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Brown's representability theorem and branched coverings

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ABSTRACT. It is well known that Brown's Representability Theorem has many applications. The proof of the existence of a universal bundle for a topological group (see [4]) and the proof that any space can be approximated by CW-complexes (see [8]) are among those applications. Here we prove the existence of classifying spaces for branched coverings over CW-complexes using Brown's Theorem as the main tool.

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1. Introduction

In 1920 ALEXANDER (see [1]) pointed out the connection between the study of manifolds and of branched coverings over spheres. Actually, his results imply that information about the homotopy groups of classifying spaces of branched coverings translates into information about manifolds. Our principal goal in this paper is to prove the existence of classifying spaces for k-fold branched coverings over CW-complexes for which the branch set is a stratified set. To accomplish this purpose, we will use Brown's Representability Theorem as the key tool. Unfortunately, our proof is not constructive. In [3] we give an explicit construction of classifying spaces for branched coverings over manifolds.

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First of all, we recall some basic definitions and notations. Let X and Y be topological spaces, $A \subset X$ and I the unit interval $[0,1]$. We say that the pair (X, A) has the *homotopy extension property* if any continuous map $f : A \times I \to Y$ admits a continuous extension $F : X \times I \to Y$. Now, let $\mathcal{C}$ be the category of topological spaces with base point which admit a CW-complex structure, and the morphisms being the continuous maps preserving base points. A triple (X, X_1, X_2) will be called a *proper triad* of $\mathcal{C}$ if $X = X_1 \cup X_2, X_1, X_2$ and $X_1 \cap X_2$ are all in $\mathcal{C}$, all have the same base point, and $(X_1, X_1 \cap X_2)$ and $(X_2, X_1 \cap X_2)$ both have the homotopy extension property. If X and Y belong to $\mathcal{C}$, $[X, Y]$ denotes the set of homotopy classes of maps of X into Y with respect to homotopies which leave the base point of X fixed. $[\bullet\,, Y]$ will denote the functor from $\mathcal{C}$ to $\mathcal{S}$ (where $\mathcal{S}$ is the category of sets with a distinguished element and maps preserving distinguished elements) which assigns to each X in $\mathcal{C}$ the set $[X, Y]$ with the class of the constant map as distinguished element, and to each map $f : X \to X'$, the map $F : [X', Y] \longrightarrow [X, Y]$ defined by $F[g] = [g \circ f]$, where $[g]$ denotes the homotopy class of g.

Now we state Brown's Theorem (see [4] and [8]), which guarantees the existence of classifying spaces for functors that verify some special conditions.

Brown's Representability Theorem. *If $H : \mathcal{C} \to \mathcal{S}$ is a contravariant functor, and H satisfies the conditions A, B, C, D listed below, there is a space Y in $\mathcal{C}$, unique up to homotopy type, such that the functors $[\bullet\,, Y]$ and H are naturally equivalent.*

A. *If $f, g : X \to Y$ are homotopic, $H(f) = H(g)$.*

B. *(1) If p is a point, $H(p)$ contains only one element. (2) Suppose (X, X_1, X_2) is a proper triad , $A = X_1 \cap X_2$, and $j_i : A \to X_i$ and $k_i : X_i \to X$ are the inclusion maps, $i = 1, 2$. If $u_1 \in H(X_1)$ and $u_2 \in H(X_2)$ are such that $H(j_1)u_1 = H(j_2)u_2$, then there is a v in $H(X)$ such that $H(k_1)v = u_1$ and $H(k_2)v = u_2$. Furthermore, if A is a point, v is unique.*

C. *Suppose S^n_α is a collection of disjoint n-spheres whose wedge product $\vee S^n_\alpha$ is in $\mathcal{C}$. Let $i_\beta : S^n_\beta \longrightarrow \vee S^n_\alpha$ be the inclusion map. Then the map $\prod H(i_\alpha) : H(\vee S^n_\alpha) \longrightarrow \prod H(S^n_\alpha)$ is bijective.*

D. *Suppose $X_1 \subset X_2 \subset \cdots \subset X_n \cdots$ is a collection of subcomplexes of $X = \bigcup X_n$ in $\mathcal{C}$ with respect to some CW-complex structure on X such that $X^n_n = X^n$ (where X^n is the n-skeleton of X). Let $i_n : X_n \to X$ be the inclusion map. Let $\varprojlim H(X_n)$ be the inverse limit of $H(X_n)$ with respect to the maps induced by the inclusions of X_n to X_m. Then the function*

$$\varprojlim H(i_n) : H(X) \longrightarrow \varprojlim H(X_n)$$

is an epimorphism.

Section 2 focuses on the basic notions about k-fold branched coverings over manifolds. There we identify some contravariant functors from the category of (pointed) smooth manifolds to the category of sets. In Section 3 we show that the concepts and properties developed in Section 2 give a natural framework for the definitions of k-fold branched coverings over simplicial complexes and over CW-complexes. They also allow us to construct a functor from the category of CW-complexes to the category of sets. Then, we prove that this functor verifies all the conditions demanded by Brown's Representability Theorem. In this way we show the existence of a classifying space for k-fold branched coverings over CW-complexes.

2. Branched coverings over manifolds

In this section we study k-fold branched coverings over manifolds and some of their properties. The definitions and propositions exposed in this section are essentially the base for the rest of the paper.

Let N^n be a smooth manifold (the index n means that N has dimension n), and let l be a natural number. We say that $K \subset N$ is a *stratified set in* N *of type* l if there is a sequence $\varnothing = K_{l+1} \subset K_l \subset K_{l-1} \subset K_{l-2} \subset \cdots \subset K_3 \subset K_2 = K$ of closed sets in N such that $(K_j - K_{j+1})$ is a smooth manifold without boundary of codimension j and $\overline{K_j - K_{j+1}} = K_j$ for every j, $j = 2, \cdots, l$. The manifolds $(K_j - K_{j+1})$ are called the *strata* of K.

Given M, N manifolds, K a stratified set in N of type l, and $f : M \to N$ a smooth function, we say that f is *transverse to* K if f is transverse ([6], p. 28) to the strata.

If M^n, N^n are smooth manifolds and $K \subset N$ is a stratified set in N of type l, using induction over the number of strata it is possible to prove that if $g_0 : M \to N$ is any smooth function, there exists $g : M \to N$, homotopically equivalent to g_0, such that g is transverse to K ([9], p. 12). Consequently, $g^{-1}(K)$ is a stratified set in M of type l ([9], p. 14).

Now, we define a branched covering over a manifold. Let $\widetilde{N}^n$, N^n be smooth manifolds, and k, l be natural numbers, and let $f : \widetilde{N} \to N$ be a smooth function. We say that f is a *k-fold branched covering of type l over N* if f satisfies:

(*i*) The *branch set* K is stratified in N of type l.

(*ii*) f is transverse to each stratum of K.

(*iii*) The set $f^{-1}(K)$ is also a stratified set of type l whose strata are the submanifolds $f^{-1}(K_j - K_{j+1})$, $j = 2, \cdots, l$.

(*iv*) $f|_{f^{-1}(N-K)}$ is a k-fold covering.

(*v*) $f|_{f^{-1}(K_l)}, f|_{f^{-1}(K_{l-1}-K_l)}, \cdots, f|_{f^{-1}(K_2-K_3)}$ are k_s-coverings over their respective components, where k_s is less than k.

Let $f_1 : \widetilde{M}_1 \to M$ and $f_2 : \widetilde{M}_2 \to M$ be k-fold branched coverings of type l over a manifold M. We say that f_1 and f_2 are *equivalent up to homeomorphism* if there is a homeomorphism $h : \widetilde{M_1} \to \widetilde{M_2}$ such that $f_2 \circ h = f_1$. Now, we say that f_1 and f_2 are *concordant* (of type l) if there is a k-fold branched covering of type l, $F : \widetilde{W}^{n+1} \longrightarrow M^n \times I$ (I is the closed interval [0,1]), such that $F|_{F^{-1}(M\times\{0\})}$ is equivalent up to homeomorphism to f_1 and $F|_{F^{-1}(M\times\{1\})}$ is equivalent up to homeomorphism to f_2. We will denote with $B_{k,l}(M)$ the set of all concordance classes of k-fold branched coverings over M of type l.

A triple (E, B, γ) is called a *k-fold universal branched covering of type l* if $\gamma : E \to B$ is a k-fold branched covering of type l and for any k-fold branched covering $f : \widetilde{N} \to N$ (of type l) there is a continuous function $\mathbf{c} : N \to B$ such that the pullback of γ under $\mathbf{c}$, namely $\mathbf{c}^*\gamma$, is a k-fold branched covering that is concordant with f. Moreover, it is also required that if $c_1 : N \to B$, $c_2 : N \to B$ are maps such that the pullbacks $\mathbf{c}_1^*\gamma$ and $\mathbf{c}_2^*\gamma$ give concordant branched coverings, then the maps $\mathbf{c}_1$ and $\mathbf{c}_2$ are homotopic. B is called a *classifying space* and $\mathbf{c}$ is called a *classifying function.* It is not difficult to see that there is a bijection between the set $B_{k,l}(N)$ of concordance classes of k-fold branched coverings of type l over N and the set $[N, B]$ of homotopic classes of continuous functions from N to B.

Now, we proceed to define a branched covering over a manifold with boundary. Let $\widetilde{N}$, N be smooth manifolds with boundary and $f : \widetilde{N} \to N$ be a smooth function. We say that f is a *k-fold branched covering of type l over a manifold with boundary* if

(i) $f|_{(N-\partial N)} : (\widetilde{N} - \partial\widetilde{N}) \longrightarrow (N - \partial N)$ and $f|_{\partial N} : \partial\widetilde{N} \to \partial N$ are k- fold branched coverings of type l over a manifold without boundary.

(ii) the following diagram is commutative

$$\begin{array}{ccc} \partial\widetilde{N} \times I & \xrightarrow{\ i\ } & \widetilde{N} \\ {\scriptstyle f|_{\partial N} \times id}\big\downarrow & & \big\downarrow{\scriptstyle f} \\ \partial N \times I & \xrightarrow[\ j\ ]{} & N \end{array} \tag{2.1}$$

where i and j are the embeddings obtained by "collaring" $\partial\widetilde{N}$ and ∂N in $\widetilde{N}$ and N, respectively (see [5]), and I is the closed interval [0,1] in $\mathbb{R}$.

Remarks

(1) If no ambiguity arises, we will omit the words "with boundary" or "without boundary".

(2) If H is a set in N, $f^{-1}(H)$ will be denoted by $\widetilde{H}$: for example, $\widetilde{K}_j = f^{-1}(K_j)$.

Roughly speaking, the next proposition says that the pullback of a k-fold branched covering is also a k-fold branched covering.

Proposition 2.1. *Let $f : \widetilde{N} \to N$ be a k-fold branched covering of type l with branched set K. Let M be a smooth manifold. If $g : M \to N$ is any smooth map transverse to K, and $\widetilde{M}$ is the fiber product in the diagram*

$$\begin{array}{ccc} \widetilde{M} & \longrightarrow & \widetilde{N} \\ {\scriptstyle f_1}\downarrow & & \downarrow{\scriptstyle f} \\ M & \underset{g}{\longrightarrow} & N \end{array} \tag{2.2}$$

then the pullback $g^*(f) = f_1 : \widetilde{M} \to M$ is a k-fold branched covering of type l.

Proof. In fact, since g is transverse to K, $g^{-1}(K)$ is a stratified set in M of type l. Moreover, the restriction of f to the preimages of the strata of K are finite coverings, and their pullbacks under g are also finite coverings.

We have to show that the branch set of f_1 is just $g^{-1}(K)$. If x belongs to $M - g^{-1}(K)$, then x is not a branched point, because the restriction of f_1 to $f_1^{-1}(M - g^{-1}(K))$ is a k-covering. Let $x \in g^{-1}(K)$. To show that x is a branched point we need only to observe that the cardinality of the set $f_1^{-1}(x)$ is different from the cardinality of the set $f_1^{-1}(x_0)$ when x_0 is not a branched point. In fact, if z is any element in M, the cardinality of $f_1^{-1}(z)$ is equal to the cardinality of $f^{-1}(g(z))$, because of the definition of the fiber product. □

Recall that $B_{k,l}(N)$ is the set of concordance classes of k-fold branched coverings of type l over the manifold N.

Proposition 2.2. *$B_{k,l}(\bullet)$ is a contravariant functor from the category of smooth manifolds to the category of sets.*

Proof. Let $M, N, \widetilde{N}$ be smooth manifolds (with or without boundary). Let $g : M \to N$ be a smooth map and let $f : \widetilde{N} \to N$ be a k-fold branched covering of type l. Without loss of generality, we may asssume that g is transverse to K, where K is the branch set of f. The pullback $f_1 : \widetilde{M} \to M = g^*(f)$ is a k-fold branched covering of type l. Let us define $B_{k,l}(f)$ as the concordance class of $f_1 = g^*(f)$. We need to show that $B_{k,l}$ is well defined.

First of all, let us prove that if $f' : \widetilde{N}' \to N$ is a branched covering of type l concordant to $f : \widetilde{N} \to N$, then $B_{k,l}(f) = B_{k,l}(f')$. Without loss of generality, we may assume that g is also transverse to K', the branch set of f' (actually, we can change g up to homotopy so that it is transverse to K'; see [9], p. 12). Let $F : W^{n+1} \to N \times I$ be a branched covering (of type l) that makes f and f' concordant. Consider the pullback of F under $g \times id$. It is clear that this pullback makes the pullback of f concordant with the pullback of f' under g, i.e., $B_{k,l}(f) = B_{k,l}(f')$.

Let $g_1, g_2 : M \to N$ be homotopic functions (both transverse to K) with pullbacks f_1 and f_2, respectively, under g. Let $H : M \times I \to N$ be a homotopy from g_1 to g_2. Without loss of generality, we may assume that H is transverse to K. Taking W_1 as the fiber product in the diagram

$$\begin{array}{ccc} W_1 & \longrightarrow & \widetilde{N} \\ {\scriptstyle F_1}\big\downarrow & & \big\downarrow{\scriptstyle f} \\ M \times I & \underset{H}{\longrightarrow} & N \end{array} \tag{2.3}$$

we get the branched covering $F_1 : W_1 \to M \times I$, which implies the concordance between f_1 and f_2. □

Now we define a pointed branched covering over a manifold and modify the definitions of equivalence up to homeomorphism and concordance according to this new definition. Let $\widetilde{M}, M$ be manifolds. A *pointed branched covering* is a k-fold branched covering $f : \widetilde{M} \to M$ together with a base point $*$ in $M - K$ (where K is the branch set of f) and a one–to–one correspondence of $\{1, 2, \ldots, k\}$ with the set $f^{-1}(*)$. This correspondence is denoted by $c(f)_i$, where $1 \leq i \leq k$. Two pointed branched coverings f_1 and f_2 are *equivalent up to homeomorphism* if there is a branched covering homeomorphism h which preserves the labeling, that is, such that $h(c(f_1)_i) = c(f_2)_i$. Two pointed k-fold branched coverings are *concordant* if there is a branched covering concordance $F : \widetilde{W}^{n+1} \to M \times I$ such that $* \times I$ does not intersect the branch set of F, and $c(f_1)_i$ and $c(f_2)_i$ are in the same components of $F^{-1}(* \times I)$ when $F|_{F^{-1}(M\times\{0\})}$ and $F|_{F^{-1}(M\times\{1\})}$ are equivalent up to homeomorphism to f_1 and f_2, respectively.

Let us denote with $B_{k,l}(M, *)$ the set of all concordance classes of pointed k-fold branched coverings over the manifold M. It is not difficult to see that $B_{k,l}(\bullet, *)$ is a contravariant functor from the category of pointed smooth manifolds to the category of sets with a distinguished element.

3. Branched coverings over simplicial complexes

In this section we establish the existence of a functor that operates over the category of CW-complexes and verifies all the hypotheses of Brown's Theorem. Since every CW-complex is homotopically equivalent to a simplicial complex, we first work on simplicial complexes. Then, at the end of the section, we proceed to generalize the results to CW-complexes.

Let Δ_m be an m-simplex. For every face Δ_{m-1} of Δ_m there is a one-to-one function $sl : \Delta_{m-1} \times I \to \Delta_m$ such that $(\Delta_m - sl(\Delta_{m-1} \times [0, 1))$ is homeomorphic to Δ_m. We call this kind of function a *"slice"*.

Now, we extend the definition of branched coverings over manifolds to branched coverings over simplicial complexes. Let Δ_m be an m-simplex. We say that $f : \widetilde{\Delta}_m \to \Delta_m$ (where $\widetilde{\Delta}_m$ is not necessarily an m-simplex) is a *k-fold* *branched covering of type l over the simplex* Δ_m if for every face Δ_s of Δ_m the function $f|_{f^{-1}(\mathrm{int}\Delta_s)} : f^{-1}(\mathrm{int}\Delta_s) \to \mathrm{int}\Delta_s$ (where int means interior) is a k-fold branched covering of type l over the manifold $\mathrm{int}\Delta_s$, and for each face Δ_{s-1} of Δ_s, we have the following commutative diagram

$$\begin{array}{ccc} f^{-1}(\Delta_{s-1}) \times I & \longrightarrow & f^{-1}(\Delta_s) \\ {\scriptstyle f\times I}\downarrow & & \downarrow{\scriptstyle f} \\ \Delta_{s-1} \times I & \underset{slice}{\longrightarrow} & \Delta_s \end{array} \tag{3.1}$$

where I is the closed interval [0,1] and $f^{-1}(\Delta_{s-1}) \times I \to f^{-1}(\Delta_s)$ is some embedding.

Let Σ be a simplicial complex and let $f : \widetilde{\Sigma} \to \Sigma$ be a continuous function (where $\widetilde{\Sigma}$ is not necessarily a simplicial complex). We say that f is a *k-fold branched covering of type l over* Σ, if for every m-simplex Δ_m contained in any subdivision of Σ, the function $f : \widetilde{\Delta}_m \to \Delta_m$ is a k-fold branched covering of type l over Δ_m.

The next proposition is in the spirit of Proposition 2.1.

Proposition 3.1. *Let Σ_1, Σ_2 be simplicial complexes, $f : \widetilde{\Sigma}_2 \to \Sigma_2$ be a k-fold branched covering of type l and $g : \Sigma_1 \to \Sigma_2$ be a simplicial function. If $\widetilde{\Sigma}_1$ is the fiber product in the diagram*

$$\begin{array}{ccc} \widetilde{\Sigma}_1 & \longrightarrow & \widetilde{\Sigma}_2 \\ {\scriptstyle f_1}\downarrow & & \downarrow{\scriptstyle f} \\ \Sigma_1 & \underset{g}{\longrightarrow} & \Sigma_2 \end{array} \tag{3.2}$$

then the pullback f_1 is a k-fold branched covering of type l over the simplicial complex Σ_1.

Proof. Let Δ_n be any n-simplex contained in Σ_1. Since $g : \Sigma_1 \to \Sigma_2$ is a simplicial function, $g(\Delta_n)$ is a simplex in Σ_2. Moreover, for every $\Delta_s \subset \Delta_n$, $g(\mathrm{int}\Delta_s)$ is equal to the interior of the simplex $g(\Delta_s)$. The linearity of g in Δ_s now implies that g is transverse to the branch set of

$$f|_{f^{-1}(\mathrm{int}\, g(\Delta_s))} : f^{-1}(\mathrm{int}\, g(\Delta_s)) \to \mathrm{int}\, g(\Delta_s),$$

and by Proposition 2.1, the restriction $f_1|_{f^{-1}(\mathrm{int}\Delta_s)} : f^{-1}(\mathrm{int}\Delta_s) \to \mathrm{int}\Delta_s$ is a k–fold branched covering of type l.

Now, consider the *slice* : $\Delta_{s-1} \times I \subset \Delta_s$, for every $\Delta_{s-1} \subset \Delta_s$. The linearity of g in Δ_n implies that $g(\Delta_{s-1}) \times I$ is also a *slice* inside $g(\Delta_s)$, and for some embedding $f_1^{-1}(\Delta_{s-1}) \times I \to f_1^{-1}(\Delta_s)$ the following diagram commutes:

$$\begin{array}{ccc} f_1^{-1}(\Delta_{s-1}) \times I & \longrightarrow & f_1^{-1}(\Delta_s) \\ {\scriptstyle f_1|_{f_1^{-1}(\Delta_{s-1})} \times id} \downarrow & & \downarrow {\scriptstyle f_1} \\ \Delta_{s-1} \times I & \underset{slice}{\longrightarrow} & \Delta_s \end{array} \tag{3.3}$$

Therefore, $f_1 : \widetilde{\Sigma}_1 \to \Sigma_1$ is a k-fold branched covering of type l over the simplex Δ_n. □

Let us extend the definition of concordance for branched coverings over simplicial complexes. We say that two k-fold branched coverings f_1, f_2 over a simplicial complex Σ are *concordant* if there is a branched covering $F : W \to \Sigma \times I$ over the simplicial complex $\Sigma \times I$ such that $F|_{F^{-1}(\Sigma \times \{j\})}$ is equivalent up to homeomorphism with f_{j+1}, for $j = 0, 1$.

The proofs of the following lemmas are not difficult ([9], p. 22, 23).

Lemma 3.2. *Let Σ_1, Σ_2 be simplicial complexes and let $f_2 : \widetilde{\Sigma}_2 \to \Sigma_2$ and $f_2' : \widetilde{\Sigma}_2' \to \Sigma_2$ be two concordant branched coverings over Σ_2, and $g : \Sigma_1 \to \Sigma_2$ a simplicial function. Then the pullbacks $g^* f_2$ and $g^* f_2'$ given in the diagrams below are concordant:*

$$\begin{array}{ccc} \widetilde{\Sigma}_1 & \longrightarrow & \widetilde{\Sigma}_2 \\ {\scriptstyle g^* f_2} \downarrow & & \downarrow {\scriptstyle f_2} \\ \Sigma_1 & \underset{g}{\longrightarrow} & \Sigma_2 \end{array} \tag{3.4}$$

$$\begin{array}{ccc} \widetilde{\Sigma}_1' & \longrightarrow & \widetilde{\Sigma}_2' \\ {\scriptstyle g^* f_2'} \downarrow & & \downarrow {\scriptstyle f_2'} \\ \Sigma_1 & \underset{g}{\longrightarrow} & \Sigma_2 \end{array} \tag{3.5}$$

Lemma 3.3. *Let Σ_1, Σ_2 be simplicial complexes and let $g_1 : \Sigma_1 \to \Sigma_2$ and $g_2 : \Sigma_1 \to \Sigma_2$ be homotopic simplicial functions. Also let $f_2 : \widetilde{\Sigma}_2 \to \Sigma_2$ be a branched covering over the simplicial complex Σ_2. Then, the branched coverings $g_1^* f_2$ and $g_2^* f_2$ are concordant.*

Let Σ be a simplicial complex and denote with $B_k(\Sigma)$ the set of all concordance classes of k-fold branched coverings over Σ.

Proposition 3.4. *$B_k(\bullet)$ is a contravariant functor from the category of simplicial complexes to the category of sets.*

Proof. It comes straightforwardly from Proposition 3.1 and Lemmas 3.2 and 3.3. □

The notions of pointed branched coverings and of concordance for pointed branched coverings over manifolds are easily extended to the case of simplicial complexes. Similarly, $B_k(\Sigma, *)$ denotes the set of all concordance classes of pointed k-fold branched coverings over the simplicial complex Σ. In this context, Proposition 3.1 and Lemmas 3.2 and 3.3 extend to pointed branched coverings over simplicial complexes, and the following proposition is straightforward.

Proposition 3.5. *$B_k(\bullet, *)$ is a contravariant functor from the category of pointed simplicial complexes to the category of sets with distinguished element the constant function.*

So far we have proved the existence of the functor $B_k(\bullet, *)$. Now we proceed to verify Conditions A, B, C and D, of Brown's Theorem for this functor. Actually, the extension of Lemma 3.3 to the case of pointed branched coverings proves that the functor $B_k(\bullet, *)$ verifies Condition A, and the first part of Condition B is straightforward. The following propositon establishes the second part of Condition B.

Proposition 3.6. *Let $(\Sigma, \Sigma_1, \Sigma_2)$ be a proper triad in the category of pointed simplicial complexes $\mathcal{C}$. Let $A = \Sigma_1 \cap \Sigma_2$ and $j_i : A \to \Sigma_i$, $k_i : \Sigma_i \to \Sigma$ be the canonical inclusions. If $f_1 \in B_k(\Sigma_1, *)$ and $f_2 \in B_k(\Sigma_2, *)$ are such that the pullbacks $j_1^* f_1$ and $j_2^* f_2$ are concordant, then there is $f \in B_k(\Sigma, *)$ such that $k_i^* f$ is concordant to f_i, $i = 1, 2$. Furthermore, if A is a point, then f is unique.*

Proof. First of all, let us take subdivisions in A, Σ_1, Σ_2 such that every simplex in A is a simplex in Σ_1 and in Σ_2. Let $F : W \to A \times I$ be such that $F|_{F^{-1}(A\times\{0\})}$ is equivalent up to homeomorphism to $j_1^* f_1|_{(j_1^* f_1)^{-1}(A)}$ and $F|_{F^{-1}(A\times\{1\})}$ is equivalent up to homeomorphism to $j_2^* f_2|_{(j_2^* f_2)^{-1}(A)}$. Without loss of generality, we may assume that $F|_{F^{-1}(A\times\{0\})}$ and $F|_{F^{-1}(A\times\{1\})}$ are exactly the functions $j_1^* f_1|_{(j_1^* f_1)^{-1}(A)}$ and $j_2^* f_2|_{(j_2^* f_2)^{-1}(A)}$, respectively. Consider the quotient space $((A \times I) \cup \Sigma_1) \,/\!\sim$ (where $a \sim (a, 0)$, for all a in A). Call this space X_1. Since (Σ_1, A) has the homotopy extension property, there exists a homotopy $H : \Sigma_1 \times I \to X_1$ such that $H|_{X_1} = id_{X_1}$, where id_{X_1} is the identity function of X_1.

If $id_i : \Sigma_i \to \Sigma_i$ is the identity function, then $id_i^* f_i$ coincides with $j_i^* f_i$ in $(j_i^* f_i)^{-1}(A)$, for $i = 1, 2$. For $i = 1, 2$, let us denote $(j_i^* f_i)^{-1}(A)$ and $(id_i^* f_i)^{-1}(\Sigma_i)$ by $\widetilde{A}_i$ and $\widetilde{\Sigma}_i$, respectively. Now, consider the function G, $G : (W \cup \widetilde{\Sigma}_1) \longrightarrow X_1$ defined by $G(x) = F(x)$ if $x \in W$ and $G(x) = (id_1^* f_1)(x)$ if $x \in \widetilde{\Sigma}_1$; G is well defined, because F coincides with $id_1^* f_1$ in $\widetilde{A}_1$.

Let $H^* G$ be the pullback in the diagram

$$
\begin{array}{ccc}
(H^*G)^{-1}(\Sigma_1 \times I) & \longrightarrow & W \cup \widetilde{\Sigma}_1 \\
{\scriptstyle H^*G}\downarrow & & \downarrow{\scriptstyle G} \\
\Sigma_1 \times I & \xrightarrow[H]{} & X_1
\end{array}
\tag{3.8}
$$

Then, $H^*G|_{(H^*G)^{-1}(A\times\{0\})}$ is equivalent up to homeomorphism to $j_1^*f_1$ and $H^*G|_{(H^*G)^{-1}(A\times\{1\})}$ is equivalent up to homeomorphism to $j_2^*f_2$.

Construct the following space: $X = ((H^*G)^{-1}(\Sigma_1 \times \{1\}) \cup \widetilde{\Sigma}_2) \,/\!\sim$ where we use the homeomorphism that makes $H^*G|_{(H^*G)^{-1}(A\times\{1\})}$ equivalent to $j_2^*f_2$ to identify the elements in $f_2^{-1}(A)$ with those in $(H^*G)^{-1}(A \times \{1\})$. Now let us define $f : X \to ((\Sigma_1 \times \{1\}) \cup \Sigma_2) \,/\!\sim$ (where $a \sim (a,1)$ for every $a \in A$) by $f(x) = H^*G(x)$ if $x \in (H^*G)^{-1}(\Sigma_1 \times \{1\})$ and $f(x) = id_2^*f_2(x)$ if $x \in \widetilde{\Sigma}_2$. Clearly f is a well defined branched covering over $((\Sigma_1 \times \{1\}) \cup \Sigma_2) \,/\!\sim$.

Since $((\Sigma_1 \times \{1\}) \cup \Sigma_2) \,/\!\sim$ is isomorphic to $\Sigma = \Sigma_1 \cup \Sigma_2$, we can say that $f \in B_k(\Sigma, *)$. Moreover, the construction of f implies that k_i^*f is concordant to f_i, $(i = 1, 2)$. Hence, f is the desired function.

It is also straightforward that if A reduces to a single point the construction of f is unique up to concordance. □

The next proposition verifies Condition D for the functor $B_k(\bullet, *)$.

Proposition 3.7. *Suppose $\Sigma_1 \subset \Sigma_2 \subset \cdots \subset \Sigma_n \subset \cdots$ is a collection of simplicial subcomplexes of $\Sigma = \bigcup_n \Sigma_n \in \mathcal{C}$ with respect to some simplicial-complex structure on Σ such that $\Sigma_n^n = \Sigma^n$ (where X^n is the n-skeleton of X). Let $i_n : \Sigma_n \to \Sigma$ be the inclusion map and $\varprojlim B_k(\Sigma_n, *)$ be the inverse limit of $B_k(\Sigma_n, *)$ with respect to the maps induced by the inclusions ψ_m^n of Σ_n into Σ_m. Then $\varprojlim(i_n^*) : B(\Sigma, *) \longrightarrow \varprojlim B(\Sigma_n, *)$ is an epimorphism.*

Proof. First of all, we recall that the inverse limit of $B_k(\Sigma_n, *)$ is

$$\varprojlim B(\Sigma_n, *) = \left\{ (f_n) \in \prod_n B_k(\Sigma_n, *) \mid f_n = (\psi_m^n)^* f_m, m \le n \right\}$$

(see [7]). Let $(f_n) \in \varprojlim B_k(\Sigma_n, *)$. We want to find $f \in B_k(\Sigma, *)$ such that $\varprojlim(i_n^*)(f) = (f_n)$.

Notice that $f_1 = (\psi_1^2)^* f_2$; in other words, that f_1 and the restriction $f_2|_{\Sigma_1}$ are concordant. Using this concordance we construct a function $f_2' \in B_k(\Sigma_2, *)$ such that $f_2'|_{(f_2')^{-1}(\Sigma_1)} = f_1$. Inductively, we construct $f_n' \in B_k(\Sigma_n, *)$ such that $f_n'|_{(f_n')^{-1}(\Sigma_{n-1})} = f_{n-1}'$. Hence, we have a collection of functions $f_1 = f_1', f_2', f_3', \ldots$ such that f_n' is an extension of f_{n-1}', for all n in $\mathbb{N}$.

Let $\widetilde{\Sigma} = \bigcup_n \widetilde{\Sigma}_n$, where $\widetilde{\Sigma}_n = (f_n')^{-1}(\Sigma_n)$, and define $f : \widetilde{\Sigma} \to \Sigma$ by $f(x) = f_n'(x)$ if $x \in (f_n')^{-1}(\Sigma_n)$. It is obvious that f is well defined. Therefore,

$f \in B_k(\Sigma, *)$. Because of the construction of f'_n, we have that f'_n is concordant to f_n. Therefore, $\varprojlim(i_n^*)(f) = (f_n)$, which implies that

$$\varprojlim(i_n^*) : B(\Sigma,^*) \longrightarrow \varprojlim B(\Sigma_n, *)$$

is an epimorphism. □

The following proposition proves condition C for the functor $B_k(\bullet, *)$. Here we regard the boundary of an $(n+1)$–simplex as an n–sphere.

Proposition 3.8. *Suppose $\{S^n_\alpha\}$ is a collection of disjoint n-spheres whose wedge product $\bigvee S^n_\alpha$ is in the category of simplicial complexes $\mathcal{C}$. Let i_β : $S^n_\beta \to \bigvee S^n_\alpha$ be the inclusion map. Then the function*

$$\prod i^*_\beta : B_k\Big(\bigvee S^n_\alpha, *\Big) \to \prod B_k(S^n_\alpha, *)$$

is bijective.

Proof. Let $(f_\alpha) \in \prod B_k(S^n_\alpha, *)$, where $f_\alpha \in B(S^n_\alpha, *)$. Recall that $*$ never is a branched point of f_α and that there is a one-to-one correspondence of $\{1, 2, \ldots, k\}$ with $f_\alpha^{-1}(*)$. This correspondence is denoted by $c(f_\alpha)_i$, for every α.

Let $X = (\bigcup_\alpha f_\alpha^{-1}(S^n_\alpha)) \;/\!\sim$, where $x_\alpha \sim x_\beta$ if $f_\alpha(x_\alpha) = f_\beta(x_\beta) = *$ and the corresponding $i \in \{1, 2, \ldots, k\}$ under $c(f_\alpha)_i$ and $c(f_\beta)_i$ is the same for both elements.

If $f : X \to \bigvee S^n_\alpha$ is such that $f(x) = f_\alpha(x)$ for $x \in f_\alpha^{-1}(S^n_\alpha)$, f is a well defined branched covering in $B_k(\bigvee S^n_\alpha, *)$. Moreover, $\prod i^*_\beta(f) = (f_\alpha)$. Therefore $\prod i^*_\beta$ is surjective.

Let us prove the injectivity. Let $f, g \in B_k(\bigvee S^n_\alpha, *)$ and assume that $\prod i^*_\beta(f) = (f_\alpha)$ and $\prod i^*_\beta(g) = (g_\alpha)$. Here $(f_\alpha) = (g_\alpha)$ means that f_α is concordant to g_α for each α. So, there exists a branched covering $F_\alpha : W_\alpha \to S^n_\alpha \times I$ such that $F_\alpha|_{F_\alpha^{-1}(S^n_\alpha \times \{1\})}$ is equivalent up to homeomorphism to f_α and $F_\alpha|_{F_\alpha^{-1}(S^n_\alpha \times \{0\})}$ is equivalent up to homeomorphism to g_α. Recall that $* \times I$ does not intersect the branch set. Therefore $F_\alpha^{-1}(* \times I)$ contains exactly k copies of I, and $c(f_\alpha)_i$ belongs to the same component of $F_\alpha^{-1}(* \times I)$ that contains $c(g_\alpha)_i$. Let us denote by $F_\alpha^{-1}(* \times I)_i$ the copy of I that contains $c(f_\alpha)_i$. Without loss of generality, we may assume that if $\alpha \neq \beta$ then $W_\alpha \neq W_\beta$. Let $\widetilde{X} = (\bigcup_\alpha W_\alpha) \;/\!\sim$, where $\sim$ is defined in the following way: $x \sim y$ if there are α, β and i ($i = 1, \ldots, k$) such that $x \in F_\alpha^{-1}(* \times I)_i$, $y \in F_\beta^{-1}(* \times I)_i$, and $F_\alpha(x) = F_\beta(y)$. Consider $F : \widetilde{X} \to \bigvee S_\alpha \times I$ defined by $F(x) = F_\alpha(x)$ if $x \in W_\alpha$. Clearly F is a well defined function such that $F|_{F^{-1}(\vee\, S^n_\alpha \times \{1\})}$ is equivalent up to homeomorphism to f and $F|_{F^{-1}(\vee\, S^n_\alpha \times \{0\})}$ is equivalent up to homeomorphism to g. Hence, f is concordant to g, *i.e.*, $\prod i^*_\beta$ is injective. □

Now recall that every CW-complex is homotopically equivalent to a simplicial complex, *i.e.*, that if X is a CW-complex there exist a simplicial complex Σ and maps $h : X \to \Sigma$, $h' : \Sigma \to X$ such that $h \circ h'$ is homotopic to the identity of Σ and $h' \circ h$ is homotopic to the identity of X. We will say that a function $f : \widetilde{X} \to X$ is a *k-fold branched covering of type l over the CW-complex X*, if the function $(h')^* f : \widetilde{\Sigma} \to \Sigma$ is a k-fold branched covering over the simplicial complex Σ. In a natural way we extend all definitions, Lemmas and propositions given for (resp. pointed) branched coverings over simplicial complexes to (resp. pointed) branched coverings over CW-complexes. Actually, we prove that if $\mathcal{C}$ is the category of CW-complexes (not necessarily finite), the functor $B_k(\bullet, *)$ satisfies Conditions A, B, C, D of Brown's Representability Theorem. Hence, the following theorem holds.

Theorem 3.9. *Let $\mathcal{C}$ be the category of spaces with base point $*$ which has as objects all topological spaces admitting a CW-complex structure. Let $\mathcal{S}$ be the category of sets with a distinguished element. Let $B_k(\bullet, *) : \mathcal{C} \to \mathcal{S}$ be the functor defined earlier. Then, there is a space Y in $\mathcal{C}$, unique up to homotopy type, such that the functors $[\bullet, Y]$ and $B_k(\bullet, *)$ are naturally equivalent.*

In other words, we have obtained the existence of a classifying space Y for k-fold branched coverings over CW-complexes.

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