THE CONJUGATES AND PRECONJUGATES OF LINEAR OPERATORS

by

KATHY K. FORBES

(Universidad de Maryland, E. U. A.)

I. THE PRECONJUGATE

1. *Definition*. With the aid of a systematic study by A. E. Taylor and S. Goldberg, one can gain knowledge about a linear operator T by studying its conjugate operator T'. However, it is at times difficult to study T' when T is a map from ℓ_∞ into ℓ_∞, since $(\ell_\infty)'$ is rather complicated. While the conjugate is a map between dual spaces, Y' and X', the preconjugate operates between spaces, X and Y. For example, the preconjugate of an operator from ℓ_∞ to ℓ_∞ is a map between ℓ_1 and ℓ_1. These remarks are also applicable to operators on $\mathcal{L}_\infty$, the Lebesgue space of functions bounded almost everywhere.

In the following discussion X and Y will always be normed linear spaces and X' and Y' their dual or adjoint spaces. T will represent a linear operator and $\mathcal{D}(T)$ will be the domain of T. Unless otherwise noted x, y, x' and y' will be elements of X, Y, X' and Y' respectively.

Definition. A subset F of X' is total in X' if $x'x = 0$ for all $x' \in F$ implies $x = 0$.

Let A be a linear operator mapping X to Y with $\mathcal{D}(A)$ dense in X. Let T be its conjugate operator mapping Y' to X'. In other words $\mathcal{D}(T) = \{ y' \in Y' \,/\, y'A$ is continuous on $\mathcal{D}(A)\}$, and Ty' is the unique continuous extension of y'A to the whole of X. We would like to define the preconjugate, 'T, of T mapping X to Y. 'T should be equal to A or at least an extension of A. If $x \in \mathcal{D}(A)$, then $y'Ax = A'y'x = Ty'x$ for all $y' \in \mathcal{D}(T)$. Thus motivated, we say $x \in \mathcal{D}('T)$ if there exists $y \in Y$ such that $y'y = Ty'x$ for all $y' \in \mathcal{D}(T)$, and we define $'Tx = y$. In order for $'T$ to be well defined we require that the domain of T be total in Y. We give the following

Definition. Let X and Y be normed linear spaces and T a linear operator mapping Y' to X' with $\mathcal{D}(T)$ total in Y'. Then the preconjugate, 'T, of T maps X to Y and is defined as follows. An element $x \in X$ lies in the domain of 'T if there exists a $y \in Y$ such that $y'y = Ty'x$ for all $y' \in \mathcal{D}(T)$.
Then $'Tx = y$.

Notice that the preconjugate of T is only defined when T is a map between dual spaces and the domain of T is total. The restriction that $\mathcal{D}(T)$ be total is necessary to insure that 'T is well-defined, for suppose y and y_0 are elements of Y with the property that $y'y = y'y_0 = Ty'x$ for all $y' \in \mathcal{D}(T)$. Then by the totality of $\mathcal{D}(T)$, $y = y_0$.

It follows immediately from the definition that $Ty'x = y'('Tx)$ if and only if $x \in \mathcal{D}('T)$ and $y' \in \mathcal{D}(T)$. A common mistake is to assert that $Ty'x = y'('Tx)$ when it is not necessarily true that $x \in \mathcal{D}('T)$ or that $y' \in \mathcal{D}(T)$.

2. *Properties of the Preconjugate.* Let A be a linear operator mapping X to Y. The graph of A, $G(A)$, equals $\{(a, Ax) \in X \times Y \,/\, x \in \mathcal{D}(A)\}$. G(A) is a subspace of $X \times Y$. We say that A is a closed operator if its graph is closed in $X \times Y$ and that A is closeable if it has a closed extension. If A is closeable, let $\bar{A}$ denote the minimal closed extension of A; $G(\bar{A}) = \overline{G(A)}$.
Suppose the conjugate, A', of A exists; i.e., $\mathcal{D}(A)$ is dense in X. Then A is closeable if and only if $\mathcal{D}(A')$ is total in Y'.

If M is a subset of X, then
$M^0 = \{ x' \in X' \,/\, x'x = 0 \text{ for all } x \in M\}$.

Likewise if N is a subset of X', then

$$^{\circ}N = \{ x \in X \ / \ x'x = 0 \text{ for all } x' \in N \}.$$

We first prove an elementary consequence of the definition of the preconjugate. Unless otherwise noted we assume T maps Y' to X' and that $\mathcal{D}(T)$ is total in Y'.

Theorem 1. The preconjugate of T is a closed linear operator.

Proof. Clearly 'T is linear. Suppose that $\{x_n\}$ is a sequence in $\mathcal{D}('T)$ and that $x_n \to x$ and $'Tx_n \to y$. To show 'T is closed we must show $x \in \mathcal{D}('T)$ and $'Tx = y$. Let $y' \in \mathcal{D}(T)$. Ty' is continuous, hence $Ty'x = \lim_{n\to\infty} Ty'x_n$; likewise $y'y = \lim_{n\to\infty} y'('Tx_n)$. But $x_n \in \mathcal{D}('T)$ implies $y'('Tx_n) = Ty'x_n$. Hence $Ty'x = y'y$. Since y' was an arbitrary element in $\mathcal{D}(T)$, this equality holds for all $y' \in \mathcal{D}(T)$. Thus $x \in \mathcal{D}('T)$ and $'Tx = y$. ∎

In the following discussion we assume that the domain of the preconjugate is dense in X. This will enable us to consider the conjugate of the preconjugate, i.e. ('T)' exists. We now want to study the relation between T and ('T)'.

Theorem 2. Let T be a linear operator from Y' to X' such that the domain of T is total and the domain of its preconjugate is dense in X. Then T is closeable and ('T)' is an extension of $\bar{T}$.

Proof. Let $y' \in \mathcal{D}(T)$. Then if $x \in \mathcal{D}('T)$, we have $y'(Tx) = Ty'x$. In other words, $y' \circ {'T} = Ty'$ on $\mathcal{D}('T)$. Thus $y' \in \mathcal{D}(('T)')$ and $('T)'y' = Ty'$. Therefore ('T)' is a closed extension of T and so T is closeable. Clearly ('T)' is an extension of $\bar{T}$. ∎

We have at our disposal theorems concerning linear operators and their conjugates, see [1]. In the special cases where ('T)' = T We can use these diagrams by letting 'T be the operator and T its conjugate. We will show that $\bar{T}$ = ('T)' whenever both X and Y are reflexive or T is the conjugate of some operator. For our first result we will need the following lemmas. Some of the lemmas are well known results, and will be stated without proof.

Lemma 1. Let X and Y be normed linear spaces. Let $X \times Y$ have the norm $\|(x,y)\| = \|x\| + \|y\|$ and $X' \times Y'$ the norm $\|(x',y')\| = \max\{\|x'\|, \|y'\|\}$. Then there exists a linear isometry i between $X' \times Y'$ and $(X \times Y)'$ defined by

$$[i(x', y')](x,y) = x'x + y'y$$

where $(x',y') \in X' \times Y'$ and $(x,y) \in X \times Y$.

This lemma permits $Y' \times X'$ and $(Y \times X)'$ to be identified, which we shall do. Notice that under the identification $z' \in (X \times Y)'$ implies $i^{-1}z' = (x',y')$ where $x'x = z'(x,0)$ and $y'y = z'(0,y)$.

Lemma 2. If T is a linear operator from Y' to X' and $\mathcal{D}(T)$ is total, then $(y,x) \in {}^{\circ}G(T)$ if and only if $x \in \mathcal{D}('T)$ and $'Tx = -y$; i. e., $(x,-y) \in G('T)$.

Proof. An element (y,x) is in ${}^{\circ}G(T)$ if and only if $(y', Ty')(y,x) = y'y + Ty'x = 0$ for all $y' \in \mathcal{D}(T)$. But this statement is equivalent to $x \in \mathcal{D}('T)$ and $'Tx = -y$. ∎

Lemma 3. If T maps Y' to X' and the domain of T is total in Y' and the domain of $'T$ is dense in X, then

$$({}^{\circ}G(T))^{\circ} = G('T)'.$$

Proof. An element (y_0', x_0') is in $({}^{\circ}G(T))^{\circ}$ if and only if $(y_0', x_0')(y,x) = y_0'y + x_0'x = 0$ for all $(y,x) \in ({}^{\circ}G(T))$. But by lemma 2 we have the equivalent statement that $y_0'(-'Tx) + x_0'x = 0$ for all $x \in \mathcal{D}('T)$. This, however, is a necessary and sufficient condition for (y_0', x_0') to be in $G('T)'$. (Recall that $A'y' = x'$ if and only if $y'(Ax) = x'x$ for all $x \in \mathcal{D}(A)$). ∎

Lemma 4. If X is a reflexive and N a closed subspace of X', then $N = ({}^{\circ}N)^{\circ}$.

Proof. If x' is in N then clearly x' annihilates all the elements which are annihilated by N. In other words x' is in $({}^{\circ}N)^{\circ}$. Thus we have $N \subset ({}^{\circ}N)^{\circ}$.

Suppose x_0' is in $({}^{\circ}N)^{\circ}$ but not in N. N is closed, hence by the Hahn-Banach theorem, there exists $x_0'' \in (X')'$ such that $x_0''(x_0') \neq 0$ and $x_0''(x') = 0$ for all $x' \in N$. Since X is reflexive, there exists $x_0 \in X$ such that $x_0' x_0 \neq 0$ and $x' x_0 = 0$ for all $x' \in N$. This implies $x_0 \in {}^{\circ}N$, but by assumption $x_0' \in ({}^{\circ}N)^{\circ}$. Thus $x_0' x_0 = 0$, which is a contradiction. ∎

Lemma 5. If X and Y are reflexive then XxY is reflexive.

We are now ready to prove the following.

Theorem 3. Let T be an operator from Y' to X' such that ('T)' exists; i.e., $\mathfrak{D}(T)$ is total in Y' and $\mathfrak{D}('T)$ is dense in X. If X and Y are reflexive then $('T)' = \bar{T}$.

Proof. We shall show that $G(\bar{T}) = G('T)'$. By lemma 5, Y' x X' is reflexive; hence we can apply lemma 4 which tells us that $({}^{\circ}G(T))^{\circ} = \overline{G(T)} = G(\bar{T})$. By lemma 3, $({}^{\circ}G(T))^{\circ} = G('T)'$, and so $G(\bar{T}) = G('T)'$. ∎

Theorem 4. If A is a linear operator mapping X to Y such that $\overline{\mathfrak{D}(A)} = X$ and $\mathfrak{D}(A')$ is total in Y', then $\bar{A} = '(A')$.

Proof. If $x \in \mathfrak{D}(A)$ and $y' \in \mathfrak{D}(A')$ then $y'(Ax) = (A'y')x$. Hence $x \in \mathfrak{D}('(A'))$ and $'(A')x = Ax$. Thus $'(A')$ is an extension of A, hence of $\bar{A}$, since by theorem 1 the preconjugate is a closed operator.

To complete the proof we will show that G('(A')) is contained in $\overline{G(A)} = G(\bar{A})$. If not, there exists $x_0 \in \mathfrak{D}('(A'))$ such that $(x_0, '(A')x_0) \notin \overline{G(A)}$. By the Hahn Banach Theorem there exists

$(x', y') \in (X \times Y)'$ (we identify $(X \times Y)'$ and $X' \times Y'$ as before) such that (1) $(x', y')(x_0, '(A')x_0) = x'x_0 + y'('(A')x_0) \neq 0$ and (2) $(x', y')(x, Ax) = x'x + y'(Ax) = 0$ for all $x \in \mathcal{D}(A)$.

We first note that y' is in $\mathcal{D}(A')$ since by (2) $y' \cdot A = -x'$ on $\mathcal{D}(A)$. $\overline{\mathcal{D}(A)} = X$ so there exists a sequence $\{x_n\}$ in $\mathcal{D}(A)$ which converges to x_0. From (2) we have

$$0 = x'x_n + y'Ax_n = x'x_n + (A'y')x_n \rightarrow x'x_0 + (A'y')x_0 =$$

$$= x'x_0 + y'('(A')x_0).$$

Thus we have contradicted (1). ∎

Corollary. If T maps Y' to X' with $\mathcal{D}(T)$ total in Y' and $T = A'$ for some operator A, then $T = ('T)'$.

Proof. T is closed since the conjugate of a linear operator is closed. By the above theorem, $\bar{A} = '(A') = 'T$. Hence $('T)' = \bar{A}'$. Therefore to complete the proof we need only show that $A' = \bar{A}'$.

Suppose $y' \in \mathcal{D}(\bar{A}')$ and $\bar{A}'y' = x'$. Then $y'\bar{A}x = x'x$ for all $x \in \mathcal{D}(A)$. Thus $y'Ax = x'x$ for all $x \in \mathcal{D}(A)$ which implies $y' \in \mathcal{D}(A')$ and $A'y' = x'$.

Suppose $y' \in \mathcal{D}(A')$ and $\bar{A}'y' = x'$. Let x_0 be in $\mathcal{D}(\bar{A})$ and let $\{x_n\}$ be a sequence in $\mathcal{D}(A)$ such that $(x_n, Ax_n) \rightarrow (x_0, \bar{A}x_0)'$. Then since $y'Ax = x'x$ for all $x \in \mathcal{D}(A)$ we have

$$x'x_0 = \lim_{n \to \infty} x'x_n = \lim_{n \to \infty} y'Ax_n = y'\bar{A}x_0.$$

Hence $y'\bar{A} = x'$ on $\mathcal{D}(\bar{A})$. Thus $y' \in \mathcal{D}(\bar{A})$ and $\bar{A}'y' = x'$. ∎

If we add the hypothesis that X and Y are reflexive we get another representation for $\bar{A}$. For simplicity we shall denote $(X')'$ by X'' and $(A')'$ by A''.

Theorem 5. Suppose X and Y are reflexive and J_X and J_Y are the respective isometries onto their second duals. Let A be a linear operator from X to Y with the domain of A dense in X and the domain of A' total in Y'. Then $\mathcal{D}(A')$ is dense in Y' and $J_Y^{-1}(A'')J_X = \bar{A}$.

Proof. We first note that $\mathcal{D}(A')$ is dense in Y' since $\mathcal{D}(A')$ is total and X' is reflexive. Recall that $(J_X x)x' = x'x$ and $y'(J_Y^{-1}y'') = y''y'$.

We first show that $J_Y^{-1}(A'')J_X$ is an extension of A. Let $x \in \mathcal{D}(A)$. To show $x \in \mathcal{D}(J_Y^{-1}(A'')J_X)$ it suffices to show that $J_X x \in \mathcal{D}(A'')$; i.e., we must show $(J_X x)A'$ is continuous on $\mathcal{D}(A')$. If $y' \in \mathcal{D}(A')$, then

$$\|(J_X x)A'y'\| = \|A'y'x\| = \|y'Ax\| \leq \|y'\| \|Ax\| .$$

If $y' \in \mathcal{D}(A')$, then

$$y'J_Y^{-1}(A''J_X x) = A''(J_X x)y' = ((J_X x)A'y') = A'y'x = y'Ax.$$

Hence since $\mathcal{D}(A')$ is total we have that $Ax = J_Y^{-1}A''J_X x$. This holds for all $x \in \mathcal{D}(A)$.

We now complete the proof by showing that $G(\bar{A}) = G(J_Y^{-1}(A'')J_X)$. To see this, one need only observe the following equivalent statements:

i) $(x, y) \in G(\bar{A})$

ii) there exists a sequence $\{x_n\}$ in $\mathcal{D}(A)$ such that $(x_n, Ax_n) \to (x, y)$

iii) $(x_n, J_Y^{-1}(A'')J_X x_n) \to (x, y)$ (This follows from the preceeding paragraph)

iv) $(J_X x_n, A''J_X x_n) \to (J_X x, J_Y y)$

v) $(J_X x, J_Y y) \in G(A'')$ since A'' is closed

vi) $A''(J_X x) = J_Y y$ or $J_Y^{-1}(A'')J_X x = y$

vii) $(x, y) \in G(J_Y^{-1}(A'')J_X x)$.

Hence $(x, y) \in G(\bar{A})$ if and only if $(x, y) \in G(J_Y^{-1}(A'')J_X)$. ∎

3. THE STATE OF A LINEAR OPERATOR.

Let A be a linear operator from X to Y. If A is one-to-one, then A^{-1} is a linear operator from the range of A into X. The state of a linear operator shall be described in terms of the following :

I. $R(A) = Y$,

II. $R(A) \neq Y$, but $\overline{R(A)} = Y$,

III. $\overline{R(A)} \neq Y$.

1. A has a bounded inverse,

2. A has an unbounded inverse,

3. A has no inverse.

By the various pairings of I, II, or III, with 1, 2, 3, nine conditions can thus be described relating to $R(A)$ and A^{-1}. For instance, it may be that $R(A) = Y$, and that A has a bounded inverse. This we will describe by saying that A is in state I_1, (written $A \in I_1$). A operator in state I we shall call surjective.

We shall use the above classification for both T and 'T. To the ordered pair of operators (T, 'T) we now make correspond an ordered pair of conditions which we call the "state" of (T, 'T). Thus if $T \in I_3$ and $'T \in III_1$, we say that (T, 'T) is in state (I_3, III_1) (written $(T, 'T) \in (I_3, III_1)$).

At times we shall use a notation such as $(T, 'T) \in (I_2, 3)$ to mean that $T \in I_2$ and 'T has no inverse.

We shall now exhibit several theorems which will enable us to determine which states can or cannot occur for the pair (T, 'T). The attention of the reader is called to the symmetry between theorems 2 and 3, 4 and 5, and 6 and 7. Remember T will always map Y' to X'; hence 'T maps X to Y.

THEOREM 6. If the range of T is total in X' then 'T is one-to-one. In particular, 'T $\in$ 3 implies T $\in$ III.

Proof. If 'Tx = 0, then for $y' \in \mathcal{D}(T)$, $0 = y'('Tx) = (Ty')x$, and so $x = 0$. ∎

THEOREM 7. If the range of 'T is dense in Y, then T has an inverse.

Proof. Suppose T has no inverse. Then there exists a $y_o' \in \mathcal{D}(T)$ such that $y_o' \neq 0$ and $Ty_o' = 0$. Say $y_o'y \neq 0$. The range of 'T is dense so we can find a sequence $\{x_n\}$ in $\mathcal{D}('T)$ such that $'Tx_n \to y$. But then $0 = (Ty_o')x_n = y_o'('Tx_n) \to y_o'y \neq 0$. ∎

We state without proof the following well known

LEMMA. A linear operator A does not have a bounded inverse if and only if there exists a sequence $\{x_n\}$ in the domain of A such that $\|x_n\| \to \infty$ and $Ax_n \to 0$.

THEOREM 8. If R(T) = X', then 'T has a bounded inverse.

Proof. In the theorem were not true, then by the lemma there would exist a sequence $\{x_n\}$ in $\mathcal{D}('T)$ such that $\|x_n\| \to \infty$ and $'Tx_n \to 0$. Let $'Tx_n = y_n$; then for all $y' \in \mathcal{D}(T)$, $y'y_n = y''Tx_n = Ty'x_n \to 0$. Hence since T is surjective, $x'x_n \to 0$ for all $x' \in X$. As a consequence of the Uniform Boundedness Principle, $\|x_n\| < M$ for some M. We have thus reached a contradiction. ∎

THEOREM 9. If Y is complete and 'T is surjective, then T has a bounded inverse.

Proof. Suppose the theorem is false, then by the lemma there exists a sequence $\{y_n'\}$ in $\mathcal{D}(T)$ such that $Ty_n' \to 0$ and $\|y_n'\| \to \infty$. If $x \in \mathcal{D}('T)$ we have $(Ty_n')x = y_n'('Tx) \to 0$. R('T) = Y, hence $y_n'y \to 0$ for all $y \in Y$. Y is complete, therefore by Uniform boundedness Principle, the sequence $\{y_n'\}$ is bounded which is a contradiction. ∎

Theorem 10. If the range of 'T is dense in Y and 'T has a continuous inverse, then T has a continuous inverse.

Proof. By Theorem 7, T^{-1} exists. We shall show it is bounded. First note that since 'T has a continuous inverse,

$$\frac{1}{\|'Tx\|} \leq \frac{\|('T)^{-1}\|}{\|x\|},$$

for all $x \neq 0 \in \mathcal{D}('T)$. This gives us the following expression :

$$\|T^{-1}x'\| = \sup_{y \neq 0} \frac{|(T^{-1}x')y|}{\|y\|} = \sup_{\substack{x \in \mathcal{D}('T) \\ x \neq 0}} \frac{|(T^{-1}x')'Tx|}{\|'Tx\|} =$$

$$\sup_{\substack{x \in \mathcal{D}('T) \\ x \neq 0}} \frac{|T(T^{-1}x')x|}{\|'Tx\|} \leq \sup_{\substack{x \in \mathcal{D}('T) \\ x \neq 0}} \frac{\|x'\| \, \|x\|}{\|'Tx\|} \leq \sup_{\substack{x \in \mathcal{D}('T) \\ x \neq 0}} \frac{\|x'\| \|x\| \, \|('T)^{-1}\|}{\|x\|}$$

$$= \|x\| \, \|('T)^{-1}\|. \blacksquare$$

Observe that the second equality is valid because $\overline{R('T)} = Y$.

Theorem 11. If the range of T is dense in X' and T^{-1} exists and is continuous, then $('T)^{-1}$ is continuous. (If $T \in I_1$ or II_1, then $'T \in 1$.)

Proof. This proof is analogous to the one above. Notice that since T^{-1} is bounded we have for $y \neq 0$ in $\mathcal{D}(T)$

$$\frac{1}{\|Ty'\|} \leq \frac{\|T^{-1}\|}{\|y'\|}$$

and that if $x \in X$, then $x = \sup_{x' \neq 0} \frac{|x'x|}{\|x'\|}$

By Theorem 2, $('T)^{-1}$ exists. Hence if $y \in R('T)$, then

$$\|('T)^{-1} y\| = \sup_{x' \neq 0} \frac{|x'('T)^{-1}y|}{\|x'\|} = \sup_{\substack{y' \in \mathcal{D}(T) \\ y' \neq 0}} \frac{|Ty'('T)^{-1}y|}{\|Ty'\|} =$$

$$= \sup_{\substack{y' \in \mathcal{D}(T) \\ y' \neq 0}} \frac{|y'('T('T)^{-1}y)|}{\|Ty'\|} \leq \sup_{\substack{y' \in \mathcal{D}(T) \\ y' \neq 0}} \frac{\|y'\| \cdot \|y\|}{\|Ty'\|} \leq \sup_{\substack{y' \in \mathcal{D}(T) \\ y' \neq 0}} \frac{\|y'\| \cdot \|y\| \cdot \|T^{-1}\|}{\|y'\|}$$

$= \|y\| \, \|T^{-1}\|$. Hence $'T^{-1}$ is bounded. ▮

Lemma. If A is a closed, continuous linear operator from a normed linear space X into a Banach space Y, then $\mathcal{D}(A)$ is closed.

Proof. Suppose x is a limit point of $\mathcal{D}(A)$ and $\{x_n\}$ is a sequence contained in $\mathcal{D}(A)$ converging to x. A is continuous, hence $\{Ax_n\}$ is a Cauchy sequence in Y. Y is complete, therefore there exists a $y \in Y$ such that $Ax_n \to y$. Then since A is a closed operator x must be in the domain of A. ▮

Theorem 12. If X is complete and 'T has a continuous inverse then R('T) is closed. (If X is complete then $'T \notin II_1$).

Proof. By the Theorem 1, 'T is closed, hence $('T)^{-1}$ is also closed. Thus we can apply the lemma to $('T)^{-1}$ and the result follows. ▮

Theorem 13. If X is reflexive and $\overline{R(T)} \neq X'$, then 'T is not one-to-one.

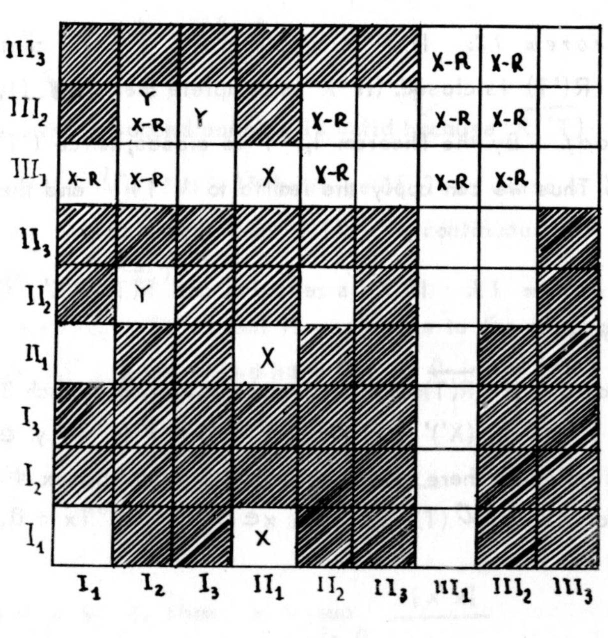
Proof. If $\overline{R(T)} \neq X'$, then by the Hahn-Banach Theorem there exists $x'' \neq 0$ in $(X')'$ such that $x''(Ty') = 0$ for all $y' \in \mathcal{D}(T)$. Since X is reflexive there exists an $x \in X$ such that $x \neq 0$ and $Ty'x = 0$ for all $y' \in \mathcal{D}(T)$. But then $x \in \mathcal{D}('T)$ and $'Tx = 0$. ▮

4. THE STATE DIAGRAM OF PAIRS (T, 'T).

Similar pairings were first done by A. E. Taylor for a bounded operator and its conjugate. In order to present systematically which states can or cannot occur for T and its conjugate, a "state diagram" was constructed. This diagram is a large square divided into 81 congruent smaller squares arranged in rows and columns. Each column is labeled at the bottom denoting a given state for T, and the rows represent states for T'. The small square which is the intersection of a certain column and row denotes the state of the pair (T, T'). Squares belonging to states which cannot exist are blacked out.

Based on the theorems of the last section we have constructed such a state diagram for T and its preconjugate. A square is crossed out if the corresponding state is imposible. If a square contains X then the corresponding state cannot occur if X is complete, likewise for Y. X-R in the square implies that the corresponding state will not exist if X is reflexive.

STATE DIAGRAM FOR AN OPERATOR AND ITS PRECONJUGATE



2 - EXAMPLES OF ADMISSIBLE STATES

We have presented a state diagram for the linear operator and its preconjugate. The obvious question now is whether any more of the squares can be crossed out. In this chapter we will show this is not possible by exhibiting examples of operators along with their conjugates which have states corresponding to the empty squares.

In all of the examples T will map between the infinite sequence spaces ℓ_1, ℓ_2 and ℓ_∞. Thus 'T maps between the spaces such as C_0, ℓ_2, ℓ_1 and in some cases a dense subspace of C_0, ℓ_2 or ℓ_1.

We begin by proving some propositions concerning various linear operators between sequence spaces and some theorems on linear operators in general.

Propositon 1. If $Ty' = y'$ then $'Tx = x$ and $\mathcal{D}('T) = \{ x \in X / x \in Y \}$.

Proof. Given $x \in X$, then $Ty'x = y'x$ for all $y' \in Y'$, hence for all $y' \in \mathcal{D}(T)$, Thus $x \in \mathcal{D}('T)$ whenever $x \in Y$ and $'Tx = x$. ∎

Proposition 2. If $T(u_1, u_2, \ldots) = (u_1, 2u_2, 3u_3 \ldots)$ then $'T(x_1, x_2, \ldots) = (x_1, 2x_2, 3x_3, \ldots)$ and

$$\mathcal{D}('T) = \{ (x_1, x_2, \ldots) \in X / (x_1, 2x_2, 3x_3, \ldots) \in Y \} .$$

We shall denote this operator by U.

Proof. If $x \in X$ then

$Ty'x = (u_1, 2u_2, 3u_3, \ldots)(x_1, x_2, \ldots) = u_1x_1 + 2u_2x_2 + 3u_3x_3 + \ldots = (u_1, u_2, \ldots)(x_1, 2x_2, 3x_3, \ldots) = y'('Tx)$

for all $y' \in Y$. Thus the desired result follows. ∎

Proposition 3. If $T(u_1, u_2, \ldots) = (u_1, 1/2u_2, 1/3u_3, \ldots)$ then 'T is the same operator with $\mathcal{D}('T) =$

$$\{ (x_1, x_2, \ldots) \in X / (x_1, 1/2x_2, 1/3x_3, \ldots) \in Y \} .$$

We shall denote this operator by D.

Proof. $Ty'x = (u_1, 1/2u_2, \ldots)(x_1, \ldots) = u_1x_1 + 1/2u_2x_2 + \ldots = (u_1, u_2, \ldots)(x_1, 1/2x_2, \ldots) = y'('Tx)$ for all $y' \in Y'$. The proposition follows. ∎

Proposition 4. If $T(u_1, u_2, \ldots) = (u_2, u_3, \ldots)$ then $'T(x_1, x_2, \ldots) = (0, x_1, x_2, \ldots)$ and

$$\mathcal{D}('T) = \{ (x_1, x_2, \ldots) \in X / (0, x_1, x_2, \ldots) \in Y \} .$$

Proof. $Ty'x = (u_2, u_3, \ldots)(x_1, x_2, \ldots) = u_2x_1 + u_3x_2 + \ldots = (u_1, u_2, \ldots)(0, x_1, x_2, \ldots) = y'('Tx)$ for all $y' \in Y$. Hence when T is a shift to the left 'T is a shift to the right. ∎

Proposition 5. If $T(u_1, u_2, \ldots) = (0, u_1, u_2, \ldots)$ then $'T(x_1, x_2, \ldots) = (x_2, x_3, \ldots)$ and

$$\mathcal{D}('T) = \{ (x_1, x_2, \ldots) \in X/(x_2, x_3, \ldots) \in Y \},$$

Proof. $Ty'x = (0, u_1, u_2, \ldots)(x_1, x_2, \ldots) = u_1x_2 + u_2x_3 + \ldots = (u_1, u_2, \ldots)(x_2, x_3, \ldots) = y'('Tx)$ for all $y' \in Y$. The desired result follows. ∎

We shall denote the operator which shifts to the right by R and the one that shifts to the left by L. So far we have shown that 'D = D, 'U = U, 'R = L, and 'L = R.

THEOREM 1. Suppose A maps Z' to X' and B maps Y' to Z'. Then if $\mathcal{D}$(A) is total in Z' and $\mathcal{D}$(B) is total in Y', '(AB) is an extension of 'B'A.

Proof. Let $x \in \mathcal{D}$('B'A) and y = ('B'A)x. Then

i) $x \in \mathcal{D}$('A) and z'('Ax) = (Az') x for all $z' \in \mathcal{D}$(A) and

ii) $'Ax \in \mathcal{D}$('B) and y'('B('Ax)) = By'('Ax) for all $y' \in \mathcal{D}$(B).

We must show that y'y = (AB) y'x for all $y' \in \mathcal{D}$(AB). But if $y' \in \mathcal{D}(AB)$ then $y' \in \mathcal{D}(B)$ and $By' \in \mathcal{D}(A)$. Hence by i) and ii) $y'y = y'('B('Ax)) = By'('Ax) = A(By')x$. ∎

Proposition 6. If $T(u_1, u_2, \ldots) = (1/2u_2, 1/3u_3, \ldots)$ and the set

$$S = \{ Dx \ / \ x \in X \}$$

is contained in Y, then

$$'T(x_1, x_2, \ldots) = (0, 1/2x_1, 1/3x_2, \ldots) \text{ and } \mathcal{D}('T) = X.$$

Proof. $T = LD$ where $D : Y' \longrightarrow X'$, $\mathcal{D}(D) = \mathcal{D}(T)$ and $L : X' \longrightarrow X'$ with $\mathcal{D}(L) = X'$. Hence by the above theorem $'T$ is an extension of $'D'L$. By propositions 3 and 4 and the fact that $S \subset Y$, it is easy to see that $\mathcal{D}('D'L) = X$.

Thus $'T = 'D'L = DR$ and $\mathcal{D}('T) = X$. ∎

Proposition 7. If $T(u_1, u_2, \ldots) = (0, u_1, 1/2u_2, \ldots)$ and S is as described above, then $'T(x_1, x_2, \ldots) = (x_2, 1/2x_3, \ldots)$ and $\mathcal{D}('T) = X$.

Proof. The proof is similar to the one above. Here $T = RD$ and $'T = DL$. ∎

Proposition 8. If $D : \ell_p \rightarrow \ell_q$ and if all (except perhaps a finite number) of the coordinate unit vectors ε_i are contained in the domain of D, then D has an unbounded inverse.

Proof. Clearly D is one-to-one. The norms of the $n\varepsilon_n$ go to infinity in ℓ_p but the norm of $T(n\varepsilon_n)$ in ℓ_q is one. Hence T^{-1} is unbounded. ∎

Corollary. If D is followed or preceeded by a right or left shift then if the inverse of the composite map exists, it will be unbounded.

Theorem 2. Let A be a dense subspace of X and $\mathcal{A}$ the isometry from X' onto A' defined by $\mathcal{A}x' = x'$ restricted to A. If T maps Y' to X' with $\mathcal{D}(T)$ total, then $'(\mathcal{A}T) = 'T$ restricted to A.

Proof. $\mathcal{I}: A \to X$ is the identity on A for if $x \in A$ and $x' \in X'$ then $x'x = x'x$. We know by theorem 1 that $'(\mathcal{I}T)$ is an extension of $'T\mathcal{I} = 'T$ restricted to A.

Since $'(\mathcal{I}T)$ maps A to Y, to complete the proof we need to show that if $x \in \mathcal{D}'(\mathcal{I}T)$, then $x \in \mathcal{D}('T)$ and $'(\mathcal{I}T)x = 'Tx$. Let $x \in \mathcal{D}'(\mathcal{I}T)$ and $y = '(\mathcal{I}T)x$. Then $\mathcal{I}(Ty')x = y'x$ for all $y' \in \mathcal{D}(\mathcal{I}T) = \mathcal{D}(T)$. But since $x \in \mathcal{D}'(\mathcal{I}T)$ implies $x \in A$, $(Ty')x = Ty'x$. Therefore $Ty'x = y'x$ for all $y' \in \mathcal{D}(T)$. ∎

Theorem 3. Let A be a dense subspace of Y and $\mathcal{I}$ the linear isometry mapping A' onto Y' defined by $\mathcal{I}a'$ the unique continuous extension of a' to all of Y Suppose T maps Y' to X' with $\mathcal{D}(T)$ total in Y'. Then $'(T\mathcal{I})$ mapping X to A is the restriction of $'T$ to $\{x \in \mathcal{D}('T) \,/\, 'Tx \in A\}$.

Proof. We first look at $'\mathcal{I}$ which maps Y to A. If $y \in A$ then $\mathcal{I}a'y = a'y$ for all $a' \in A'$. Hence A is contained in the domain of $'\mathcal{I}$ and $'\mathcal{I}y = y$ for all $y \in A$. Since $R(\mathcal{I}) = Y'$ we see by the state diagram that $'\mathcal{I}$ is one-to-one. Therefore $\mathcal{D}('\mathcal{I})$ must equal A and $'\mathcal{I}$ is the identity map.

By theorem 1 we know that $'(T\mathcal{I})$ is an extension of $'\mathcal{I}'T = 'T$ restricted to $\{x \in \mathcal{D}('T) \,/\, 'Tx \in A\}$. Suppose $x \in \mathcal{D}'(T)$. Then there exists $a \in A$ such that $(T\mathcal{I})a'x = a'a$ for all $a' \in \mathcal{D}(T\mathcal{I})$. But since $\mathcal{D}(T\mathcal{I}) = \mathcal{I}^{-1}(\mathcal{D}(T))$, $(T\mathcal{I})(\mathcal{I}^{-1}y')x = (\mathcal{I}^{-1}y)a'$ for all $y' \in \mathcal{D}(T)$. Hence $Ty'x = y'a$ for all $y' \in \mathcal{D}(T)$. Thus $x \in \mathcal{D}'(T)$ and $'Tx = '(T\mathcal{I})x$. Since $'(\mathcal{I}T)$ is only defined on A, the theorem is proved. ∎

In all examples A will be the subspace spanned by coodinate vectors. If it is not clear which norm is relative to A, then we will let A_1 be the subspace of ℓ_1, and A_2 of ℓ_2 and A_0 will have the "maximum norm".

We are now in a position to examine efficiently examples which will show that the thirty-three remaining squares cannot be crossed out without strengthening the hypothesis. It should be noticed that in the theorems in section 3 we did not require that the domain of the preconjugate be dense in X. One might think that with this added hypotheses fewer states could exist. It turns out that this is not the case, for in each one of the examples the domain of the preconjugate is dense in X.

When the details of the verification of the examples are obvious or follow immediately from the proposition, the operator and its preconjugate will just be listed.

(I_1, I_1) $T: \ell_2 \to \ell_2$, $\mathcal{D}(T) = \ell_2$ and $Ty' = y'$.

$'T: \ell_2 \to \ell_2$, $\mathcal{D}('T) = \ell_2$ and $'Tx = x$.

(II_1, I_1) $T: \ell_2 \to \ell_2$, $\mathcal{D}(T) = A$ and $Ty' = y'$.

$'T$ is the identity operator on ℓ_2.

(III_1, I_1) $T: \ell_\infty \to \ell_\infty$, $\mathcal{D}(T) = C_0$ and $Ty' = y'$.

$'T$ is the identity map from ℓ_1 to ℓ_1.

(III_2, I_2) Let $\mathcal{L}$ be the isometry from A_0' onto $C_0' = \ell_1$, and let T_0 be the identity map from ℓ_1 into ℓ_∞. Then $T: A_0' \to \ell_\infty$ with $\mathcal{D}(T) = A_0'$ and $T = T_0\mathcal{L}$. Clearly $T \in III$ and is one-to-one. To show T^{-1} is unbounded we exhibit a sequence in $\mathcal{D}(T)$ such that the norms of the elements in the sequence go to infinity but their images under T have norm one. Define the sequence $\{y_n'\}$ in ℓ_1 by $y_n' = \sum_{i=1}^{n} (1/i)\varepsilon_i$. Then $\|y_n'\| = \sum_{i=1}^{n} 1/i \to \infty$ and $\|T_0 y_n'\| = \max\{1, 1/2, \ldots, 1/n\} = 1$. The sequence that we want is $\{\mathcal{L}^{-1} y_n'\}$.

$'T: \ell_1 \to A_0$, $'Tx = x$ and $\mathcal{D}('T) = A_1$. $'T$ has an unbounded inverse for the same reasons that T has an unbounded inverse.

(I_1, II_1) Let $T: \ell_1 \to A_0'$ be the linear isometry from ℓ_1 onto A_0'

By the proof of theorem 2, we see that $'T: A_0 \to C_0$ is the identity map on A_0.

(II_1, II_1) Let T be the isometry in the above example, only let $\mathcal{D}(T) = A_1$. 'T is the identity map from A_0 to C_0.

(III_1, II_1) Let T be the isometry from ℓ_∞ onto A_1' restricted to C_0. 'T is the identity map from A_1 to ℓ_1.

(I_3, III_1) Let $T: \ell_2 \to \ell_2$, $\mathcal{D}(T) = \ell_2$ and T is a left shift.

$'T: \ell_2 \to \ell_2$, $\mathcal{D}('T) = \ell_2$ and 'T is a right shift.

(See proposition 4).

(II_3, III_1) Let T be the above example restricted to A.

(III_3, III_1) Let $T: \ell_\infty \to \ell_\infty$ with $\mathcal{D}(T) = C_0$ be a left shift.

$'T: \ell_1 \to \ell_1$ is a right shift with $\mathcal{D}('T) = \ell_1$.

(III_1, II_3) Let $T: \ell_2 \to A_2'$, $\mathcal{D}(T) = \ell_2$ and $T = \mathcal{J}R$ where R is a right shift from ℓ_2 to ℓ_2 and $\mathcal{J}$ is a linear isometry from ℓ_2 onto A_2.

$'T: A_2 \to \ell_2$ and by theorem 2, 'T is the restriction of 'R to A_2. Hence 'T is a left shift. (See proposition 5 and theorem 2).

(III_1, I_3) $T: \ell_2 \to \ell_2$, $\mathcal{D}(T) = \ell_2$ and T is a right shift. 'T is a left shift from ℓ_2 onto ℓ_2.

(II_2, II_2) Let $T: \ell_2 \to \ell_2$, $\mathcal{D}(T) = \ell_2$, and $T = D$. Recall that $D: (u_1, u_2, \ldots) \to (u_1, 1/2u_2, 1/3u_3 \ldots)$. The element $(1, 1/2, 1/3, \ldots)$ is in ℓ_2 but not in R(T). By proposition 8, $T \in 2$.

$'T: \ell_2 \to \ell_2$, $\mathcal{D}(T) = \ell_2$, and $'T = D$, by proposition 3.

(III_2, II_2) Let $T: \ell_1 \to \ell_\infty$, $\mathcal{D}(T) = \ell_1$ and $T = D$.

$'T: \ell_1 \to C_0$, $\mathcal{D}('T) = \ell_1$ and $'T = D$.

(II_2, I_2) Let $T: A' \to \ell_2$, $\mathcal{D}(T) = A'$ and $T = D\mathcal{J}$ where $\mathcal{J}$ is the linear isometry from A' onto ℓ_2.

$'T: \ell_2 \to A$ and by theorem 3, $'T$ is the restriction of $'D = D$ to $\{x \in X / 'Dx \in A\}$. Hence $'T = D$, $\mathcal{D}('T) = A$.

(III_2, II_3) Let $T: \ell_2 \to \ell_2$ with $\mathcal{D}(T) = \ell_2$ be defined by $T = RD$. (Recall that R is a right shift).

$'T: \ell_2 \to \ell_2$, and by proposition 7, $'T = DL$ and $\mathcal{D}('T) = \ell_2$.

(III_2, I_3) Let $T: A' \to \ell_2$, be defined by $T = T_0 \mathcal{J}$, where T_0 is the operator of the previous example and $\mathcal{J}$ is the isometry from A' onto ℓ_2. $\mathcal{D}(T) = A'$.

$'T: \ell_2 \to A$ and by theorem 3, $\mathcal{D}('T) = A$ and $'T = DL$.

(II_3, III_2) Let $T: \ell_2 \to \ell_2$, $\mathcal{D}(T) = \ell_2$ and $T = LD$.
$'T: \ell_2 \to \ell_2$ and by proposition 6, $\mathcal{D}('T) = \ell_2$ and $'T = DR$.

(III_3, III_2) Let $T: \ell_2 \to \ell_\infty$, $\mathcal{D}(T) = \ell_2$ and let $T = LD$.
$'T: \ell_1 \to \ell_2$, $\mathcal{D}('T) = \ell_1$ and $'T = DR$.

(III_3, III_3) Let $T: \ell_2 \to \ell_2$ and $Ty' = 0$ for all $y' \in \ell_2$
$'T: \ell_2 \to \ell_2$ is also the zero operator on ℓ_2 for if $x \in \ell_2$ then $Ty'x = (0')x = y'0$ for all $y' \in \ell_2$.

(III_1, III_1) Let $T: \ell_\infty \to \ell_\infty$, $\mathcal{D}(T) = A$ and

$T(u_1, u_2, \ldots) = (u_2 - u_1, u_3 - u_1, u_4 - u_1, \ldots)$.

It is clear that Ty' is in ℓ_∞.

First we show that T has a bounded inverse. Let

$$y' = \sum_{i=1}^{n} u_i \varepsilon_i$$

If $u_1 \geqslant 1/2 \, \| y' \|$, then

$$\|Ty'\| = \max\{|u_2 - u_1|, \ldots, |u_n - u_1|, |u_1|\} \geqslant$$

$$\geqslant |u_1| \geqslant 1/2 \cdot \| y' \|.$$

If $|u_1| < 1/2 \cdot \|y'\|$ then there exists an integer i between 2 and n such that $\|y'\| = |u_i|$. Hence $\|Ty'\| > |u_i - u_1| \geq \|y'\| - 1/2 \cdot \|y'\|$. Thus $\|Ty'\| \geq 1/2 \cdot \|y'\|$ for all $y' \in \mathcal{D}(T)$.

To see that $T \in III$, one need only note that T is continuous $\mathcal{D}(T)$ is separable, and ℓ_∞ is not.

$'T : \ell_1 \to \ell_1$, $\mathcal{D}('T) = \ell_1$ and $'T(x_1, x_2, \ldots) =$

$= (-\sum_{i=1}^{\infty} x_i, x_1, x_2, \ldots)$, for if $x \in \ell_1$

$$Ty'x = (u_2 - u_1, u_3 - u_1, \ldots)(x_1, x_2, \ldots) =$$
$$= u_1(-\sum_{i=1}^{\infty} x_i) + u_2 x_1 + u_3 x_2 + \ldots =$$
$$= (u_1, u_2, \ldots)(-\sum_{i=1}^{\infty} x_i, x_1, x_2, \ldots) = y'('Tx).$$

Proposition 9. $H = \{ (x_1, x_2, \ldots) / \sum_{i=1}^{\infty} x_i = 0 \}$ is a closed subspace of ℓ_1.

Proof. Clearly H is a subspace. Suppose the sequence $\{x_n\}$, where $x_n = (x_1^n, x_2^n, \ldots)$, is in H and converges to $x = (x_1, x_2, \ldots)$. Given $\varepsilon > 0$, there exists an integer N such that

$$\|x - x_N\| = \sum_{i=1}^{\infty} |x_i - x_i^N| < \varepsilon.$$

Hence

$$|\sum_{i=1}^{\infty} x_i| = |\sum_{i=1}^{\infty} x_i - 0| = |\sum_{i=1}^{\infty} x_i - \sum_{i=1}^{\infty} x_i^N| \leq \sum_{i=1}^{\infty} |x_i - x_i^N| < \varepsilon.$$

Thus $|\sum_{i=1}^{\infty} x_i| = 0.$ ∎

(III_1, III_3) Let $T: \ell_\infty \to \ell_\infty$ with $\mathcal{D}(T) = A$, $T = RT_0$ where T_0 is the operator in the preceding example and R is a right shift. Clearly $T \in III$. $T \in 1$ since both T_0^{-1} and R^{-1} are continuous.

$'T : \ell_1 \to \ell_1$ and by theorem 1, $'T$ is an extension of $'T_0{}'R$. But since $\mathcal{D}('T_0{}'R) = \ell_1$, $'T = {}'T_0{}'R$ and $\mathcal{D}('T) = \ell_1$. Hence

$'T(x_1, x_2, \ldots) = (-\sum_{i=2}^{\infty} x_i, x_2, x_3, \ldots)$.

$'T \in III$ for $R('T) = H$ and $'T \in 3$ for $'T\varepsilon_1 = 0$.

(III_2, III_3) Let $T: \ell_\infty \to \ell_\infty$ with $\mathcal{D}(T) = A$ and $T = DT_1$, T_1 is the operator described above. T is one-to-one since it is the composite of one-to-one maps. The sequence $\{n\varepsilon_n\} \subset \mathcal{D}(T)$ and

$\|n\varepsilon_n\| = n$ but for $n > 1$, $\|Tn\varepsilon_n\| = \|Dn\varepsilon_n\| = \|\varepsilon_n\| = 1$.

Hence T has an unbounded inverse.

$'T: \ell_1 \to \ell_1$, $\mathcal{D}('T) = \ell_1$ and $'T = 'T_1 D$. In other words

$'T(x_1, x_2, \ldots) = (-\sum_{i=1}^{\infty} 1/ix_i, 1/2x_2, 1/3x_3, \ldots)$.

Proposition 10. The set $\{\lambda_i \,/\, \lambda_i = \varepsilon_i - \varepsilon_{i-1}\}$ is total in ℓ_p for $1 \leq p \leq \infty$.

Proof. Let $x = (x_1, x_2, \ldots)$ be in C_0 or ℓ_p, $1 \leq p < \infty$. Suppose $\lambda_i x = 0$ for $i = 1, 2, \ldots$ Then $\varepsilon_i x = \varepsilon_{i-1} x$ or $x_i = x_{i-1}$ for $i = 1, 2, \ldots$. But since x is in C_0 or ℓ_p for $1 \leq p < \infty$, $x_i \to 0$. Hence $x_i = 0$ for each i ∎

Proposition 11. If $T: \ell_p \to \ell_p$, $1 \leq p < \infty$, is a left shift and if $\mathcal{D}(T) = H$, then T is one-to-one. H is the set in proposition 9.

Proof. Suppose $T(x_1, x_2, \ldots) = (x_2, x_3, \ldots) = 0$. Then $x_i = 0$ for $i = 2, 3, \ldots$. But if $(x_1, x_2, \ldots) \in H$, then

$$x_1 = -\sum_{i=1}^{\infty} x_i.$$

Hence $x_1 = 0$.

(I_1, III_1) Let $T: \ell_1 \to \ell_1$, $\mathcal{D}(T) = H$ and T is a left shift. T is surjective for if $(w_1, w_2, \ldots) \in \ell_1$, then

$$T(-\sum_{i=1}^{\infty} w_i, w_1, w_2, \ldots) = (w_1, w_2, \ldots) \text{ and}$$

$(-\sum_{i=1}^{\infty} w_i, w_2, \ldots) \in H$. $T \in 1$ for if $y' = (u_1, u_2, \ldots) \in H$,

then $\|Ty'\| = |u_2| + |u_3| + \ldots = 1/2(|u_2| + |u_3| + \ldots + u_2 + u_3 + \cdots)$

$1/2(\sum_{i=1}^{\infty} u_i + u_2 + u_3 + \ldots = 1/2(u_1 + u_2 + \ldots) = 1/2\|y'\|$.

We first note that by proposition 10, H is total in ℓ_1, hence 'T exists. $'T: C_0 \to C_0$, $\mathcal{D}('T) = C_0$, and 'T is a right shift.

(II_1, III_1) Let $T: \ell_1 \to \ell_1$ be the operator in the above example restricted to $A \cap H$. Note that $A \cap H$ is total in $\ell_1 = C'_0$ by proposition 10.

$'T: C_0 \to C_0$ and $\mathcal{D}('T) = C_0$ and 'T is a right shift.

(I_2, III_1) Let $T: \ell_2 \to \ell_2$, $\mathcal{D}(T) = H$ and T is again a left shift. As is the example (I_1, III_1), T is surjective and one-to-one. T^{-1} is unbounded for the sequence $\{x_n\}$ where

$$x_n = \sqrt{n}\,\varepsilon_1 - \sum_{i=2}^{n+1} (1/\sqrt{n})\varepsilon_i = (\sqrt{n}, -1/\sqrt{n}, \ldots, -1/\sqrt{n}, 0, 0, \ldots)$$

is contained in $H = \mathcal{D}(T)$ and $\|x_n\| = \sqrt{n+1}$, but $\|Tx_n\| =$
$= \sqrt{\sum_{i=1}^{n} (1/n)} = 1$.

$'T: \ell_2 \to \ell_2$, $\mathcal{D}('T) = \ell_2$ and 'T is a right shift.

In the rest of the examples we shall use the subspace B described as follows. If S is a subset of a linear space then $sp\langle S \rangle$ is the subspace generated by S or equivalently the smallest subspace containing S. Let $x_0 = (1, 1/2, 1/2^2, 1/2^3, \ldots)$. Then

$$B = sp\langle x_0 \rangle \oplus sp\langle \{\varepsilon_2, \varepsilon_3, \ldots\} \rangle$$

where $\oplus$ is the algebraic direct sum.

B is total in ℓ_p for B contains the set

$$\{x_0, \varepsilon_2, \varepsilon_3, \ldots\}$$

which is total.

Proposition 12. If $T: \ell_p \to \ell_p$ is a left shift and if

$\mathcal{D}(T) = B$ then T is one-to-one.

Proof. Suppose $Tx = 0$. If $x \in B$ then x is of the form $(k, x_1 + k/2, \ldots, x_n + k/2^n, k/2^{n+1}, \ldots)$. Hence if

$$0 = Tx = (x_1 + k/2, \ldots, x_n + k/2^n, k/2^{n+1}, \ldots)$$

then $k = 0$. Thus $x_i = 0$ for $i = 1, 2, \ldots, n$. Hence $x = 0$. ∎

(II_2, III_1) Let $T: \ell_1 \to \ell_1$, $\mathcal{D}(T) = B$ and T be a left shift. T is not surjective for all of the elements in $R(T)$ are of the form $(u_1 + k/2, \ldots, u_n + k/2^n, k/2^{n-1}, \ldots)$. $R(T) = T(sp\langle\{\varepsilon_2, \varepsilon_3, \ldots\}\rangle) \oplus T(sp\langle x_0 \rangle) = A \oplus sp\langle Tx \rangle$ which contains A. Hence $T \in II$. By the above proposition, T is one-to-one. $T \in 2$ for sequence $\{x_n\}$ where

$$x_n = 2^n \varepsilon_1 - \sum_{i=n+1}^{\infty} (2^n/2^{i+1})\varepsilon_i = 2^n x_0 + (0, -2^n/2, \ldots, -2^n/2^{n+1}, 0, \ldots)$$

is contained in $B = \mathcal{D}(T)$ and $\|x_n\| = 2^{n+1}$, but

$$\|Tx_n\| = \|(0, \ldots, 0, 0, 2^n/2^{n+1}, 2^n/2^{n+2}, \ldots)\| = 1.$$

Hence T^{-1} is unbounded.

$'T: C_0 \to C_0$, $\mathcal{D}('T) = C_0$ and $'T$ is a right shift.

(III_2, III_1) Let $T: \ell_\infty \to \ell_\infty$, $\mathcal{D}(T) = B$ and T is a left shift. To see that $T \in III$ we note that T is continuous and $\mathcal{D}(T)$ is separable. Hence $\overline{R(T)}$ is separable.

By proposition 12, T is one-to-one. Let $\{x_n\}$ be the sequence of the previous example. We see that the norm of x_n in ℓ_∞ is 2^n and the norm of Tx_n is $2^n/2^{n+1}$ which is less than 1. Hence T has an unbounded inverse.

$'T: \ell_1 \to \ell_1$, $\mathcal{D}('T) = \ell_1$ and $'T$ is a right shift.

(III_2, III_2) Let $T: \ell_\infty \to \ell_\infty$, $\mathcal{D}(T) = B$ and $T = LD$. T is continuous, hence $\overline{T(B)} = \overline{R(T)}$ is separable, thus $T \in III$. T is one-to-one since it is the composite of two one-to-one maps.

$T \in 2$ by the corollary to proposition B.

$'T: \ell_1 \to \ell_1$, $\mathcal{D}(T) = \ell_1$ and $'T = DR$.

(II_2, III_2) $T: \ell_1 \to \ell_1$, $\mathcal{D}(T) = B$ and $T = LD$.

$'T: C_0 \to C_0$, $\mathcal{D}('T) = C_0$ and $'T = DR$.

Proposition 13. B is dense in C_0.

Proof. Given $x = (u_1, u_2, \ldots) \in C_0$, the sequence $\{x_n\}$ defined by

$$x_n = u_1 x_0 + \sum_{i=1}^{n} (u_i - u_1 \, 1/2^{i-1}) \varepsilon_i = (u_1, \ldots, u_n, u_1 \, 1/2^n, \ldots)$$

is in B.

Suppose $\varepsilon > 0$ is given. Since $x \in C_0$, there exists a positive integer N' such that if $n > N'$, then $|u_n| < \varepsilon/2$. There also exists a positive integer N'' such that if $n > N''$ then

$1/2^n < \frac{\varepsilon}{2|u_1|}$. Let $N = \max\{N', N''\}$, Then if $n > N$

$$\|x_n - x\| = \left\|\sum_{i=n-1}^{\infty} (u_1/2^{i-1} - u_i)\right\| = \max_{i \geq n} |u_1/2^i - u_{i+1}| \leq$$

$$\max_{i \geq n} |u_1| / 2^{i+1} + |u_{i+1}| \leq |u_1| \frac{\varepsilon}{2|u_1|} + \varepsilon/2 = \varepsilon \quad \blacksquare$$

(III_1, II_2) By the previous proposition B' is isomorphic to $C'_0 = \ell_1$. Let $\mathcal{D}$ be this isometry mapping ℓ_1 onto B'. Let $T: \ell_1 \to B'$, $\mathcal{D}(T) = \ell_1$, and $'T = \mathcal{D} R$ where R is a right shift. $T \in 1$ since it is the composite of functions which have continuous inverses.

$'T: B \to C_0$, $\mathcal{D}('T) = B$ and $'T$ is a left shift. To see that $'T \in 2$ see the first half of example (III_2, III_1).

(III_1, I_2) Let $B_0 = A \oplus \text{sp}\langle(1/2, 1/2^n, \ldots)\rangle$. It is clear that B_0 is dense in C_0 since it contains A. Hence B'_0 is equivalent to ℓ_1. Let $\mathcal{D}_0$ be this isometry from ℓ_1 onto B'_0 and as above let $\mathcal{D}$ be the isometry from ℓ_1 onto B'.

Let $'T: B'_0 \to B'$, $\mathcal{D}(T) = B'_0$ and $T = \mathcal{D} R \mathcal{D}_0^{-1}$.

$'T: B \to B_0$, $\mathcal{D}('T) = B$ and $'T$ is a left shift. $'T$ is surjective for $R(T) = T(B) = B_0$.

3. APPLICATIONS

1. The Conjugate and Preconjugate of a Differential Operator.

$L_p(S)$, $1 \leq p \leq \infty$ shall denote the set of complex-valued functions f on the set S with the property that $|f|^p$ is Lebesgue integrable. $\mathcal{L}_p(S)$ is the set of equivalence classes of L_p under the relation $\sim$ defined by, $f \sim g$ if and only if $f = g$ a. e. (almost everywhere). The equivalence class in $\mathcal{L}_p$ containing f shall be represented by $\dot{f}$. $\mathcal{L}_p$ is a Banach space under the norm $\|\dot{f}\|_p = (\int_S |f|^p)^{1/p}$.

$L_\infty(S)$ is the set of complex-valued functions which are bounded a. e. If f is in $L_\infty(S)$ and K is the set on which f is bounded, then the least upper bound on K of $|f|$ is called the essential bound of that function. $\mathcal{L}_\infty$ is the spaced of equivalenceclasses of L_∞ under $\sim$, defined above. $\mathcal{L}_\infty$ is a Banach space with the norm $\|\dot{f}\|$ = essential bound of f.

Let p' be the codimension of p, $1/p + 1/p' = 1$ $(1/\infty = 0)$. Then for $1 \leq p < \infty$ the map $\dot{\mathcal{J}}: \mathcal{L}_{p'} \to (\mathcal{L}_p)'$ defined by $\dot{\mathcal{J}}\dot{g}(\dot{f}) = \int gf$, $\dot{g} \in \mathcal{L}_{p'}$, $\dot{f} \in \mathcal{L}_p$, is an isometry between $\mathcal{L}_p$ and $(\mathcal{L}_{p'})'$.

A complex-valued function is absolutely continuous if its real and imaginary parts are absolutely continuous. A function which has a continuous derivative is absolutely continuous.

I shall denote a real interval, not necessarily bounded. $C_n(I)$ is the set of all complex-valued functions defined on I whose n^{th} derivative exists and is continuous. Let $C_\infty(I) = \bigcap_{n=1}^{\infty} C_n(I)$. With D as the derivative operator, we define

$$A_n(I) = \{ f \in L_p(I) \ / \ D^{n-1}f \text{ is absolutely continuous on every compact subinterval of } I \}.$$

We represent the formal differential expression

$$a_n D^n + a_{n-1} D^{n-1} + \ldots - a_1 D + a_0 ,$$

$a_i \in C_\infty(I)$, by τ. We shall assume that $a_n(t) \neq 0$ for all $t \in I$. T is the differential operator mapping $\mathcal{L}_p(I)$ to $\mathcal{L}_q(I)$ defined by $T\dot{f} = \tau^{\bullet} f$ (where $f \in A_n \cap \dot{f}$). The domain of T is

$$\mathcal{D}(T) = \left\{ \dot{f} \in \mathcal{L}_p(I) \,/\, f \in A_n(I) \text{ and } \tau^{\bullet} f \in \mathcal{L}_q(I) \right\}.$$

The purpose of this chapter is to find a restriction of T which is surjective and has a continuous inverse.

A function defined on a subset A of the real line has compact support in the interior of A if there exists a compact set K contained in the interior of A such that $f(x) = 0$ for all $x \notin K$. Let

$$A_n^c(I) = \left\{ f \in A_n(I) \,/\, f \text{ has compact support in the interior of } I \right\}$$

and

$$C_\infty^c(I) = \left\{ f \in C_\infty(I) \,/\, f \text{ has compact support in the interior of } I \right\}.$$

Then T_c is the restriction of T to the set

$$\mathcal{D}(T_c) = \left\{ \dot{f} \in \mathcal{L}_p(I) \,/\, f \in A_n^c(I) \text{ and } \tau^{\bullet} f \in \mathcal{L}_q(I) \right\}.$$

The set, $\{ \dot{f} \,/\, f \in C_\infty^c(I) \}$, is dense in $\mathcal{L}_p(I)$, $1 \leq p < \infty$ and total in $(\mathcal{L}_1(I))' \cong \mathcal{L}_\infty(I)$. Thus T_c' exists for $1 \leq p < \infty$ and $'T_c$ exists for $1 < p \leq \infty$.

2. THE FORMAL ADJOINT τ^* AND ITS CORRESPONDING OPERATOR T^*

From now on we shall omit I from expressions such as $A_n(I)$. In order to investigate the preconjugate of T_c, we consider T_c as an operator from $(\mathcal{L}_{p'})'$ to $(\mathcal{L}_{q'})'$ when $1 < p,\ q \leq \infty$. We let $\mathcal{J}_p : \mathcal{L}_p \to (\mathcal{L}_{p'})'$ be the isometry defined above by $\mathcal{J}_p \dot{g}(\dot{f}) = \int fg$ for $\dot{g} \in \mathcal{L}_p$ and $\dot{f} \in \mathcal{L}_{p'}$. $\mathcal{J}_q : \mathcal{L}_q \to (\mathcal{L}_{q'})'$ is defined similarly. Then

$$\mathcal{D}(T_c) = \left\{ \mathcal{J}_p \dot{f} \,/\, \dot{f} \in \mathcal{L}_p,\ f \in A_n^c \text{ and } \tau^{\bullet} f \in \mathcal{L}_q \right\}$$

and $T_c \mathcal{J}_p \dot{f} = \mathcal{J}_q \tau^{\bullet} f$. Thus for all $\mathcal{J}_p \dot{f} \in \mathcal{D}(T_c)$ and all $\dot{h} \in \mathcal{L}_{q'}$

$$(T_c \mathcal{D}_p f)^{\cdot}(h) = (\mathcal{D}_q \tau^{\cdot} f)^{\cdot}(h) = \int_I h \tau f$$

We now let $I = I_0$ be the compact interval $[a, b]$. Then $h \in A_n(I_0)$ is bounded, say by K. If $f \in A_n(I_0)$, then $|\tau f|$ is integrable, and thus $|h \tau f| \leq |\tau f| K$, implies $h \tau f$ is integrable. We have

$$\int_a^b h \tau f = \int_a^b a_0 hf + \int_a^b a_1 h\, Df + \ldots + \int_a^b a_n h\, D^n f.$$

Since $D^k f$ is integrable and $a_k h$ is absolutely continuous, we can integrate by parts and for $k = 1, 2, \ldots, n$ obtain

$$\int_a^b a_k h\, D^k f = a_k h\, D^{k-1} f \Big]_a^b - \int_a^b D(a_k h)\, D^{k-1} f.$$

If $k-1 \geq 1$, we can integrate by parts again, and after repeated integrations we obtain

$$\int_a^b a_k h\, D^k f = a_k h\, D^{k-1} f \Big]_a^b - D(a_k h)\, D^{k-2} f \Big]_a^b + \ldots$$

$$+ (-1)^{k-1} D^{k-1}(a_k h)\, f \Big]_a^b + (-1)^k \int_a^b D^k(a_k h)\, f.$$

Thus

$$\int h \tau f = \sum_{k=1}^{n} \sum_{i=1}^{k} (-1)^{i-1} D^{i-1}(a_k h)\, D^{k-i} f \Big]_a^b$$

$$\sum_{k=0}^{n} \int_a^b (-1)^k D^k(a_k h)\, f.$$

Let us denote the double sum by $\sum\sum$.

In the following lemma we shall show that if f has compact support (which it does in $\mathcal{D}(T_c)$) then $\sum\sum = 0$. We define τ^* the formal adjoint of τ, by

$$\tau^* h = \sum_{k=0}^{n} (-1)^k D^k(a_k h).$$

The following properties are well known.

(1) If h and $f \in A_n$, then $\int_a^b h\tau f - \int_a^b f\tau^* g = \sum\sum$,

(2) $\tau^*(ag + bf) = a\tau^* g + b\tau^* h$,

(3) $\tau^{**} = \tau$

(4) The leading coefficient of τ^* is a constant multiple of a_n.

We define T^* mapping $\mathcal{L}_{q'}$ to $\mathcal{L}_{p'}$ by $T^* f = \tau^* f$ with

$$\mathcal{D}(T^*) = \{ f \in \mathcal{L}_{q'} / f \in A_n \text{ and } \tau^* f \in \mathcal{L}_p \}.$$

Lemma 1. As above let $I_o = [a, b]$. If $f \in A_n^c(I_o)$ and $h \in A_n(I_o)$, then $h\tau f$ and $f\tau^* h$ are integrable and

$$\int_a^b h\tau f = \int_a^b f\tau^* h$$

Proof. Since $f \in A_n^c(I_o)$, $D^k f(a) = D^k f(b) = 0$ for $k = 0, 1, \ldots, n$. Thus $\sum\sum = 0$. ∎

3. RELATIONS BETWEEN T_o, T_c' and T^*. The following theorems stated, without proof, are well-known. It is not necessarily compact.

Theorem 1. Suppose g is complex-valued and integrable over every compact subinterval of I, i. e., g is locally integrable. As usual $\tau = \sum_{i=1}^n a_i D^i$ and $a_n(t) \neq 0$ for all $t \in I$. Then given $t_o \in I$ and n arbitrary complex constants $c_o, \ldots, c_{n-1}$, there exists a unique $f \in A_n$ such that $\tau f = g$ a. e. and
$D^K f(t_o) = c_K$, $k = 0, 1, \ldots, n-1$.

If f is continuous, g is absolutely continuous, and $Dg = f$ a. e., then $Dg = f$. If $h \in \mathcal{L}_p(I)$, $1 \leq p < \infty$ then h is locally integrable. This follows from the fact that for a compact interval and for $1 \leq p$, $q \leq \infty$, $\mathcal{L}_q(I_o) \subset \mathcal{L}_p(I_o)$.

Theorem 2. The set of solutions in A_n to the differential equation $\tau f = 0$ is an n-dimensional subspace of C_∞.

Proof. Let t_o be a fixed point in I. By the previous theorem there exists unique functions $f_1, \ldots, f_n$ in A_n such that

$\tau f_j = 0$ a. e. and for $j = 1, \dots, n$.

$$D^i f_{j+1}(t_0) = \delta_{ij} \quad \text{(Kronecker delta)}.$$

To see that the f_i's are linearly independent, observe that if

$\sum_{i=1}^{n} \alpha_i f_i = 0$. Then $0 = D^i \sum_{j=1}^{n} \alpha_j f_j(t) = \alpha_{i+1}$, for

$i = 0, \dots, n-1$.

Suppose $f \in A_n$ and $\tau f = 0$. Let

$$g = \sum_{j=0}^{n} \left[D^j f(t_0) \right] f_{j+1}.$$

Using the uniqueness established in theorem 1, we shall show that $g - f = 0$. We first observe that for $i = 0, \dots, n-1$

$$D^i(f-g)(t_0) = D^i f(t_0) - \sum_{j=0}^{n-1} \left[D^j f(t_0) \right] D^i f_{j+1}(t_0) =$$

$$= D^i f(t_0) - D^i f(t_0) = 0.$$

Also since $\tau f_K = 0$ a. e., we have $\tau(f - g) = 0$ a. e. Hence $f - g$ and the zero function are both in A_n and satisfy the conditions of theorem 1, so they must be equal. Thus f is a linear combination of the f_i's.

To complete the proof we must show that all of the solutions are in C_∞. First note that if $\tau f = 0$, then

$$D^n f = -(1/a_n) \sum_{i=0}^{n-1} a_i D^i f.$$

Hence $D^n f$ is continuous and differentiable. We take the derivative of both sides and then $D^{n+1} f$ is a linear combination of the first n derivatives of f. Thus $D^{n-1} f$ is continuous and differentiable.

Theorem 3. If $1 \le p, q < \infty$, then $T_c' = T^*$ and if $1 < p, q \le \infty$, then $'T_c = T^*$.

Proof. We shall only prove the theorem for the preconjugate.

The proof for the conjugate follows similarly.

We have $T_c : (\mathcal{L}_{p'})' \to (\mathcal{L}_{q'})'$ with

$$\mathcal{D}(T_c) = \{ \mathcal{D}_p \dot{f} / f \in \mathcal{L}_p, f \in A_n^c \text{ and } \tau^{\bullet} f \in \mathcal{L}_q \}$$

and $T_c \mathcal{D}_p \dot{f} = \mathcal{D}_q \tau^{\bullet} f$. Thus for $1 < p, q \leq \infty$, $'T_c : \mathcal{L}_{q'} \to \mathcal{L}_{p'}$ exists and

$$\mathcal{D}('T_c) = \{ \dot{g} \in \mathcal{L}_{q'} / \ T_c \mathcal{D}_p \dot{f} (\dot{g}) = \mathcal{D}_p \dot{f} (\dot{h}) \text{ for some } \dot{h} \in \mathcal{L}_{p'}$$

$$\text{and all } \mathcal{D}_p \dot{f} \in \mathcal{D}(T_c) \}.$$

Also $T^* : \mathcal{L}_{q'} \to \mathcal{L}_{p'}$ and $T^* \dot{g} = \tau^{*\bullet} g$ with

$$\mathcal{D}(T^*) = \{ \dot{g} \in \mathcal{L}_{q'} / \dot{g} \in A_n \text{ and } \tau^{*\bullet} g \in \mathcal{L}_{p'} \}.$$

Suppose $\dot{g} \in \mathcal{D}(T^*)$. We must show that $T_c \mathcal{D}_p \dot{f}(\dot{g}) =$ $= \mathcal{D}_p \dot{f} (\tau^{*\bullet} g)$ for all $\mathcal{D}_p \dot{f} \in \mathcal{D}(T_c)$. If $\mathcal{D}_p \dot{f} \in \mathcal{D}(T_c)$, then $f \in A_n^c$ and thus by lemma 1

$$\int g \tau f = \int f \ \tau^* g.$$

Hence $T_c \mathcal{D}_p \dot{f} (\dot{g}) = \mathcal{D}_q \tau^{\bullet} f (\dot{g}) = \int g \tau f = \int f \tau^* \ g = \mathcal{D}_p \dot{f} (\tau^{*\bullet} g)$

We have shown $\mathcal{D}(T^*) \subset \mathcal{D}('T_c)$ and $T^* = {}'T_c$ on $\mathcal{D}(T^*)$.

Now suppose $\dot{g} \in \mathcal{D}('T_c)$.

Letting $\dot{h} = {}'T_c \dot{g}$, we have $\int fh = \mathcal{D}_p \dot{f} (\dot{h}) = \mathcal{D}_p \dot{f} ('T_c \dot{g}) = T_c \mathcal{D}_p \dot{f} (\dot{g}) =$ $\mathcal{D}_q \tau^{\bullet} f (\dot{g}) = \int g \tau f$ for all $\mathcal{D}_p \dot{f} \in \mathcal{D}(T_c)$. Thus for $\mathcal{D}_p \dot{f}$ in $\mathcal{D}(T_c)$,

$$(1) \qquad \int_I (\tau f) \, g = \int_I f h.$$

To show $\dot{g} \in \mathcal{D}(T^*)$, it suffices to show that for any compact interval $I_0 = [a, b]$ contained in I, that g is equal a. e. to a function in $A_n(I_0)$ and $\tau^* g$ is equal to h a. e. on I_0.

Define $D_0 = \{ f / \mathcal{D}_p \dot{f} \in \mathcal{D}(T_c)$ and f has support in $I_0 \}$. For $f \in D_0$, it follows that $D^k f(a) = D^k f(b) = 0$, $0 \leq k \leq n-1$, and therefore successive integration by parts yields the formula

$$(2)\quad D^k f(t) = \int_a^t \frac{(t-s)^{n-k-1}}{(n-k-1)!} D^n f(s)\, ds \qquad t \in [a,b].$$

Since f vanishes outside of I_o, it follows from (1) and (2) that

$$(3)\quad \int_a^b a_n(s)\, g(s)\, D^n f(s)\, ds + \sum_{k=0}^{n-1} \int_a^b dt \int_a^t a_k(t)\, g(t) \frac{(t-s)^{n-k-1}}{(n-k-1)!} D^n f(s)\, ds$$

$$= \int_a^b dt \int_a^t h(t) \frac{(t-s)^{n-1}}{(n-1)!} D^n f(s)\, ds.$$

Each of the integrands in (3) is in $L_1(I_o \times I_o)$ since a_k is continuous on I, $g \in L_{q'}(I_o) \subset L_1(I_o)$ and $f \in L_p(I_o) \subset L_1(I_o)$. Thus by Fubini's theorem we may change the order of integration in (3) and obtain

$$(4)\quad 0 = \int_a^b D^n f(s) \left[a_n(s)\, g(s) + \sum_{k=0}^{n-1} \int_a^b \frac{(t-s)^{n-k-1}}{(n-k-1)!} a_k(t)\, g(t)\, dt - \int_s^b \frac{(t-h)^{n-1}}{(n-1)!} h(t)\, dt \right] ds$$

for all $f \in D_o$.

Let F(s) be the expresion inside the square brackets in (4). We show that f is equivalent on I_o to a polynomial of degree at most n-1.

Given $Q \in L_q(I_o)$ such that Q is orthogonal to the subspace $\mathcal{P}$ of $L_{q'}(I_o)$ of polynomials of degree at most n-1, the function r defined by

$$r(t) = \int_a^t \frac{(t-s)^{n-1}}{(n-1)!} Q(s)\, ds, \quad t \in I_o$$

and equal to 0 outside of I_o is easily seen to be in D_o with $D^n h = Q$ a.e. on I_o. Thus setting $r = f$ in equation (4), we have

$$(5)\qquad 0 = \int_a^t Q(s)\, F(s)\, ds$$

for all $Q \in L_q(I_o)$ orthogonal to $\mathcal{P} \subset L_{q'}(I_o)$, i.e., for all $Q \in \mathcal{P}^o$ when $1 < q \leq \infty$ or for all $Q \in {}^o\mathcal{P}$ when $1 = q$. Since $\mathcal{P}$ is of dimension n, we have from (5)

$$F \in {}^{\circ}(\mathcal{P}^{\circ}) = \mathcal{P}, \; 1 < q \leq \infty \;, \; \mathcal{P} \subset L_{q'}(I_0)$$

$$F \in ({}^{\circ}\mathcal{P})^{\circ} = \mathcal{P}, \; 1 = q, \; \mathcal{P} \subset L_{\infty}(I_0)$$

Thus F is equivalent on I_0 to a polynomial p of degree at most n - 1 or

$$(6) \quad a_n(s)\; g(s) = p(s) - \sum_{k=0}^{n-1} \int_s^b \frac{(t-s)^{n-k-1}}{(n-k-1)!} a_k(t)\, g(t)\, dt + \int_a^b \frac{(t-s)^{n-1}}{(n-1)} h(t)\, dt, \text{ a.e.}$$

Since the right hand side of (6) and $1/a_n$ are absolutely continuous on I_0, we redefine g on a set of measure zero so as to be absolutely continuous on I_0. Differentiating, we obtain

$$D(a_n g)(s) = a_n(s)\; Dg(s) + g(s)\; Da_n(s) =$$

$$(7) \quad Dp(s) + a_{n-1}(s)\, g(s) + \sum_{k=0}^{n-2} \int_s^b \frac{(t-s)^{n-k-1}}{(n-k-2)!}\; a_k(t)\; g(t)\, dt - \int_s^b \frac{(t-s)^{n-2}}{(n-2)!}\, h(t)\, dt$$

Since g, Da_n, $1/a_n$, and the right hand of (6) are absolutely continuous on I_0, it follows that Dg is also absolutely continuous on I_0. Repeated differentiation of both sides of (7) shows that $D^{n-1}g$ is absolutely continuous and $\tau^* g = h$ a.e. on I_0 (Recall that p is a polynomial of degree at most n-1.) ■

Corollary. T is a closed operator; hence T_c is closable.

Proof. Let T_c^* be the restriction of T^* to the equivalence classes containing functions with compact support. We then have for $1 \leq p', q' < \infty$, $(T_c^*)' = (T^*)_* = T$ which is closed, and for $1 < p', q' \leq \infty$, $'(T_c^*) = (T^*)^* = T$ which is closed. $T = (T^*)^*$ since $(\tau^*)^* = \tau$. ■

4. ACKNOWLEDGEMENTS.

I wish to acknowledge to my advisor Seymour Goldberg for ideas and advice, to Juan Horváth for encouragement and to the National Science Foundation for financial support.

REFERENCES

1 GOLDBERG, SEYMOUR.
Linear operators and their conjugates, Pacific J. of Math., vol. 9 Nº. 1, 1959.

2 TAYLOR, ANGUS E.
Functional Analysis, Wiley, 1958.

(Recibido en abril de 1964)

SUMARIO :

Este artículo trata de operadores lineales (no necesariamente acotados) entre espacios lineales normados. El conjugado T' de un operador T es cosa bien conocida. El preconjugado $'T$ se define como sigue :

Suponemos que el domino de $T, \mathcal{D}(T)$, es total en el espacio dual Y' y que toma valores en el espacio dual X' Entonces $'T : X \rightarrow Y$ y $'Tx$ se define como el elemento de Y para el cual $y'('Tx) = (Ty')x$ para todo $y' \in \mathcal{D}(T)$. Si $'Tx$ existe es único dada la totalidad de $\mathcal{D}(T)$. $\mathcal{D}('T)$ consiste de los $x \in X$ para los cuales $'Tx$ existe.

$'T$ es operador lineal cerrado y si $('T)'$ existe es entonces una extensión de $\overline{T}$, donde $\overline{T}$ es la extensión cerrada minimal de T. Si X y Y son reflexivos o si T es el conjugado de algún operador entonces $('T)' = \overline{T}$.

El "estado" de un operador lineal $A: X \rightarrow X$ se describe en términos de lo siguiente : I. $R(A)$ (rango de A) $= Y$, II. $R(A) \neq Y$ y $\overline{R(A)} = Y$, III. $\overline{R(A)} \neq Y$; 1. A^{-1} existe y es continuo, 2. A^{-1}

existe y no es continuo, 3. A^{-1} no existe. Luego se muestra en cuánto el estado de un operador determina (es determinado por) el estado de su preconjugado. Un "diagrama de estado" se construye para mostrar que parejas de estado son inadmisibles.

En la segunda parte damos ejemplos de operadores lineales de ℓ_{p_1} a ℓ_{q_1} y de sus preconjugados mostrando que las parejas de estado no clasidicadas como inadmisibles pueden existir.

Finalmente, en la última parte se determina el conjugado y el preconjugado de un operador diferencial que aplica ℓ_p en ℓ_q.