On two properties of the numerical range of a bounded Hilbert space operator

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ABSTRACT. Necessary and sufficient conditions are given for the numerical range of a bounded Hilbert space operator to have an empty interior. A sufficient condition for this set to be open is also established.

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1. Introduction

Let $H$ be a complex vector space. Endowed with an inner Hermitian product $\langle , \rangle$, $H$ will be called a pre-Hilbert space. The norm of $H$ is $\|x\| = \sqrt{\langle x, x \rangle}$. If $H$ is complete as a normed space (i.e., if $H$ is a Banach space for $\| \|$, $H$ will be called a Hilbert space.

By an operator on $H$ we mean a linear map $T$ of a subspace $D(T)$ of $H$, called the domain of $T$, into $H$. If $D(T) = H$ and there is a constant $C > 0$ such that $\|Tx\| \leq C\|x\|$, $T$ will be called a bounded operator on $H$.

An operator $T$ on $H$ is symmetric if $\langle Tx, y \rangle = \langle x, Ty \rangle$ for all $x, y$ in $D(T)$. An operator $T$ is symmetric if and only if $\langle Tx, x \rangle$ is a real number for all $x$ in $D(T)$. 

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If $T$ is an operator and $D(T)$ is a dense subset of $H$, the adjoint $T^*$ of $T$ can be defined: it is the operator of $D(T^*)$ into $H$ such that $(Tx, y) = (x, T^*y)$ for all $x \in D(T)$ and all $y \in D(T^*)$. It can be shown that $D(T^*)$ is also a dense subspace of $H$ and that $D(T^*) = H$ if $D(T) = H$. If $T$ is symmetric with dense domain then $D(T) \subseteq D(T^*)$. If $T$ is symmetric and $D(T) = D(T^*)$, $T$ is called a self-adjoint or Hermitian operator on $H$. If $T$ is symmetric and $D(T) = H$, $T$ is self-adjoint and bounded.

Let $T$ be a bounded operator on $H$ and let

$$T_1 = \frac{1}{2}(T + T^*), \quad T_2 = \frac{1}{2i}(T - T^*)$$  \hspace{1cm} (1.1)

Then $T_1$ and $T_2$ are bounded self-adjoint operators on $H$, and

$$T = T_1 + iT_2$$  \hspace{1cm} (1.2)

The operators $T_1$ and $T_2$ are called the Cartesian coordinates of $T$, and the decomposition (1.2), its Cartesian decomposition. Observe that

$$\Re\langle Tx, x \rangle = \langle T_1 x, x \rangle, \quad \Im\langle Tx, x \rangle = \langle T_2 x, x \rangle$$  \hspace{1cm} (1.3)

for all $x$ in $H$.

We also recall that a bounded operator $T$ on $H$ is normal if $\|T^*x\| = \|Tx\|$ for all $x$ in $H$. If $T$ is an operator, the set

$$W(T) = \{ \langle Tx, x \rangle \mid x \in D(T), \|x\| = 1 \},$$

which is a subset of the set $\mathbb{C}$ of complex numbers, is the numerical range of $T$. In recent literature much attention has been paid to topological and geometric properties of the numerical range. It is known for example that $W(T)$ is a convex set [2], [3], [5], [11], [12], [13], that the closure of $W(T)$ contains the spectrum of $T$ and, moreover, that if $T$ is normal, it really is the closed convex hull of the spectrum (see [12]). Topological properties of $W(T)$ are extremely important. For example, if $T$ is normal and $W(T)$ is closed, the extreme points of $W(T)$ are eigenvalues of $T$ (see [6]). Many of these basic results have been extended one way or another to more general classes of operators (hyponormal ([9],[10]), quasihyponormal ([1], [7], [8]) and the like). All this constitutes a very active field of research in operator theory. In this paper we give necessary and sufficient conditions for the interior of the numerical range of a bounded operator in Hilbert space to be empty. These conditions refer to the structure of the operator and to the Jacobian matrix of a certain $C^1$-function related to it. Sufficient conditions also involving that matrix and some other simple properties of $T$ are established for $W(T)$ to be open (see [4], problem 168).
2. Main Results

We first give necessary and sufficient conditions for the numerical range of a bounded operator on a Hilbert space to have an empty interior. Then, sufficient conditions for that set to be open will also be given. Some preliminary results will be needed. With \( \mathbb{R} \) we denote the set of real numbers.

**Lemma 2.1.** Let \( T \) be a symmetric operator on a pre-Hilbert space \( H \). Then, for each pair \( x, y \) in \( D(T) \), the map \( f : \mathbb{R}^2 \to \mathbb{R} \) given by

\[
f(s, t) = \langle T(\tau x + (1 - \tau)y), \tau x + (1 - \tau)y \rangle, \quad \tau = s + it \tag{2.1}
\]

is in \( C^1 \). Furthermore

\[
\frac{\partial f}{\partial s}(0, 0) = 2\Re\langle T(x - y), y \rangle, \quad \frac{\partial f}{\partial t}(0, 0) = -2\Im\langle T(x - y), y \rangle \tag{2.2}
\]

**Proof.** From

\[
\frac{1}{h}(f(s + h, t) - f(s, t)) = \langle T(\tau x + (1 - \tau)y), x - y \rangle
\]

\[
+ \langle T(x - y), \tau x + (1 - \tau)y \rangle + h\langle T(x - y), x - y \rangle
\]

and

\[
\frac{1}{h}(f(s, t + h) - f(s, t)) = \langle T(\tau x + (1 - \tau)y), i(x - y) \rangle
\]

\[
+ \langle T(i(x - y)), \tau x + (1 - \tau)y \rangle + h\langle T(x - y), x - y \rangle
\]

it follows, letting \( h \to 0 \), that

\[
\frac{\partial f}{\partial s}(s, t) = 2\Re\langle T(x - y), \tau x + (1 - \tau)y \rangle \tag{2.3}
\]

and

\[
\frac{\partial f}{\partial t}(s, t) = -2\Im\langle T(x - y), \tau x + (1 - \tau)y \rangle \tag{2.4}
\]

which are continuous functions of \( \tau \). Relations (2.2) follow from (2.3) and (2.4) with \( \tau = 0 \). \( \square \)

**Lemma 2.2.** Let \( x, y \) be vectors in a pre-Hilbert space \( H \) and let \( g : \mathbb{R}^2 \to \mathbb{R} \) be the map

\[
g(s, t) = \|\tau x + (1 - \tau)y\|^2, \quad \tau = s + it. \tag{2.5}
\]

Then \( g \) is in \( C^1 \) and

\[
\frac{\partial g}{\partial s}(s, t) = 2\Re\langle x - y, \tau x + (1 - \tau)y \rangle \tag{2.6}
\]

\[
\frac{\partial g}{\partial t}(s, t) = -2\Im\langle x - y, \tau x + (1 - \tau)y \rangle.
\]

**Proof.** Let \( T \) be the identity operator in Lemma 2.1. \( \square \)
Lemma 2.3. Let $T$ be a bounded operator on a pre-Hilbert space $H$ and let $x, y$ with $\|y\| = 1$ be linearly independent vectors in $H$. Denote with $F$ the map of $\mathbb{R}^2$ into $\mathbb{R}^2$ given by

$$F(s, t) = \left( \frac{f_1(s, t)}{g(s, t)}, \frac{f_2(s, t)}{g(s, t)} \right)$$

(2.7)

where

$$f_i(s, t) = \langle T_i(\tau x + (1 - \tau)y), \tau x + (1 - \tau)y \rangle, \quad i = 1, 2$$

(2.8)

with $T_1, T_2$ as in (1.1) and $g$ as in (2.5). Then $F$ is in $C^1$, and the Jacobian matrix of $F$ at $(0, 0)$ is

$$J_T(x, y) = \begin{bmatrix} 2\Re \langle (T_1 - (T_1y, y))(x - y), y \rangle & -2\Im \langle (T_1 - (T_1y, y))(x - y), y \rangle \\ 2\Re \langle (T_2 - (T_2y, y))(x - y), y \rangle & -2\Im \langle (T_2 - (T_2y, y))(x - y), y \rangle \end{bmatrix}$$

(2.9)

Proof. This follows from $T_1, T_2$ being self-adjoint (so that $f_1, f_2$ are $C_1$ of $\mathbb{R}^2$ into $\mathbb{R}$), from observing that $g(s, t)$, which is also $C^1$, never vanishes, and from relations (2.2) and (2.6). \qed

Lemma 2.4. Let $T, T_1$ be commuting bounded operators on a Hilbert space $H$. Assume that $T_1$ is self-adjoint and there is $f: H \to \mathbb{C}$ such that

$$Tx = f(x)T_1x$$

(2.10)

for all $x \in H$. Then $T = \beta T_1$ for some $\beta$ in $\mathbb{C}$.

Proof. If $T = 0$, let $\beta = 0$. Now assume there is $x_0 \in H$ such that $Tx_0 \neq 0$ and let $\beta = f(x_0)$. If

$$H_0 = \{ x \in H \mid Tx = \beta T_1x \}$$

(2.11)

then $H_0$ is a non trivial closed subspace of $H$. We claim that $H_0 = H$. Since $T(T_1x) = T_1(Tx) = T_1(\beta T_1x) = \beta T_1(T_1x)$ for all $x \in H_0$, it follows that $T_1(H_0) \subseteq H_0$ and, $T_1$ being self-adjoint, also $T_1(H_0^\perp) \subseteq H_0^\perp$. Hence, from (2.10), $T(H_0) \subseteq H_0$ and $T(H_0^\perp) \subseteq H_0^\perp$.

Now let $x \in H_0, y \in H_0^\perp$. Since $T(x_0 + y) = f(x_0 + y)T_1(x_0 + y) = f(x_0 + y)T_1x_0 + f(x_0 + y)T_1y = Tx_0 + Ty$, we get

$$(f(x_0 + y) - \beta)T_1x_0 = (f(y) - f(x_0 + y))T_1y \in H_0 \cap H_0^\perp;$$

and, since $T_1x_0 \neq 0$, that $\beta = f(x_0 + y)$. Then

$$T(x + y) = T(x - x_0) + T(x_0 + y) = \beta T_1(x - x_0) + \beta T_1(x_0 + y) = \beta T_1(x + y),$$

which implies that $H = H_0 \cap H_0^\perp \subseteq H_0$, and completes the proof. \qed
Lemma 2.5. Let $T$ be a bounded operator on a Hilbert space $H$ and let $T_1, T_2$ as in (1.1) be the self-adjoint operators in the Cartesian decomposition of $T$. Assume there is a function $f : H \to \mathbb{R}$ such that

$$T_2 y = \langle T_2 y, y \rangle y + f(y)(T_1 y - \langle T_1 y, y \rangle y) \quad (2.12)$$

for each $y \in H$ with $\|y\| = 1$. Then, $T$ is a normal operator on $H$.

Proof. Since $T = T_1 + iT_2$ and $T^* = T_1 - iT_2$ then

$$Ty = (1 + if(y))T_1 y + i(\langle T_2 y, y \rangle y - f(y)\langle T_1 y, y \rangle y)$$

and

$$T^* y = (1 - if(y))T_1 y - i(\langle T_2 y, y \rangle y - f(y)\langle T_1 y, y \rangle y)$$

for each $y$ with $\|y\| = 1$. Let

$$\alpha(y) = 1 + if(y), \quad \beta(y) = \langle T_2 y, y \rangle - f(y)\langle T_1 y, y \rangle, \quad y \in H,$$

then

$$(1/\alpha(y)) Ty = T_1 y + i(\beta(y)/\alpha(y)) y$$

and

$$\left(1/\alpha(y)\right) T^* y = T_1 y - i\left(\beta(y)/\alpha(y)\right) y$$

whenever $\|y\| = 1$. But, $T_1$ being self-adjoint, we have $\|(T_1 + \alpha)y\| = \|(T_1 + \overline{\alpha})y\|$ for all $y \in H$ and $\alpha \in \mathbb{C}$. Thus

$$\|1/\alpha(y)Ty\| = \|(1/\alpha(y))T^*y\|, \quad \|y\| = 1$$

which implies that $\|Ty\| = \|T^*y\|$ for all $y$ in $H$. Hence, $T$ is normal. □

Lemma 2.6. Let $H$ be a Hilbert space and let $T, T_1$ and $T_2$ be as in Lemma 2.5. Assume that the determinant $|J_T(x, y)|$ of the Jacobian matrix $J_T(x, y)$ vanishes for all linearly independent vectors $x, y$ in $H$ with $\|y\| = 1$. Then, $T$ is a normal operator on $H$.

Proof. The assumptions imply that $|J_T(x + y, y)| = 0$ whenever $x, y$ are linearly independent and $\|y\| = 1$. On the other hand, (1.2) yields

$$\Re \langle Tx, y \rangle = \Re \langle T_1 x, y \rangle - \Im m \langle T_2 x, y \rangle$$

$$\Im m \langle Tx, y \rangle = \Im m \langle T_1 x, y \rangle + \Re \langle T_2 x, y \rangle,$$

so that

$$|\langle Tx, y \rangle|^2 = |\langle T_1 x, y \rangle|^2 + |\langle T_2 x, y \rangle|^2 + RT(x, y)$$

$$\quad (2.14)$$

where

$$RT(x, y) = 2\left\{\Re \langle T_2 x, y \rangle \Im m \langle T_1 x, y \rangle - \Re \langle T_1 x, y \rangle \Im m \langle T_2 x, y \rangle\right\}$$

$$\quad (2.15)$$
Since
\[ (T - \langle Ty, y \rangle) x = (T_1 - \langle T_1 y, y \rangle)x + i(T_2 - \langle T_2 y, y \rangle)x, \quad (2.16) \]
(2.14), with \( T - \langle Ty, y \rangle I \) in the place of \( T \), gives
\[ |\langle (T - \langle Ty, y \rangle)x, y \rangle|^2 = |\langle (T_1 - \langle T_1 y, y \rangle)x, y \rangle|^2 + |\langle (T_2 - \langle T_2 y, y \rangle)x, y \rangle|^2 \]
\[ + \frac{1}{2} |J_T(x + y, y)| \quad (2.17) \]

Hence, if \( x \) and \( y \) are linearly independent with \( ||y|| = 1 \), in which case \( |J_T(x + y, y)| = 0 \), we have that \( x \) is orthogonal to \( (T^* - \overline{T y, y})y \) if and only if \( x \) is orthogonal to both \( (T_1 - \langle T_1 y, y \rangle)y \) and \( (T_2 - \langle T_2 y, y \rangle)y \), i.e.,
\[ \{(T^* - \overline{T y, y})y\}^\perp = \{(T_1 - \langle T_1 y, y \rangle)y\}^\perp \cap \{(T_2 - \langle T_2 y, y \rangle)y\}^\perp \quad (2.18) \]

Since all three spaces in (2.18) are closed hyperplanes, this relationship is possible if and only if those spaces coincide, which ensures the existence of \( f(y) \in \mathbb{C} \) such that
\[ (T_2 - \langle T_2 y, y \rangle)y = f(y)(T_1 - \langle T_1 y, y \rangle)y, \quad ||y|| = 1. \quad (2.19) \]

We claim that \( f(y) \) can be taken to be real. Indeed, let \( y \in H \) with \( ||y|| = 1 \). If \( (T_1 - \langle T_1 y, y \rangle)y = 0 \), take \( f(y) = 0 \). If \( (T_1 - \langle T_1 y, y \rangle)y \neq 0 \), take \( x \) linearly independent of \( y \) and such that \( \langle (T_1 - \langle T_1 y, y \rangle)y, x \rangle \neq 0 \) (for instance, \( x = (T_1 - \langle T_1 y, y \rangle)y \) will do). From \( |J_T(x + y, y)| = 0 \) it follows that for some \( \alpha \in \mathbb{R} \),
\[ \left( \Re \langle (T_2 - \langle T_2 y, y \rangle)x, y \rangle, \Im \langle (T_2 - \langle T_2 y, y \rangle)x, y \rangle \right) = \alpha \left( \Re \langle (T_1 - \langle T_1 y, y \rangle)x, y \rangle, \Im \langle (T_1 - \langle T_1 y, y \rangle)x, y \rangle \right), \]
so that
\[ \langle (T_2 - \langle T_2 y, y \rangle)x, y \rangle = \alpha \langle (T_1 - \langle T_1 y, y \rangle)x, y \rangle, \]
which is the same as
\[ \langle (T_2 - \langle T_2 y, y \rangle)y, x \rangle = \alpha \langle (T_1 - \langle T_1 y, y \rangle)y, x \rangle. \quad (2.20) \]

This, together with (2.19) and the assumption \( \langle (T_1 - \langle T_1 y, y \rangle)y, x \rangle \neq 0 \), ensures that \( f(y) = \alpha \). The conclusion now follows from Lemma 2.5. \( \square \)
Theorem 2.1. Let $T$ be a bounded operator on a Hilbert space $H$ and let $T = T_1 + iT_2$ be its Cartesian decomposition. The following assertions are equivalent:

1. There are $\alpha, \beta$ in $\mathbb{C}$ and a bounded self-adjoint operator $B$ on $H$ such that
   \[ T = \alpha I + \beta B \]  

2. \( \text{Int} (W(T)) = \emptyset \).

3. For any pair of vectors $x, y$ in $H$ with $\|y\| = 1,$
   \[ |J_T(x, y)| = 0 \]  

4. For any pair of vectors $x, y$ in $H$ with $\|y\| = 1,$
   \[ |\langle (T - \langle Ty, y \rangle)x, y \rangle|^2 = |\langle (T_1 - \langle T_1 y, y \rangle)x, y \rangle|^2 + |\langle (T_2 - \langle T_2 y, y \rangle)x, y \rangle|^2. \]  

Proof. If (2.21) holds then $W(T) = \alpha + \beta W(B)$, so that $\text{Int} W(T) = \alpha + \beta \mathbb{S} = \emptyset$ (as $W(B) \subseteq \mathbb{R}$). Hence, (1) $\implies$ (2).

To prove that (2) $\implies$ (3), assume (2.22) does not hold for a couple of vectors $x, y$ in $H$ with $\|y\| = 1$. Then, $x, y$ are linearly independent. Let $F : \mathbb{R}^2 \to \mathbb{R}^2$ be given by (2.7). Since $F(0, 0) = (\langle T_1 y, y \rangle, \langle T_2 y, y \rangle) = \langle Ty, y \rangle$, the Inverse Function Theorem, guaranties the existence of open sets $U, V$ of $\mathbb{R}^2$ with $(0, 0) \in U$ and $F(0, 0) \in V$ such that $F(U) = V$. Since $F(\mathbb{R}^2) \subseteq W(T)$, $\text{Int} (W(T)) \neq \emptyset$.

To see that (3) $\implies$ (4), just observe that (3) implies that $|J_T(x + y, y)| = 0$, and, from (2.17), this is equivalent to (2.23).

Now assume that (4) holds. Lemma 2.6 and (2.23), which imply (2.17) to hold, ensure that $T$ is normal and that for each $y$ with $\|y\| = 1$ the three vectors $T_2 y, T y$ and $T^* y$ are in the subspace spanned by $y$ and $T_1 y$. Thus, for each $y$ in $H$ with $\|y\| = 1$, there are $\alpha(y)$ and $\beta(y)$ in $\mathbb{C}$ such that

\[ Ty = \alpha(y)y + \beta(y)T_1 y \]  

(2.24)

Now, if $y$ belongs to the subspace spanned by $T_1 y$, for all $y$ with $\|y\| = 1$, (2.24) and Lemma 2.4 apply to give that $T = \alpha I$ for some $\alpha$ in $\mathbb{C}$. If on the contrary there is $x_0$ in $H$ not belonging to the subspace spanned by $Tx_0$, and if $\alpha = \alpha(x_0), \beta = \beta(x_0)$ and

\[ H_0 = \{ x \in H \mid Tx = \alpha x + \beta T_1 x \}, \]

then $H_0$ is a non trivial closed subspace of $H$ which is readily seen to be, as well as $H_0^\perp$, invariant under $T_1, T_2, T,$ and $T^*$. Since $H = H_0 + H_0^\perp$, this implies, exactly as in Lemma 2.4, that $H_0 = H$. Hence $T = \alpha I + \beta T_1$ and, since $T_1$ is self-adjoint, this shows that (4) $\implies$ (1) and completes the proof of the theorem.
Corollary 2.1. Let $T$ be a bounded operator on the Hilbert space $H$ and $T = T_1 + iT_2$ be its Cartesian decomposition. Each of the following conditions is sufficient for $W(T)$ to be open:

1. For each $y$ in $H$ with $||y|| = 1$, there is $x \in H$ such that
   \[ |J_T(x + y, y)| \neq 0. \]  
   (2.25)

2. For each $y$ in $H$ with $||y|| = 1$, there is $x \in H$ such that
   \[ |\langle (T - \langle Ty, y \rangle)x, y \rangle|^2 \neq |\langle (T_1 - \langle T_1y, y \rangle)x, y \rangle|^2 
   + |\langle (T_2 - \langle T_2y, y \rangle)x, y \rangle|^2 \]  
   (2.26)

3. For each $y$ in $H$ with $||y|| = 1$, there is $\alpha \in \mathbb{C}$ such that
   \[ ||(T - \alpha)y|| \neq \|(T - \alpha^*)y|| \]  
   (2.27)

4. For all $y \in H$ with $||y|| = 1$, the vectors $y$, $Ty$ and $T^*y$ are linearly independent.

Proof. Since $|J_T(x+y, y)| = |J_T(x, y)|$, it is clear, from (2.17), that (1) $\iff$ (2). Now, if (2.25) holds, $x$ and $y$ are linearly independent, and arguing as in the proof of (2) $\implies$ (3) in Theorem 2.1, we conclude that $\langle Ty, y \rangle$ is interior to $W(T)$. Now let ||$y|| = 1$ and assume $|J_T(x, y)| = 0$ for all $x$ linearly independent of $y$. The argument in the proof of Lemma 2.6 shows, on the one hand, that $y$, $Ty$ and $T^*y$ are linearly dependent, and, on the other hand, that $T_2y = \langle T_2y, y \rangle y + \beta(T_1y - \langle T_1y, y \rangle y)$ for some $\beta$ in $\mathbb{R}$. Then, as in the proof of Lemma 2.5, we can show that $||(T - \alpha)y|| = \|(T - \alpha^*)y||$ for all $\alpha \in \mathbb{C}$. Hence, if (3) or (4) holds, also (1) must hold, and the proof is complete. \qed

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