particular, a hyperbolically k-convex region is always hyperbolically convex and simply connected. For each $k > 2$ a hyperbolic disk of radius (1/2) artanh (2/k) is hyperbolically k-convex but not hyperbolically k'-convex for any $k' > k$. The intersection of a finite number of hyperbolically k-convex and the union of an increasing sequence of hyperbolically k-convex regions is again hyperbolically k-convex.

LEMMA 0. *Suppose that Ω is a simply connected region in $\mathbb{D}$. If at each points $c \in \partial\Omega$ there is a supporting hyperbolic geodesic for all points of Ω in a sufficiently small neighborhood of c, then Ω is hyperbolically convex.*

Proof. This proof is adaptation of the proof of the analogous result for euclidean convexity [S]. Let $a, b \in \Omega$. We want to show that the hyperbolic geodesic connecting a and b is contained in Ω. By applying a conformal automorphism of $\mathbb{D}$ to Ω if necessary, we may assume that $a = 0$. Then 0 and b can be joined by a polygonal path Π with (euclidean) straight sides which is contained in Ω. Let $0 = z_0, z_1, z_2, \ldots z_m, z_{m+1} = b$ be the vertices of Π in the order in which they are met in traversing Π from 0 to b. We show that the vertices can be removed one at time, so that eventually the polygon, while remaining inside Ω, becomes the straight line $[0,b]$, which is also the hyperbolic geodesic connecting 0 to b. Suppose that $[0,z_k]\subset\Omega$. We want to show that $[0, z_{k+1}]\subset\Omega$. If 0, z_k and z_{k+1} are collinear, we are done. Suppose $[0,z_{k+1}]\not\subset\Omega$. Consider the set of all segments $[0,p]$, where p ranges over $[z_k, z_{k+1}]$; let $\theta(p)$ denote the angle between $[0,p]$ and $[0,z_k]$. There is a smallest angle $\theta(p_0)$ such that $p_0\in(z_k, z_{k+1})$ and $[0, p_0]$ contains a point of $\partial\Omega$. Let $c\in\partial\Omega\cap[0,p_0]$ be the point in this set that is closest to the origin. Then all points of the closed euclidean triangle Δ whit vertices 0, z_k and p_0 are in Ω, except for c and possibly other points of $[c, p_0]$. But then the point c fails to have a supporting hyperbolic geodesic locally since the hyperbolic geodesic trough 0 and c contains points of Ω arbitrarily close to c and any other hyperbolic geodesic through c meets the inside of the triangle Δ.

PROPOSITION 1. *Suppose that is Ω a simply connected region on $\mathbb{D}$ bounded by a closed C^2 Jordan curve $\partial\Omega$. If $K_h(c,\partial\Omega)\geq k>.0$ for all $c\in\partial\Omega$, , then Ω is hyperbolically k-convex.*

Proof. We begin by showing Ω that is hyperbolically convex. By the preceding lemma, it is sufficient to show that there is a locally supporting hyperbolic geodesic at each point $c\in\partial\Omega$. We may assume that $c=0$. Then $k(0,\partial\Omega)=k_h(0,\partial\Omega)>0$, so $k(\zeta,\partial\Omega)>0$ for all $\zeta\in\partial\Omega$ in a neighborhood of 0. This implies that Ω has a locally supporting euclidean straight line at 0 [S, p.46]. This straight line through 0 is also a hyperbolic geodesic, so Ω has a locally supporting hyperbolic geodesic at c.

Next, we show that Ω is hyperbolically k-convex. Fix $a, b\in\Omega$. Let τ be the supremum of all $t\geq 0$ such that $H_t[a,b]\subset\Omega$. Note that $\tau\leq 2/\tanh d_{\mathbb{D}}(a,b)$. Since Ω is hyperbolically convex, we know that $\tau>0$. We want to show that $\tau>k$. In case $\tau=2/\tanh d_{\mathbb{D}}(a,b)$, Ω contains the closed hyperbolic disk with center at the midpoint of the hyperbolic geodesic joining a and b and radius $(1/2)d_{\mathbb{D}}(a,b)$. Let D be the largest hyperbolic disk with the same center that is contained in Ω. Then ∂D meets $\partial\Omega$ at some point c. By applying a conformal automorphism of $\mathbb{D}$ to Ω if necessary, we may assume $c = 0$. The comparison principle for euclidean curvature [G,p.28] implies that $k(0, \partial\Omega) \le k(0, \partial D)$, or $k \le k_h(0, \partial\Omega) \le k_h(0, \partial D) < \tau$. The remaining case is $\tau < 2/\tanh d_{\mathbb{D}}(a,b)$. Consider the two arcs Γ_1 and Γ_2 of hyperbolic curvature τ which bound $H_\tau[a,b]$. At least one of these two arcs, say Γ_1, meets $\partial\Omega$ in some point c. As before we may assume $c = 0$. The comparison principle for euclidean curvature now gives $k(0, \partial\Omega) \le k(0, \Gamma_1)$, so that $k \le \tau$. We need to show strict inequality. Because Ω is open, we can select points a' and b' in Ω so that a and b lie strictly between a' and b' on the hyperbolic geodesic in Ω joining these latter two points. Let τ' be defined relative to a' and b' in the same manner that τ was defined for a and b. Then for a' and b' near a and b, respectively, $\tau' < \tau$. Since $k \le \tau'$ just as $k \le \tau$, we obtain $k < \tau$.

PROPOSITION 2. *Suppose Ω is a hyperbolically k-convex region. Then for any $a \in \Omega$ and $c \in \Omega$, $H_k[a,c] \setminus \{c\} \subset \Omega$.*

COROLLARY. *If Ω is a hyperbolically k-convex region, then int $H_k[c,d] \subset \Omega$ for all $c, d \in \partial\Omega$.*

LEMMA 1. *Suppose D is an open hyperbolic disk of radius $(1/2)\operatorname{artanh}(2/k)$ and B is an open hyperbolic disk such that $c \in \partial B \cap \partial D$ and B and D are externally tangent at c. If $d_{\mathbb{D}}(a,c) < \operatorname{artanh}(2/k)$ and $a \notin \overline{D}$, then $H_k[a,c] \setminus \{c\} \cap B \ne \varnothing$.*

Suppose Ω is a hyperbolically k-convex region. Assume $a \in \Omega$, $c \in \partial\Omega$ and $d_{\mathbb{D}}(a,c) = \gamma_\Omega(a)$, which we assume to be finite. Let δ be the hyperbolic geodesic which is tangent to the hyperbolic circle $C_h(a,r) = \{z\colon d_{\mathbb{D}}(a,z) = \gamma_\Omega(a)\}$ at c and let H be the hyperbolic half-plane determined by δ which contains a. Then the hyperbolic convexity of Ω implies that $\Omega \subset H$, so that H is a supporting hyperbolic half-plane for Ω at c. Moreover, if $k > 0$, let δ' be the unique arc of hyperbolic curvature k that is tangent to $C_h(a,r)$ at c and is contained in $H \cup \{c\}$. (There are two arcs of constant hyperbolic curvature k tangent to $C_h(a,r)$ at c, but just one of these is contained in $H \cup \{c\}$). If $k > 2$, then δ' is a hyperbolic circle in $\mathbb{D}$, while for $0 \leq k \leq 2$ it meets $\partial\mathbb{D}$. Let D be the region in $\mathbb{D}$ which is bounded by δ' and contains a.

PROPOSITION 3. *Suppose Ω is a hyperbolically k-convex region. Assume that $a \in \Omega$, $c \in \partial\Omega$, and $d_{\mathbb{D}}(a,c) = \gamma_\Omega(a)$. If the hyperbolically k-convex region D and hyperbolic half-plane H are as in the preceding discussion, then $\Omega \subset D$.*

PROPOSITION 4. *Suppose Ω is a hyperbolically k-convex region and $k > 2$. Assume that $a \in \mathbb{D}\backslash\Omega$,, $c \in \partial\Omega$ and $d_{\mathbb{D}}(a,c) = \gamma_\Omega(a)$. If D is the hyperbolic disk with radius $(1/2)arctan(2/k)$ that is tangent to the circle $\{z \in \mathbb{D} : d_{\mathbb{D}}(z,a) = \gamma_\Omega(a)\}$ at c and that does not meet the disk, $D_h(a, \gamma_\Omega(a))$ then $\Omega \subset D$.*

§IV. Lower bound for the hyperbolic density of the hyperbolic metric in a hyperbolically k-convex region. We obtain a sharp lower bound for the hyperbolic density of a hyperbolically k-convex region Ω in terms of the hyperbolic distance to the boundary of the region. In our work we distinguish the cases $k \geq 2$ and $0 \leq k < 2$. This distinction is actually very natural. If $k > 2$, then Ω is bounded in the hyperbolic sense. In this regard, the case $k > 2$ is the analog of euclidean or spherical k-convexity where $k > 0$. The case $k = 2$ resembles euclidean or spherical convexity in many regards, while $0 < k < 2$ seems to have no analog in either euclidean or spherical k-convexity.

EXAMPLE 2. For $k \geq 0$ we consider some standard hyperbolically k-convex regions D_k. For each of these regions we explicitly calculate $v_k = v_{D\,k}$ in terms of $\Gamma_k = \Gamma_{D\,k}$. In all cases we shall show that $v_k(z) = 1/g_k(\Gamma_k(z))$, where

$$g_k(t)=\begin{cases} \dfrac{4\arctan\sqrt{\dfrac{2+k}{2-k}}}{\pi\sqrt{4-k^2}}\,\dfrac{1+t^2-kt}{1-t^2}\sin\left[\dfrac{\pi\arctan\sqrt{\dfrac{2+k}{2-k}}\left(\dfrac{1-t}{1+t}\right)}{\arctan\sqrt{\dfrac{2+k}{2-k}}}\right], & 0\le k\le 2 \\[2ex] \dfrac{t(2-kt)}{1-t^2}, & 2\le k\,. \end{cases}$$

Note g_k that is continous for $k \geq 0$, but not analytic since it is not analytic at $k = 2$.

We first assume $k > 2$. Let D_k be the hyperbolic disk of radius $(1/2)\mathrm{artanh}(2/k) = \mathrm{artanh}[(\sqrt{k^2+4} - k)/2]$ which is centered at the origin. If r denotes the euclidean radius of D_k, then $k = (1+r^2)/r$. Also, for z in $D_k, \Gamma_k,(z) = (r - |z|)/(1-r|z|)$, or $|z| = (r-\Gamma_k(z))/(1-r\,\Gamma_k(z))$, so that

$$v_k(z)=(1-|z|^2)\frac{r}{r^2-|z|^2}=\frac{r\,(1-\Gamma_k^2(z))}{\Gamma_k(z)[2r-(1+r^2)\Gamma_k(z)]}=\frac{(1-\Gamma_k^2(z))}{\Gamma_k(z)\,[2-k\,\Gamma_k(z)]}.$$

This establishes the result when $k > 2$. Because of the invariance under Aut($\mathbb{D}$) of the quantities involved, this formula is actually valid for any hyperbolic disk of the same radius.

Next, we assume $k = 2$. In this case D_2 is the horodisk $\{z : |z-1/2| < 1/2\}$. The Möbius transformation $T(z) = i(1+z)/(1-z)$ maps $\mathbb{D}$ conformally the upper half-plane $\mathbb{H} = \{w : \mathrm{Im}\, w > 0\}$ and sends D_2 onto the half-plane $S = \{w : \mathrm{Im}\, w > 1\}$. The invariance of the hyperbolic metric under conformal mappings implies that $\nu_2(z) = \gamma_S(T(z))/\lambda_{\mathbb{H}}(T(z))$. Similarly, since the hyperbolic distance, is invariant under conformal mappings, $\gamma_2(z) = \lambda_S(T(z))$, where $\gamma_S(w)$ denotes the hyperbolic distance, relative to $\mathbb{H}$, from w to ∂S. Thus, it suffices to express $\lambda_S/\lambda_{\mathbb{H}}$ in terms of λ_S. Since each vertical line in $\mathbb{H}$, is a hyperbolic geodesic and $\lambda_{\mathbb{H}}(w) = 1/2\ \mathrm{Im} w$, we have for $w = u + iv \in S$ we have for

$$\gamma_S(w) = \frac{1}{2}\int_1^v \frac{dt}{t} = \frac{1}{2}\log v .$$

Because $v = \operatorname{Im} w$, we have $\operatorname{Im} w = \exp 2\lambda_S(w)$. Thus,

$$\frac{\lambda_S(w)}{\lambda_{\mathbb{H}}(w)} = \frac{\operatorname{Im} w}{\operatorname{Im}(w) - 1} = \frac{\exp[2\gamma_S(w)]}{\exp[2\gamma_S(w)] - 1} ;$$

this gives the desired result when $k = 2$. In fact, the formula is valid for any horodisk since any two horodisks are equivalent under the group $\operatorname{Aut}(\mathbb{D})$.

Finally, we assume $0 \le k < 2$. In this situation the region D_k is the subregion of $\mathbb{D}$ which is bounded by an arc of constant hyperbolic curvature k and defined as follows. The region D_k is bounded by the arc δ of constant hyperbolic curvature k which passes through ± 1 and lies in the upper half-plane; D_k lies below δ. Let $2\varphi \in [0, \pi/2)$ be the angle that δ makes with $(-1, 1)$. We convert to a conformally equivalent situation. The function $h(z) = \log[(1+z)/(1-z)]$ maps $\mathbb{D}$ conformally onto $\Sigma = \{w : |\operatorname{Im} w| < \pi/2\}$ and D_k onto the substrip $S = \{w : -\pi/2 < \operatorname{Im} w < 2\varphi\}$. Because the hyperbolic metric is invariant under conformal mappings, $\nu_k(z) = \lambda_S(h(z))/\lambda_\Sigma(h(z))$ and $\gamma_k(z) = \gamma_S(h(z))$, where $\gamma_S(w)$ denotes the hyperbolic distance, relative to Σ, from w to ∂S. Thus, in order to express ν_k in terms of γ_k it suffices to express λ_S/λ_Σ in terms of γ_S. Since $\lambda_\Sigma(w) = 1/2 \cos(\operatorname{Im} w)$ and

$$\lambda_S(w) = \frac{\pi}{(4\varphi + \pi)\sin\dfrac{\pi(\pi + 2\operatorname{Im} w)}{4\varphi + \pi}}$$

we obtain

$$\frac{\lambda_S(w)}{\lambda_\Sigma(w)} = \frac{2\pi\cos(\operatorname{Im} w)}{(4\varphi + \pi)\sin\dfrac{\pi(\pi + 2\operatorname{Im} w)}{4\varphi + \pi}} .$$

Because each vertical line is a hyperbolic geodesic in Σ, for $w = u + iv \in S$ we have

$$\gamma_S(w) = \frac{1}{2}\int_v^{2\varphi} \frac{dt}{\cos t} = \frac{1}{2}\log \frac{\sec 2\varphi + \tan 2\varphi}{\sec v + \tan v}$$

$$= \frac{1}{2}\log\frac{1+\tan\varphi}{1-\tan\varphi} - \frac{1}{2}\log\frac{1+\tan\frac{v}{2}}{1-\tan\frac{v}{2}} = \gamma_S(0) - \frac{1}{2}\log\frac{1+\tan\frac{v}{2}}{1-\tan\frac{v}{2}}.$$

From this we get $\operatorname{Im} w = v = 2 \arctan \tanh[\gamma_S(0) - \gamma_S(w)]$. Consequently,

$$v_k(z) = \frac{2\pi \cos\{2 \arctan \tanh[\gamma_k(0) - \gamma_k(z)]\}}{(4\varphi+\pi)\sin\dfrac{\pi\{\pi + 4\arctan\tanh[\gamma_k(0) - \gamma_k(z)]\}}{4\varphi+\pi}}.$$

From Example 1 we know that $2\varphi = \arcsin(k/2)$. Hence,

$$\gamma_k(0) = \frac{1}{2}\log\frac{1+\tan\varphi}{1-\tan\varphi} = \frac{1}{2}\log\sqrt{\frac{2+k}{2-k}}.$$

By making use of this and the identity

$$\tanh(x-y) = \frac{\tanh x - \tanh y}{1-\tanh x \tanh y},$$

we obtain

$$\tanh\left(\gamma_k(0) - \gamma_k(z)\right) = \frac{k\left[1-\Gamma_k^2(z)\right] - 2\sqrt{4-k^2}\,\Gamma_k(z)}{2\left[1-\Gamma_k^2(z)\right] + \sqrt{4-k^2}\left[1+\Gamma_k^2(z)\right]}.$$

In addition, the identity

$$\arctan x + \arctan y = \arctan\frac{x+y}{1-xy}$$

implies that

$$\frac{\pi}{4} + \arctan\frac{k\left[1-\Gamma_k^2(z)\right] - 2\sqrt{4-k^2}\,\Gamma_k(z)}{2\left[1-\Gamma_k^2(z)\right] + \sqrt{4-k^2}\left[1+\Gamma_k^2(z)\right]} = \arctan\sqrt{\frac{2+k}{2-k}}\,\frac{1-\Gamma_k(z)}{1+\Gamma_k(z)},$$

and

$$\varphi - \frac{\pi}{4} = \arctan \frac{2 - \sqrt{4 - k^2}}{k} + \arctan 1 = \arctan \sqrt{\frac{2+k}{2-k}} .$$

By putting together all of these facts, we obtain the desired result. Again, this formula is valid for any region conformally equivalent to D_k under Aut($\mathbb{D}$).

THEOREM 1. *Suppose is a hyperbolically k-convex region. Then*

$$\nu_\Omega(z) \geq \frac{1}{g_k(\Gamma_\Omega(z))} ,$$

and equality holds at a point if and only if Ω is conformally equivalent to D_k under a conformal automorphism of $\mathbb{D}$.

Proof. Now consider any hyperbolically k-convex region Ω. Fix $a \in \Omega$. Select $c \in \partial\Omega$ with $\Gamma_\Omega(a) = \tanh d_{\mathbb{D}}(a,c)$. Let D be the associated hyperbolically k-convex region defined just before Proposition 3. Then Proposition 3 implies that $\Omega \subset D$; also $\Gamma_D(a) = \Gamma_\Omega(a)$. The monotonicity property of the hyperbolic metric yields $\nu_\Omega(a) \geq \nu_D(a)$ with equality if and only if $\Omega = D$. Because $\Gamma_D(a) = \Gamma_\Omega(a)$, this inequality in conjunction with the formula for $\nu_k(z)$ in Example 2 completes the proof.

COROLLARY 1. *Suppose that Ω is a hyperbolically k-convex region, f is holomorphic in $\mathbb{D}$ and $f(\mathbb{D}) \subset \Omega$. Then for $z \in \mathbb{D}$.*

$$(1 - |z|^2 f^h(z) \leq g_k(\Gamma_\Omega(f(z))) .$$

Equality holds at a point if and only if Ω is conformally equivalent to D_k under a conformal automorphism of $\mathbb{D}$ and f is a conformal mapping of $\mathbb{D}$ onto Ω.

Proof. The principle of hyperbolic metric gives $\nu_\Omega(f(z)) f^h(z) \leq \lambda_{\mathbb{D}}(z)$ for $z \in \mathbb{D}$ with equality if and only if f is a conformal mapping of $\mathbb{D}$ onto Ω. The theorem then implies that $1/g_k(\Gamma_\Omega(f(z)) \leq \nu_\Omega(f(z))$ with equality if and only if Ω is conformally equivalent to D_k under an automorphism of $\mathbb{D}$. By combining the two preceding inequalities and the necessary and sufficient conditions for equality, we obtain the corollary.

DEFINITION. Let $K_h(k, \alpha)$ denote the family of all holomorphic functions f defined on $\mathbb{D}$ such that f is univalent, $f(0) = 0$, $f'(0) = \alpha$ and $f(\mathbb{D})$ is hyperbolically k-convex region in $\mathbb{D}$.

For $k \geq 2$ we make the following observation. If $f \in K_h(k, \alpha)$ and $\Omega = f(\mathbb{D})$, then the preceding corollary with $z = 0$ produces $\alpha = |f'(0)| \leq g_k(\Gamma_\Omega(0))$. Note that $g_k(t)$ is increasing on the interval $0 \leq t \leq (\sqrt{k^2+4} - k)/2 = r$ and $g_k(r) = r$. Because Ω is hyperbolically k-convex, we know that $\Gamma_\Omega(0) \leq r$. Therefore, $\alpha \leq r$ when $f \in K_h(k, \alpha)$. Moreover, $\alpha = r$ if and only if $f(z) = rz$.

EXAMPLE 3. For $k \leq 2$, set $f_k(z) = \alpha z / (1 - \sqrt{1 - \alpha(k - \alpha)}\; z)$. Then $f_k \in K_h(k, \alpha)$ since $f_k(\mathbb{D})$ is a hyperbolic disk of radius (1/2) arctan (2/k). Note that

$$f_k(-1) = -\alpha / \left(1 + \sqrt{1 - \alpha(k - \alpha)}\right) \text{ and } f_k(1) = \alpha / \left(1 - \sqrt{1 - \alpha(k - \alpha)}\right)$$

The largest hyperbolic disk contained in $f_k(\mathbb{D})$ and centered at the origin has euclidean radius $\alpha / (1 + \sqrt{1 - \alpha(k - \alpha)}\,) = M_k(\alpha)$. Note that $g_k(M_k(\alpha)) = \alpha$.

For $0 \leq k < 2$, we consider another standard map. For $0 \leq k < 2$ the function g_k is strictly increasing on (0,1) with $g_k(0) = 0$ and $g_k(t) \to 1$ as $t \to 1$. Hence, for each number $\alpha \in (0, 1)$ there is unique root $M_k(\alpha) \in (0, 1)$ of the equation $g_k(t) = \alpha$. Let Δ_k be the subregion of $\mathbb{D}$ that contains the origin and is bounded by an arc γ of constant hyperbolic curvature k which passes through the point $iM_k(\alpha)$ and with the property that the hyperbolic geodesic in $\mathbb{D}$ through $iM_k(\alpha)$ and tangent to γ at this point lies outside Δ_k. The arc γ meets $\partial\mathbb{D}$ in two points. Let δ be the hyperbolic geodesic of $\mathbb{D}$ determined by these two points. Then $\delta \subset \Delta_k$ and meets the imaginary axis at a point $-a$. Let $F_k(z)$ map $\mathbb{D}$ conformally onto D_k. Explicitly,

$$F_k(z) = \tanh\left[\frac{\arctan\sqrt{\frac{2+k}{2-k}}}{\pi}\log\left(\frac{1+z}{1-z}\right) - \frac{i}{2}\arctan\sqrt{\frac{2-k}{2+k}}\right].$$

Let $T_a(z) = (z - a)/(1-\bar{a}z)$. Set $f_k = T_a \circ F_k \circ T_b$, where $b = F_k^{-1}(a)$. Note that T_a is a conformal mapping of D_k onto Δ_k. Then $f_k(0) = 0$. Also, from $\lambda_{\mathbb{D}}(z) = \lambda_{\Delta_k}(f_k(z))\,|f_k'(z)|$, we obtain $1/|f_k'(0)| = \lambda_{\Delta_k}(0)/\lambda_{\mathbb{D}}(0)$. Since $\Gamma_{\Delta_k}(0) = M_k(\alpha)$, we have $1/|f_k'(0)| = 1/g_k M_k(\alpha))$ from Example 2. Hence $|f_k'(0)| = \alpha$,

Now, we determine the Koebe set for the family $K_h(k,\alpha)$.

COROLLARY 2. *Suppose $f \in K_h(k, \alpha)$. Then either $\{w : |w| \leq M_k(\alpha)\}$ is contained in $f(\mathbb{D})$ or $f(z) = e^{-i\theta} f_k(e^{i\theta}z)$ for some $\theta \in \mathbb{R}$.*

Proof. Set $\Omega = f(\mathbb{D})$ and apply the preceding corollary with $z = 0$ to obtain

$$\alpha = |f'(0)| \leq g_k(\Gamma_\Omega(0)).$$

This yields $\Gamma_\Omega(0) \geq M_k(\alpha)$ with equality if and only if Ω is conformally equivalent to D_k under some conformal automorphism of $\mathbb{D}$. In the case of equality, f is a conformal mapping of $\mathbb{D}$ onto a region that is conformally equivalent to D_k under a conformal automorphism of $\mathbb{D}$ which contains the origin and whose boundary is externally tangent to the circle $\{w : |w| = M_k(\alpha)\}$. In this case it is straight forward to check that f must have the prescribed form.

§V. The hyperbolic Bloch-Landau constant for the family $K_h(k,\alpha)$. We derive a sharp lower bound for the hyperbolic density of the hyperbolic metric for a hyperbolically k-convex region ($k \geq 2$) in terms of a uniform upper bound on γ_Ω. We use an extremal region which is similar to that employed in [MM$_1$] and [MM$_2$] .

Initially, we suppose $k \geq 2$, $\theta \in (0, \pi/2)$ and $N = \tan\theta$. Let $R = \sqrt{N(2-kN)/(k-2N)}$; R is selected so that the circle through $-R$, iN and R has hyperbolic radius (1/2) artanh $(2/k)$ = artanh $[(\sqrt{k^2+4}-k)/2]$, or equivalently, has hyperbolic curvature k. Let $H = H(N) = \text{int}\,[-R, R]$. Note that for $N = (\sqrt{k^2+4} - k)/2$ the set H is actually a hyperbolic disk. In all cases, H contains the disk $\{z : |z| < N\}$, but no larger disk centered at the origin, and H is contained in the disk $D = \{z : |z| < R\}$. Each of the two circular arcs bounding H makes an angle 2φ with the segment $[-R,R]$, where $\varphi = \arctan(N/R)$.

We also introduce a certain collection of "triangular" hyperbolically k-convex regions. Let $\mathfrak{T} = \mathfrak{T}(N)$ denote the family of all hyperbolically k-convex regions that contain the disk $\{z : |z| < N\}$ and are bounded by three distinct circular arcs each of hyperbolic radius (1/2) artanh $(2/k)$ and having the property that the full circles are tangent to $|z| = N$ and contain $\{z : |z| < N\}$ in their interior. Each of these circular arcs will meet ∂D in diametrically opposite points and has euclidean radius $k' = (1+N^2)/(k-2N)$. Therefore, each region Δ in $\mathfrak{T}$ is both hyperbolically k-convex and euclidean k'-convex. From [MM1, Lemma 2] we obtain the following result.

LEMMA 2. *If $\Delta \in \mathfrak{T}$, then for $z \in \Delta$, $\nu_\Delta(z) > (\pi/4\varphi)\, \nu_D(z) \geq (\pi/4\varphi R)$.*

THEOREM 2. *Suppose Ω is a hyperbolically k-convex region, where $k \geq 2$. Let $n = \max\{\gamma_\Omega(z) : z \in \Omega\}$ and $N = \tanh n$. Then*

$$\nu_\Omega(z) \geq \frac{\pi}{4}\sqrt{\frac{k-2N}{N(2-kN)}}\;\frac{1}{\arctan\sqrt{\dfrac{N(k-2N)}{2-kN}}}.$$

Equality holds at a point $a \in \Omega$ if and only if there is a conformal automorphism T of $\mathbb{D}$ such that $\Omega = T(H)$ and $a = T(0)$.

Proof. Select $a \in \Omega$ with $\Gamma_\Omega(a) = N$. From Proposition 3 we see that

$$\gamma_\Omega(a) \leq \frac{1}{2} \text{ artanh } \frac{2}{k} = \text{artanh } \frac{\sqrt{k^2+4}-k}{2}$$

with equality if and only if Ω is a hyperbolic disk with center a and hyperbolic radius $(1/2)$ artanh $(2/k)$. Hence, $N \leq (k - \sqrt{k^2-4})/2$ with equality if and only if Ω is a hyperbolic disk with center a and radius (1/2) artanh $(2/k)$.

First, suppose $N = (k - \sqrt{k2 - 4})/2$. Then Ω is a hyperbolic disk centered at a and so from Example 2,

$$\nu_\Omega(z) = \frac{1 - \Gamma^2_\Omega(z)}{\Gamma_\Omega(z)\,[2 - k\,\Gamma_\Omega(z)]}.$$

The right-and side of this identity is strictly decreasing function of $\Gamma_\Omega(z)$, so we obtain

$$\nu_\Omega(z) \geq \frac{1 - N^2}{N\left[2 - kN\right]}$$

with strict inequality unless $z = a$. This is the desired result in this case.

Now, assume that $0 < N < (k - (\sqrt{k2 - 4}\,)/2$. We may suppose that $a = 0$ since all quantities involved are invariant under conformal automorphisms of $\mathbb{D}$. Let $I = \{z : |z| = N \text{ and } z \in \partial\Omega\}$. The set I si nonempty and closed. A result of Blaschke [B] for euclidean convexity readily extends to hyperbolic convexity and implies that I cannot be contained in a closed subarc of the circle $|z| = N$ with angular length strictly less than π. Now the proof completely parallels that of [MM_1, Thm.4], so all further details are omitted.

The function

$$h_h(t) = \sqrt{\frac{t\,(2 - kt)}{k - 2t}} \text{ arctan } \sqrt{\frac{t\,(k - 2t)}{2 - kt}}$$

is strictly increasing on $[0, (k - (\sqrt{k^2 - 4})/2]$ with maximum value

$$h_k(k - \sqrt{k^2 - 4}/2) = \frac{\pi}{4}(k - \sqrt{k^2 - 4})/2).$$

Hence, for $\alpha \in [0, (k - (\sqrt{k^2 - 4})/2]$ the equation $h_k(t) = \alpha\pi/4$ has a unique solution $N(\alpha) \in [0, (k - (\sqrt{k^2 - 4})/2]$.

COROLLARY 1. (Bloch-Landau constant for $K_h(k, \alpha)$, k ≥ 2). *Let $f \in K_h(k, \alpha)$ where $k \geq 2$. Then either $f(\mathbb{D})$ contains an open hyperbolic disk with radius strictly larger than artanh $N(\alpha)$ or else $f(z) = e^{-i\psi} F(e^{i\psi} z)$ for some $\psi \in \mathbb{R}$, where*

$$F(z) = \sqrt{\frac{N(\alpha)(2 - kN(\alpha))}{k - 2N(\alpha)}} \tanh\left(\frac{2}{\alpha}\sqrt{\frac{N(\alpha)(2 - kN(\alpha))}{k - 2N(\alpha)}} \log \frac{1+z}{1-z}\right)$$

belongs to $K_h(k, \alpha)$ and maps $\mathbb{D}$ conformally onto $H(N(\alpha))$.

Proof. Set $\Omega = f(\mathbb{D})$ and $N = \max\{\Gamma_\Omega(z) : z \in \Omega\}$. If $N > N(\alpha)$, then we are done. Assume $N \leq N(\alpha)$. Then $h_k(N) \leq h_k(N(\alpha)) = \alpha\pi/4$. Since $\lambda_\Omega(0) = 1/f'(0) = 1/\alpha$, the theorem with $z = f(0) = 0$ gives $1/\alpha \geq \pi/4h_k(N)$, or $h_k(N) \geq \alpha\pi/4$. Now $h_k(N) = \alpha\pi/4$, so $N = N(\alpha)$. Thus, equality holds in the theorem at the origin, so Ω is just the image under a conformal automorphism of $\mathbb{D}$ of $H(N(\alpha))$. Since $F \in K_h(k, \alpha)$ and maps $\mathbb{D}$ onto $H(N(\alpha))$, we conclude $f(z) = e^{-i\psi} F(e^{i\psi} z)$ for some $\psi \in \mathbb{R}$.

For $0 \leq k < 2$, the Bloch-Landau constant for $K_h(k, \alpha)$ cannot be determined for all values of α by our method. However, our method does apply to hyperbolically k-convex regions when $0 < k < 2$, and a restriction is placed on $N = \max\{\Gamma_\Omega(z) : z \in \Omega\}$. For $0 < k < 2$ we required that $0 < N < k/(2 + \sqrt{4 - k^2})$. Then if we define the region H as before, the arcs bounding H meet the interval (-1,1) at the points $\pm R$, where $R = \sqrt{N(2 - kN)/(k - 2N)}$. Because of the condition on N we have $0 < R \leq 1$. Also, these arcs make the angle 2φ with the real axis as before. We then proceed as in the case k ≥ 2 and we obtain the following corollary.

COROLLARY 2. *Suppose* $f \in K_h(k, \alpha)$ *where* $0 < k < 2$ *and the restriction* $0 < \alpha < (4/\pi)\arctan\ [k/(2 + \sqrt{4 - k^2})]$. *Then either* $f(\mathbb{D})$ *contains an open hyperbolic disk with radius strictly larger than* artanh $N(\alpha)$ *or else* $f(z) = e^{-i\psi} F(e^{i\psi} z)$ *for some* $\psi \in \mathbb{R}$.

§VI. Open Problems. The analogs of the applications of the reflection principle for the hyperbolic metric that were given in [MM_1] and [MM_2] are not given here since the method does not seem to extend to hyperbolic k-convexity. We list some of these open problems for hyperbolic k-convexity. First, what are sharp bounds for the hyperbolic curvature (relative to the unit disk) of a hyperbolic geodesic in a hyperbolically k-convex sub-region of $\mathbb{D}$? Also, determine a sharp upper bound for $|f''(0)|$ when $f \in K_h(k, \alpha)$. Another problem is to determine an analytic characterization of the family $K_h(k, \alpha)$. Finally, find the Bloch-Landau constant and the extremal functions for $K_h(k, \alpha)$ when $0 \le k < 2$ when $\alpha \ge (4/\pi)$ arctan $[k/(2 + \sqrt{4 - k^2})]$.

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Universidad Nacional de Colombia
Departamento de Matemáticas
Apartado Aéreo 568
MEDELLIN - COLOMBIA

Department of Mahtematics
University of Cincinnati
Cincinnati,Ohio 45221-0025
U. S. A.

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