

# About the generalized Hopf bifurcations at infinity for planar cubic systems

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**ABSTRACT.** In this paper we prove the existence of four infinitesimal limit cycles bifurcating at infinity for a class of planar cubic systems. Necessary and sufficient conditions to obtain a center for that class are established.

*Key words and phrases.* Limit cycles, cubic systems, center conditions.

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## 1. Introduction

In this work we study some planar cubic polynomial systems whose linear part is null, its quadratic part is in Bautin normal form, and its cubic part is a rotation (see Bautin [1954], Christopher and Lloyd [1990], James and Lloyd [1991]). These systems can be written as follows:

$$\begin{cases} \dot{x} = -\ell_3 x^2 + (2\ell_2 + \ell_5)xy + \ell_6 y^2 + (x^2 + y^2)(-y + \epsilon x) \\ \dot{y} = \ell_2 x^2 + (2\ell_3 + \ell_4)xy - \ell_2 y^2 + (x^2 + y^2)(x + \epsilon y) \end{cases} \quad (1)$$

Denote with  $X_{\theta, \epsilon}(x, y) = (P_{\theta, \epsilon}(x, y), Q_{\theta, \epsilon}(x, y))$  the vector field associated to system (1), where

$$\theta = (\ell_2, \ell_3, \ell_4, \ell_5, \ell_6) \in \mathbb{R}^5 \text{ and } 0 \leq |\epsilon| \ll 1$$

These vector fields have a center-focus in a neighbourhood of infinity in the Poincaré sphere  $S^1$  because their rotational component

$$\langle (-y, x), (x^2 + y^2)(-y + \epsilon x, x + \epsilon y) \rangle = (x^2 + y^2)^2$$

is always positive.

Let  $\pi(r) = \sum_{k=1}^{\infty} \alpha_k(2\pi) r^k$  be the Poincaré map of equations (1), defined on a transversal section  $\Sigma$  at a point  $q$  of the equator  $S^1$ . The  $\alpha_k(2\pi)$  are polynomial expressions in the coefficients of  $X_{\theta, \epsilon}(x, y)$ .

The equator is a fine focus of order  $k \geq 1$  provided  $\alpha_1(2\pi) = 1, \alpha_n(2\pi) = 0, 1 < n < 2k + 1$  and  $\alpha_{2k+1}(2\pi) \neq 0$ .

In theorem *A* we prove that for  $\epsilon = 0$  there exist three algebraic varieties in the parameter space  $\mathbb{R}^5$  such that when the parameters belong to them, the vector fields in a neighbourhood of infinity have a center.

In Theorem *B* we give algebraic conditions on the parameter space for the different orders of weakness that appear at infinity, and it is proved that the maximal weakness is four. Furthermore, in the proof of this theorem we describe a method to construct a subfamily of equations (1) with exactly four limit cycles bifurcating at infinity.

## 2. Main results

Consider the following subset of parameter space  $\mathbb{R}^5$ .

$$\Sigma_1^{\text{sig}(\eta)} = \{\theta \mid \ell_6 - \ell_3 = \text{sig}(\eta)\}, \quad \eta \in \{0, 1, -1\}$$

$$\Sigma_2^{\text{sig}(\eta)} = \{\theta \mid \ell_3 + \ell_4 - \ell_6 = \text{sig}(\eta)\}$$

$$\Sigma_3^{\text{sig}(\eta)} = \{\theta \mid 2\ell_3 + \ell_4 - 2\ell_6 = \text{sig}(\eta)\}$$

$$\Sigma_4^{\text{sig}(\eta)} = \{\theta \mid 32\ell_2^2 - \ell_3^2 + 18\ell_3\ell_2 + 15\ell_6^2 = \text{sig}(\eta)\}$$

$$\Sigma_5^{\text{sig}(\eta)} = \{\theta \mid \ell_2\ell_4^3 + 3\ell_3\ell_4^2\ell_5 + \ell_4^3\ell_5 - 3\ell_2\ell_4\ell_5^2 - \ell_3\ell_5^3 - \ell_4\ell_5^3 = \text{sig}(\eta)\}$$

**Theorem A.** *The vector field  $X_{\theta, 0}$  has a center in a neighbourhood of infinity if and only if*

$$\theta \in (\Sigma_1^0 \cap \Sigma_5^0) \cup \{\theta \mid \ell_5 = 0 \wedge (\ell_2 = 0 \vee \ell_4 = 0)\}.$$

**Theorem B.** *The equator of the Poincaré sphere of the vector field  $X_{\theta, 0}$  is a degree  $k$  weak periodic orbit if and only if:*

$$(1) \quad k = 1 \quad \wedge \quad \theta \in \Sigma_1^{\pm} \cap \{\theta \mid \ell_5 \neq 0\},$$

- (2)  $k = 2 \wedge \theta \in (\Sigma_1^\pm \cap \{\theta \mid \ell_2 \ell_4 \neq 0 \wedge \ell_5 = 0\}) \cup \Sigma_1^0 \cap \Sigma_5^\pm$ ,  
 (3)  $k = 3 \wedge \theta \in \{\theta \mid \ell_5 = 0\} \cap ([\Sigma_1^\pm \cap \Sigma_2^0 \cap \{\theta \mid \ell_2 \neq 0\}] \cup \Sigma_3^0 \cap (\Sigma_1^0 \cup \{\theta \mid \ell_2 = 0\}) \cap \Sigma_4^\pm)$ ,  
 (4)  $k = 4 \wedge \theta \in \Sigma_1^\pm \cap \Sigma_3^0 \cap \Sigma_4^0 \cap \{\theta \mid \ell_5 = 0 \wedge \ell_2 \neq 0\}$ .

**Corollary.** The vector field  $X_{\theta,\epsilon}$  has at most one configuration with four infinitesimal limit cycles bifurcating at infinity by generalized Hopf bifurcations.

### 3. Proof sketches of the results

*Proof of Theorem A.* Let  $\theta \in (\Sigma_1^0 \cap \Sigma_5^0) \cup \{\theta \mid \ell_5 = 0 \wedge (\ell_2 = 0 \vee \ell_4 = 0)\}$ .

If  $\theta \in \{\theta \mid \ell_5 = 0 \wedge (\ell_2 = 0 \vee \ell_4 = 0)\}$ , then:

- (a) If  $\ell_5 = \ell_2 = 0$ , so that  $P_{\theta,0}(-x, y) = P_{\theta,0}(x, y)$  and  $Q_{\theta,0}(-x, y) = -Q_{\theta,0}(x, y)$ , then the vector field  $X_{\theta,0}$  has a center in a neighbourhood of infinity by the symmetry principle (see Yan-Quian [1984]).  
 (b) If  $\ell_5 = \ell_4 = 0$ , so that  $\text{div}(X_{\theta,0}(x, y)) = \ell_4 x + \ell_5 y \equiv 0$ , then  $X_{\theta,0}$  is a Hamiltonian vector field with a center at infinity. This, because there are no singular points in the equator of the Poincaré sphere.  
 (c) If  $\theta \in \Sigma_1^0 \cap \Sigma_5^0$ , let  $\varphi : \mathbb{R}^2 \rightarrow \mathbb{R}^2$  be such that  $\varphi(u, v) = (hu - vk, ku + hv)$  with  $h^2 + k^2 = 1$ . The vector field  $X_{\theta,0}$  is  $\varphi$ -invariant, since

$$\begin{aligned} \varphi_* X_{\theta,0}(u, v) &= (A_{20}u^2 + A_{11}uv - A_{20}v^2, B_{20}u^2 + B_{11}uv - B_{20}v^2) \\ &\quad + (u^2 + v^2)(-v, u), \end{aligned}$$

where

$$\begin{aligned} A_{20}(h, k) &= -\ell_3 h^2 + (3\ell_2 + \ell_5)h^2 k + (3\ell_3 + \ell_4)hk^2 - \ell_2 k^3, \\ A_{11}(h, k) &= (2\ell_2 + \ell_5)h^3 + (\ell_4 + 6\ell_3)h^2 k - (\ell_5 + 6\ell_2)hk^2 - (\ell_4 + 2\ell_3)k^3, \\ B_{20}(h, k) &= \ell_2 h^3 + (3\ell_3 + \ell_4)h^2 k - (3\ell_2 + \ell_5)hk^2 - \ell_3 k^3, \\ B_{11}(h, k) &= (2\ell_3 + \ell_4)h^3 - (\ell_5 + 6\ell_2)h^2 k - (\ell_4 + 6\ell_3)hk^2 + (2\ell_2 + \ell_5)k^3. \end{aligned}$$

The vector field  $X_{\theta,0}$  with  $\theta \in \Sigma_1^0 \cap \Sigma_5^0$  is symmetric with respect to a straight line through the origin if and only if  $\varphi_* X_{\theta,0}$  is symmetric about the  $u$ -axis, i.e., if and only if the simultaneous systems of equations  $A_{11}(h, k) = 0$  and  $B_{20}(h, k) = 0$  has at least one solution. The terms  $A_{11}(h, k)$  and  $B_{20}(h, k)$  are homogeneous polynomials of degree 3 which vanish over straight lines through the origin in the  $(h, k)$  plane.

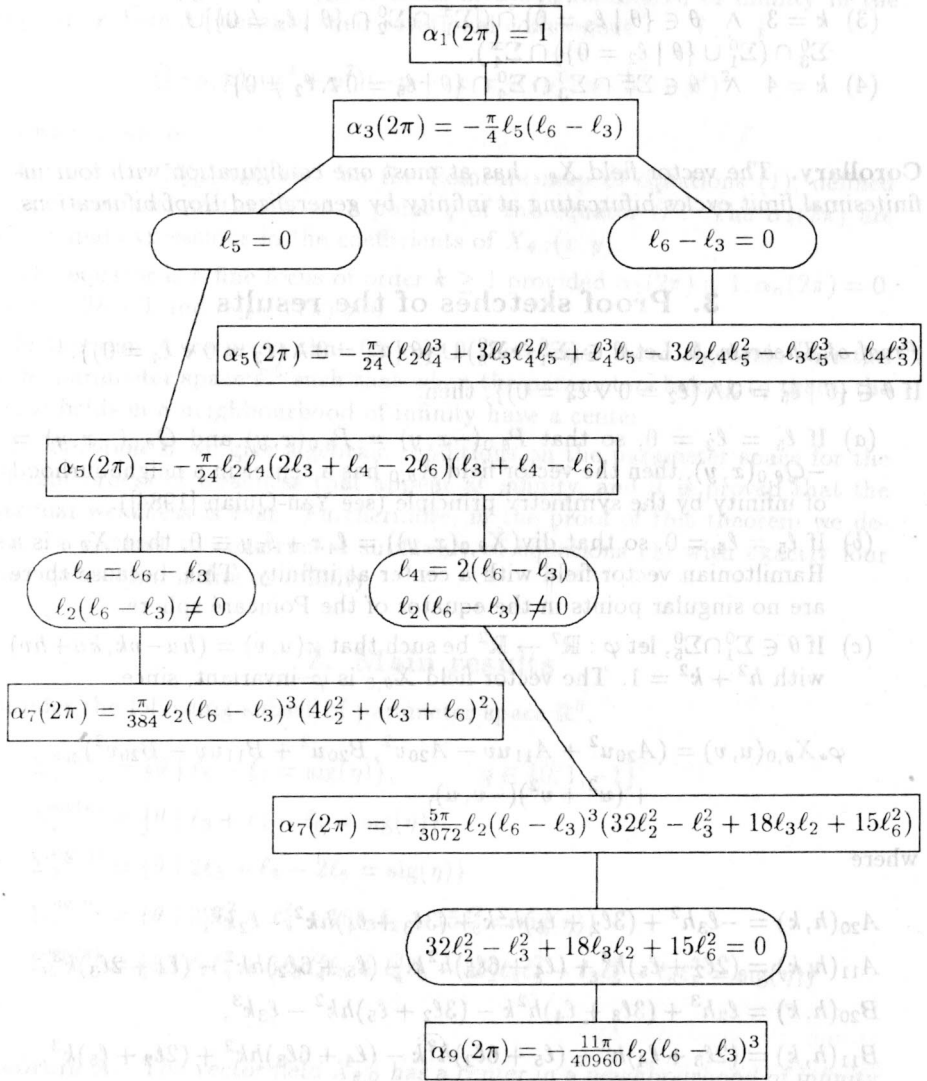


Figure 1

Let  $h = mk$ ,  $m \in \mathbb{R}$ , and let the polynomials  $p(m), q(m)$  be such that

$$A_{11}(mk, k) = k^3 p(m) \text{ and } B_{20}(mk, k) = k^3 q(m).$$

The resultant  $R(p, q)$  is

$$R(p, q) = (\ell_2 \ell_4^3 + 3\ell_3 \ell_4 \ell_5 + \ell_4^3 \ell_5 - 3\ell_2 \ell_4 \ell_5^2 - \ell_3 \ell_5^3 - \ell_4 \ell_5^3) \cdot r(\theta)$$

where  $r(\theta)$  is a polynomial with integral coefficients. Then  $R(p, q) = 0$  if  $\theta \in \Sigma_1^0 \cap \Sigma_5^0$ .

*Proof of theorem B.* We have used the MATHEMATICA computational symbolic system to evaluate the derivatives of the Poincaré map at infinity. Figure 1 displays our results inside boxes and the parameter restrictions within parenthesis.

The four statements of the theorem are straightforward conclusions from Figure 1.

*Proof of the Corollary.* From theorem B(4) we know that the circle at infinity is a degree four weak periodic orbit. Now we choose a sign for  $\alpha_9(2\pi)$  (negative or positive) and determine a perturbation of the coefficients of  $X_{\theta,0}$  in such a way that  $\alpha_7(2\pi) \cdot \alpha_9(2\pi) < 0$  for the new vector field. This implies, by classical results on Hopf bifurcations, that the resulting vector field has a unique infinitesimal limit cycle generated from the circle at infinity. Applying successively this procedure, changing the type of the stability in each step, we get the corollary.

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