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On two properties of the numerical range of a bounded Hilbert space operator

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ABSTRACT. Necessary and sufficient conditions are given for the numerical range of a bounded Hilbert space operator to have an empty interior. A sufficient condition for this set to be open is also established.

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1. Introduction

Let H be a complex vector space. Endowed with an *inner Hermitian product* $\langle\ ,\ \rangle$, H will be called a *pre-Hilbert space*. The *norm* of H is $||x|| = \sqrt{\langle x, x\rangle}$. If H is complete as a normed space (*i.e.*, if H is a Banach space for $||\ \ ||$), H will be called a *Hilbert space*.

By an *operator* on H we mean a linear map T of a subspace $D(T)$ of H, called the *domain* of T, into H. If $D(T) = H$ and there is a constant $C > 0$ such that $||Tx|| \leq C||x||$, T will be called a bounded operator on H.

An operator T on H is *symmetric* if $\langle Tx, y\rangle = \langle x, Ty\rangle$ for all x, y in $D(T)$. An operator T is symmetric if and only if $\langle Tx, x\rangle$ is a real number for all x in $D(T)$.

If T is an operator and $D(T)$ is a dense subset of H, the adjoint T^* of T can be defined : it is the operator of $D(T^*)$ into H such that $\langle Tx, y\rangle = \langle x, T^*y\rangle$ for all $x \in D(T)$ and all $y \in D(T^*)$. It can be shown that $D(T^*)$ is also a dense subspace of H and that $D(T^*) = H$ if $D(T) = H$. If T is symmetric with dense domain then $D(T) \subseteq D(T^*)$. If T is symmetric and $D(T) = D(T^*)$, T is called a *self-adjoint* or *Hermitian operator* on H. If T is symmetric and $D(T) = H$, T is self-adjoint and bounded.

Let T be a bounded operator on H and let

$$T_1 = \frac{1}{2}(T + T^*), \qquad T_2 = \frac{1}{2i}(T - T^*) \tag{1.1}$$

Then T_1 and T_2 are bounded self-adjoint operators on H, and

$$T = T_1 + iT_2 \tag{1.2}$$

The operators T_1 and T_2 are called the Cartesian coordinates of T, and the decomposition (1.2), its *Cartesian decomposition*. Observe that

$$\Re e\langle Tx, x\rangle = \langle T_1x, x\rangle, \qquad \Im m\langle Tx, x\rangle = \langle T_2x, x\rangle \tag{1.3}$$

for all x in H.

We also recall that a bounded operator T on H is *normal* if $\|T^*x\| = \|Tx\|$ for all x in H. If T is an operator, the set

$$W(T) = \{\langle Tx, x\rangle \mid x \in D(T), \|x\| = 1\},$$

which is a subset of the set $\mathbb{C}$ of complex numbers, is the *numerical range* of T. In recent literature much attention has been paid to topological and geometric propierties of the numerical range. It is known for example that $W(T)$ is a convex set [2], [3], [5], [11], [12], [13], that the *closure* of $W(T)$ contains the spectrum of T and, moreover, that if T is normal, it really is the closed convex hull of the spectrum (see [12]). Topological propieties of $W(T)$ are extremely important. For example, if T is normal and $W(T)$ is *closed*, the extreme points of $W(T)$ are eigenvalues of T (see [6]). Many of these basic results have been extended one way or another to more general classes of operators (hyponormal ([9],[10]), quasihyponormal ([1], [7], [8]) and the like). All this constitutes a very active field of research in operator theory. In this paper we give necessary and sufficient conditions for the interior of the numerical range of a bounded operator in Hilbert space to be empty. These conditions refer to the structure of the operator and to the Jacobian matrix of a certain C^1-function related to it. Sufficient conditions also involving that matrix and some other simple propierties of T are established for $W(T)$ to be open (see [4], problem 168).

2. Main Results

We first give necessary and sufficient conditions for the numerical range of a bounded operator on a Hilbert space to have an empty interior. Then, sufficient conditions for that set to be open will also be given. Some preliminary results will be needed. With $\mathbb{R}$ we denote the set of real numbers.

Lemma 2.1. *Let T be a symmetric operator on a pre-Hilbert space H. Then, for each pair x, y in $D(T)$, the map $f : \mathbb{R}^2 \to \mathbb{R}$ given by*

$$f(s,t) = \langle T(\tau x + (1-\tau)y), \tau x + (1-\tau)y\rangle, \quad \tau = s + it \tag{2.1}$$

is in C^1. Furthermore

$$\frac{\partial f}{\partial s}(0,0) = 2\Re e\langle T(x-y), y\rangle, \qquad \frac{\partial f}{\partial t}(0,0) = -2\Im m\langle T(x-y), y\rangle \tag{2.2}$$

Proof. From

$$\begin{aligned}\frac{1}{h}(f(s+h,t) - f(s,t)) =& \langle T(\tau x + (1-\tau)y), x - y\rangle \\ &+ \langle T(x-y), \tau x + (1-\tau)y\rangle + h\langle T(x-y), x-y\rangle\end{aligned}$$

and

$$\begin{aligned}\frac{1}{h}(f(s,t+h) - f(s,t)) =& \langle T(\tau x + (1-\tau)y), i(x - y)\rangle \\ &+ \langle T(i(x-y)), \tau x + (1-\tau)y\rangle + h\langle T(x-y), x-y\rangle\end{aligned}$$

it follows, letting $h \to 0$, that

$$\frac{\partial f}{\partial s}(s,t) = 2\Re e\langle T(x-y), \tau x + (1-\tau)y\rangle \tag{2.3}$$

and

$$\frac{\partial f}{\partial t}(s,t) = -2\Im m\langle T(x-y), \tau x + (1-\tau)y\rangle \tag{2.4}$$

which are continuous functions of τ. Relations (2.2) follow from (2.3) and (2.4) with $\tau = 0$. □

Lemma 2.2. *Let x, y be vectors in a pre-Hilbert space H and let $g : \mathbb{R}^2 \to \mathbb{R}$ be the map*

$$g(s,t) = \|\tau x + (1-\tau)y\|^2, \quad \tau = s + it. \tag{2.5}$$

Then g is in C^1 and

$$\begin{aligned}\frac{\partial g}{\partial s}(s,t) &= 2\Re e\langle x - y, \tau x + (1-\tau)y\rangle \\ \frac{\partial g}{\partial t}(s,t) &= -2\Im m\langle x - y, \tau x + (1-\tau)y\rangle.\end{aligned} \tag{2.6}$$

Proof. Let T be the identity operator in Lemma 2.1. □

Lemma 2.3. *Let T be a bounded operator on a pre-Hilbert space H and let x, y with $\|y\| = 1$ be linearly independent vectors in H. Denote with F the map of $\mathbb{R}^2$ into $\mathbb{R}^2$ given by*

$$F(s,t) = \left(\frac{f_1(s,t)}{g(s,t)}, \frac{f_2(s,t)}{g(s,t)} \right) \tag{2.7}$$

where

$$f_i(s,t) = \big\langle T_i(\tau x + (1-\tau)y), \tau x + (1-\tau)y \big\rangle, \quad i = 1, 2 \tag{2.8}$$

with T_1, T_2 as in (1.1) and g as in (2.5). Then F is in C^1, and the Jacobian matrix of F at $(0,0)$ is

$$J_T(x,y) = \begin{bmatrix} 2\Re e\big\langle (T_1 - \langle T_1 y, y\rangle)(x-y), y\big\rangle & -2\Im m\big\langle (T_1 - \langle T_1 y, y\rangle)(x-y), y\big\rangle \\ 2\Re e\big\langle (T_2 - \langle T_2 y, y\rangle)(x-y), y\big\rangle & -2\Im m\big\langle (T_2 - \langle T_2 y, y\rangle)(x-y), y\big\rangle \end{bmatrix} \tag{2.9}$$

Proof. This follows from T_1, T_2 being self adjoint (so that f_1, f_2 are C_1 of $\mathbb{R}^2$ into $\mathbb{R}$), from observing that $g(s,t)$, which is also C^1, never vanishes, and from relations (2.2) and (2.6). □

Lemma 2.4. *Let T, T_1 be commuting bounded operators on a Hilbert space H. Assume that T_1 is self-adjoint and there is $f : H \to \mathbb{C}$ such that*

$$Tx = f(x)T_1 x \tag{2.10}$$

for all $x \in H$. Then $T = \beta T_1$ for some β in $\mathbb{C}$.

Proof. If $T = 0$, let $\beta = 0$. Now assume there is $x_0 \in H$ such that $Tx_0 \neq 0$ and let $\beta = f(x_0)$. If

$$H_0 = \{x \in H \mid Tx = \beta T_1 x\} \tag{2.11}$$

then H_0 is a non trivial closed subspace of H. We claim that $H_0 = H$. Since $T(T_1 x) = T_1(Tx) = T_1(\beta T_1 x) = \beta T_1(T_1 x)$ for all $x \in H_0$, it follows that $T_1(H_0) \subseteq H_0$ and, T_1 being self-adjoint, also $T_1(H_0^\perp) \subseteq H_0^\perp$. Hence, from (2.10), $T(H_0) \subseteq H_0$ and $T(H_0^\perp) \subseteq H_0^\perp$.

Now let $x \in H_0, y \in H_0^\perp$. Since $T(x_0 + y) = f(x_0 + y)T_1(x_0 + y) = f(x_0 + y)T_1 x_0 + f(x_0 + y)T_1 y = Tx_0 + Ty$, we get

$$\big(f(x_0 + y) - \beta\big)T_1 x_0 = \big(f(y) - f(x_0 + y)\big)T_1 y \in H_0 \cap H_0^\perp;$$

and, since $T_1 x_0 \neq 0$, that $\beta = f(x_0 + y)$. Then

$$T(x+y) = T(x - x_0) + T(x_0 + y) = \beta T_1(x - x_0) + \beta T_1(x_0 + y) = \beta T_1(x + y),$$

which implies that $H = H_0 + H_0^\perp \subseteq H_0$, and completes the proof. □

Lemma 2.5. *Let T be a bounded operator on a Hilbert space H and let T_1, T_2 as in (1.1) be the self-adjoint operators in the Cartesian decomposition of T. Assume there is a function $f : H \to \mathbb{R}$ such that*

$$T_2 y = \langle T_2 y, y\rangle y + f(y)\big(T_1 y - \langle T_1 y, y\rangle y\big) \tag{2.12}$$

for each $y \in H$ with $\|y\| = 1$. Then, T is a normal operator on H.

Proof. Since $T = T_1 + iT_2$ and $T^* = T_1 - iT_2$ then

$$Ty = (1 + if(y))T_1 y + i\big(\langle T_2 y, y\rangle y - f(y)\langle T_1 y, y\rangle y\big)$$

and

$$T^* y = (1 - if(y))T_1 y - i\big(\langle T_2 y, y\rangle y - f(y)\langle T_1 y, y\rangle y\big)$$

for each y with $\|y\| = 1$. Let

$$\alpha(y) = 1 + if(y), \quad \beta(y) = \langle T_2 y, y\rangle - f(y)\langle T_1 y, y\rangle, \quad y \in H,$$

then

$$(1/\alpha(y))\, Ty = T_1 y + i\,(\beta(y)/\alpha(y))\, y$$

and

$$\left(1/\overline{\alpha(y)}\right) T^* y = T_1 y - i\left(\beta(y)/\overline{\alpha(y)}\right) y$$

whenever $\|y\| = 1$. But, T_1 being self-adjoint, we have $\|(T_1 + \alpha)y\| = \|(T_1 + \overline{\alpha})y\|$ for all $y \in H$ and $\alpha \in \mathbb{C}$. Thus

$$\big\|1/\alpha(y)Ty\big\| = \big\|(1/\overline{\alpha(y)})T^* y\big\|, \quad \|y\| = 1$$

which implies that $\|Ty\| = \|T^* y\|$ for all y in H. Hence, T is normal. □

Lemma 2.6. *Let H be a Hilbert space and let T, T_1 and T_2 be as in Lemma 2.5. Assume that the determinant $\big|J_T(x,y)\big|$ of the Jacobian matrix $J_T(x,y)$ vanishes for all linearly independent vectors x, y in H with $\|y\| = 1$. Then, T is a normal operator on H.*

Proof. The assumptions imply that $\big|J_T(x+y, y)\big| = 0$ whenever x, y are linearly independent and $\|y\| = 1$. On the other hand, (1.2) yields

$$\begin{aligned} \Re e\langle Tx, y\rangle &= \Re e\langle T_1 x, y\rangle - \Im m\langle T_2 x, y\rangle \\ \Im m\langle Tx, y\rangle &= \Im m\langle T_1 x, y\rangle + \Re e\langle T_2 x, y\rangle, \end{aligned} \tag{2.13}$$

so that

$$\big|\langle Tx, y\rangle\big|^2 = \big|\langle T_1 x, y\rangle\big|^2 + \big|\langle T_2 x, y\rangle\big|^2 + R_T(x,y) \tag{2.14}$$

where

$$R_T(x,y) = 2\big\{\Re e\langle T_2 x, y\rangle \Im m\langle T_1 x, y\rangle - \Re e\langle T_1 x, y\rangle \Im m\langle T_2 x, y\rangle\big\} \tag{2.15}$$

Since

$$(T - \langle Ty, y\rangle)x = (T_1 - \langle T_1 y, y\rangle)x + i(T_2 - \langle T_2 y, y\rangle)x, \tag{2.16}$$

(2.14), with $T - \langle Ty, y\rangle I$ in the place of T, gives

$$\begin{aligned}\left|\langle (T - \langle Ty, y\rangle)x, y\rangle\right|^2 = \left|\langle (T_1 - \langle T_1 y, y\rangle)x, y\rangle\right|^2 + \left|\langle (T_2 - \langle T_2 y, y\rangle)x, y\rangle\right|^2 \\ + \frac{1}{2}\left|J_T(x+y, y)\right|\end{aligned} \tag{2.17}$$

Hence, if x and y are linearly independent with $\|y\| = 1$, in which case $\left|J_T(x+y,y)\right| = 0$, we have that x is orthogonal to $(T^* - \overline{\langle Ty, y\rangle})y$ if and only if x is orthogonal to both $(T_1 - \langle T_1 y, y\rangle)y$ and $(T_2 - \langle T_2 y, y\rangle)y$, *i.e.*,

$$\left\{(T^* - \overline{\langle Ty, y\rangle})y\right\}^\perp = \left\{(T_1 - \langle T_1 y, y\rangle)y\right\}^\perp \cap \left\{(T_2 - \langle T_2 y, y\rangle)y\right\}^\perp \tag{2.18}$$

Since all three spaces in (2.18) are closed hyperplanes, this relationship is possible if and only if those spaces coincide, which ensures the existence of $f(y) \in \mathbb{C}$ such that

$$(T_2 - \langle T_2 y, y\rangle)y = f(y)(T_1 - \langle T_1 y, y\rangle)y, \quad \|y\| = 1. \tag{2.19}$$

We claim that $f(y)$ can be taken to be real. Indeed, let $y \in H$ with $\|y\| = 1$. If $(T_1 - \langle T_1 y, y\rangle)y = 0$, take $f(y) = 0$. If $(T_1 - \langle T_1 y, y\rangle)y \neq 0$, take x linearly independent of y and such that $\langle (T_1 - \langle T_1 y, y\rangle)y, x\rangle \neq 0$ (for instance, $x = (T_1 - \langle T_1 y, y\rangle)y$ will do). From $\left|J_T(x+y,y)\right| = 0$ it follows that for some $\alpha \in \mathbb{R}$,

$$\begin{aligned}\Big(\Re e\langle (T_2 - \langle T_2 y, y\rangle)x, y\rangle,\ \Im m\langle (T_2 - \langle T_2 y, y\rangle)x, y\rangle\Big) = \\ \alpha\Big(\Re e\langle (T_1 - \langle T_1 y, y\rangle)x, y\rangle,\ \Im m\langle (T_1 - \langle T_1 y, y\rangle)x, y\rangle\Big),\end{aligned}$$

so that

$$\langle (T_2 - \langle T_2 y, y\rangle)x, y\rangle = \alpha\langle (T_1 - \langle T_1 y, y\rangle)x, y\rangle,$$

which is the same as

$$\langle (T_2 - \langle T_2 y, y\rangle)y, x\rangle = \alpha\langle (T_1 - \langle T_1 y, y\rangle)y, x\rangle. \tag{2.20}$$

This, together with (2.19) and the assumption $\langle (T_1 - \langle T_1 y, y\rangle)y, x\rangle \neq 0$, ensures that $f(y) = \alpha$. The conclusion now follows from Lemma 2.5. ☐

Theorem 2.1. *Let T be a bounded operator on a Hilbert space H and let $T = T_1 + iT_2$ be its Cartesian decomposition. The following assertions are equivalent :*

(1) *There are α, β in $\mathbb{C}$ and a bounded self-adjoint operator B on H such that*

$$T = \alpha I + \beta B \tag{2.21}$$

(2) $\operatorname{Int}(W(T)) = \varnothing$.

(3) *For any pair of vectors x, y in H with $\|y\| = 1$,*

$$\big|J_T(x,y)\big| = 0 \tag{2.22}$$

(4) *For any pair of vectors x, y in H with $\|y\| = 1$,*

$$\begin{aligned} \big|\langle (T - \langle Ty, y\rangle)x, y\rangle\big|^2 &= \big|\langle (T_1 - \langle T_1 y, y\rangle)x, y\rangle\big|^2 \\ &\quad + \big|\langle (T_2 - \langle T_2 y, y\rangle)x, y\rangle\big|^2 . \end{aligned} \tag{2.23}$$

Proof. If (2.21) holds then $W(T) = \alpha + \beta W(B)$, so that $\operatorname{Int} W(T) = \alpha + \beta \operatorname{Int}(W(B)) = \alpha + \beta\varnothing = \varnothing$ (as $W(B) \subseteq \mathbb{R}$). Hence, (1) $\Longrightarrow$ (2).

To prove that (2) $\Longrightarrow$ (3), assume (2.22) does not hold for a couple of vectors x, y in H with $\|y\| = 1$. Then, x, y are linearly independent. Let $F : \mathbb{R}^2 \to \mathbb{R}^2$ be given by (2.7). Since $F(0,0) = (\langle T_1 y, y\rangle, \langle T_2 y, y\rangle) = \langle Ty, y\rangle$, the Inverse Function Theorem, guaranties the existence of open sets U, V of $\mathbb{R}^2$ with $(0,0) \in U$ and $F(0,0) \in V$ such that $F(U) = V$. Since $F(\mathbb{R}^2) \subseteq W(T)$, $\operatorname{Int}(W(T)) \neq \varnothing$.

To see that (3) $\Longrightarrow$ (4), just observe that (3) implies that $\big|J_T(x+y, y)\big| = 0$, and, from (2.17), this is equivalent to (2.23).

Now assume that (4) holds. Lemma 2.6 and (2.23), which imply (2.17) to hold, ensure that T is normal and that for each y with $\|y\| = 1$ the three vectors $T_2 y, Ty$ and $T^* y$ are in the subspace spanned by y and $T_1 y$. Thus, for each y in H with $\|y\| = 1$, there are $\alpha(y)$ and $\beta(y)$ in $\mathbb{C}$ such that

$$Ty = \alpha(y)y + \beta(y)T_1 y \tag{2.24}$$

Now, if y belongs to the subspace spanned by $T_1 y$, for all y with $\|y\| = 1$, (2.24) and Lemma 2.4 apply to give that $T = \alpha I$ for some α in $\mathbb{C}$. If on the contrary there is x_0 in H not belonging to the subspace spanned by Tx_0, and if $\alpha = \alpha(x_0), \beta = \beta(x_0)$ and

$$H_0 = \{x \in H \mid Tx = \alpha x + \beta T_1 x\},$$

then H_0 is a non trivial closed subspace of H which is readily seen to be, as well as $H_0^\perp$, invariant under T_1, T_2, T, and T^*. Since $H = H_0 + H_0^\perp$, this implies, exactly as in Lemma 2.4, that $H_0 = H$. Hence $T = \alpha I + \beta T_1$ and, since T_1 is self-adjoint, this shows that (4) $\Longrightarrow$ (1) and completes the proof of the theorem. □

Corollary 2.1. *Let T be a bounded operator on the Hilbert space H and $T = T_1 + iT_2$ be its Cartesian decomposition. Each of the following conditions is sufficient for $W(T)$ to be open:*

(1) *For each y in H with $\|y\| = 1$, there is $x \in H$ such that*

$$\left|J_T(x+y,y)\right| \neq 0. \tag{2.25}$$

(2) *For each y in H with $\|y\| = 1$, there is $x \in H$ such that*

$$\left|\langle (T - \langle Ty,y\rangle)x, y\rangle\right|^2 \neq \left|\langle (T_1 - \langle T_1y,y\rangle)x, y\rangle\right|^2 + \left|\langle (T_2 - \langle T_2y,y\rangle)x, y\rangle\right|^2 \tag{2.26}$$

(3) *For each y in H with $\|y\| = 1$, there is $\alpha \in \mathbb{C}$ such that*

$$\|(T-\alpha)y\| \neq \|(T-\alpha)^*y\| \tag{2.27}$$

(4) *For all $y \in H$ with $\|y\| = 1$, the vectors y, Ty and T^*y are linearly independent.*

Proof. Since $\left|J_T(x+y,y)\right| = \left|J_T(x,y)\right|$, it is clear, from (2.17), that (1) $\iff$ (2). Now, if (2.25) holds, x and y are linearly independent, and arguing as in the proof of (2) $\Longrightarrow$ (3) in Theorem 2.1, we conclude that $\langle Ty,y\rangle$ is interior to $W(T)$. Now let $\|y\| = 1$ and assume $\left|J_T(x,y)\right| = 0$ for all x linearly independent of y. The argument in the proof of Lemma 2.6 shows, on the one hand, that y, Ty and T^*y are linearly dependent, and, on the other hand, that $T_2y = \langle T_2y,y\rangle y + \beta\big(T_1y - \langle T_1y,y\rangle y\big)$ for some β in $\mathbb{R}$. Then, as in the proof of Lemma 2.5, we can show that $\|(T-\alpha)y\| = \|(T-\alpha)^*y\|$ for all $\alpha \in \mathbb{C}$. Hence, if (3) or (4) holds, also (1) must hold, and the proof is complete. □

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