Revista Colombiana de Matemáticas
Volumen **29** *(1995), páginas 119–126*

An improved estimate on the degree of approximation to functions of bounded variation by certain operators

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ABSTRACT. In [5] Guo introduced a modification $\hat{M}_n(f,x)$ of the Meyer-König-Zeller operators $M_n(f,x)$ and gave an estimate on the rate of convergence to f of the modified operators when f is a function of bounded variation. In the present paper we establish a sharp estimate on the operators $\hat{M}_n(f,x)$ using some results in probability theory.

Keywords and phrases. Approximation operators, rate of convergence of approximation operators, functions of bounded variation, random variables, probability distributions, moments and means, Bernstein polynomials.

1991 Mathematics Subject Classification. Primary 41A30. Secondary 41A36.

1. Introduction

For the Meyer-König-Zeller operators (see [3]) one has

$$M_n(f,x) = \sum_{k=0}^{\infty} p_{n,k}(x) f\left(\frac{k}{n+k}\right), \quad p_{n,k}(x) = \binom{n+k-1}{k} x^k (1-x)^n.$$

* Research supported by the Council of Scientific and Insdustrial Research, India.

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Recently, Guo [5] introduced an integral modification of these operators by

$$\hat{M}_n(f,x) = \sum_{k=1}^{\infty} p_{n,k+1}(x) \frac{(n+k-2)(n+k-3)}{(n-2)} \int_0^1 p_{n-2,k-1}(t) f(t)\, dt. \quad (1.1)$$

Cheng [2] studied the rate of convergence of $M_n(f,x)$ to $f(x)$ when f is a function of bounded variation. Guo ([4], [5]) estimated the rate of convergence of Durrmeyer operators and integrated Meyer-König-Zeller operators $M_n(f,x)$ using a probabilistic approach.

Guo [5] has also given an estimate for the rate of convergence to f of $\hat{M}_n(f,x)$ when f is function of bounded variation, as below

Theorem 1.1. *Let f be a function of bounded variation of [0, 1], $x \in (0,1)$. Then, for n sufficiently large,*

$$|\hat{M}_n(f,x) - \frac{1}{2}\{f(x+) + f(x-)\}| \le \frac{7}{nx} \sum_{k=1}^{n} V_{x-x/\sqrt{k}}^{x+(1-x)/\sqrt{k}}(g_x) + \frac{50}{\sqrt{n}\, x^{3/2}} |f(x+) - f(x-)|,$$

where

$$g_x(t) = \begin{cases} f(t) - f(x+), & x < t \le 1 \\ 0, & t = x \\ f(t) - f(x-), & 0 \le t < x, \end{cases}$$

and $V_a^b(g_x)$ is the total variation of g_x on $[a,b]$.

In the present paper we establish a sharper estimate than the above by using some results in probability theory.

2. Auxiliary results

The following results wil be needed to prove our main theorem.

Lemma 2.1. [1, p. 104 & 110]. *(Berry-Esseen Theorem). Let $X_1, X_2, \cdots, X_n$ be n independent and identically distributed (i. i. d) random variables with zero mean and a finite absolute third moment. If $\rho_2 \equiv E(X_1^2) > 0$, then*

$$\sup_{x \in R} |F_n(x) - \varphi(x)| \le (0.409)\ell_{3,n},$$

where F_n is the distribution function of $(X_1 + X_2 + \cdots + X_n)n\rho_2$, φ is the standard normal distribution function and $\ell_{3,n}$ is the Liapounov ratio given by

$$\ell_{3,n} = \left(\frac{\rho_3}{\rho_2^{3/2}}\right) n^{-1/2} \quad , \quad \rho_3 = E(|X_1|^3).$$

Lemma 2.2 [6]. *If $\{\xi_i\}(i=1,2,\cdots)$ are independent random variables with the same geometric distribution*

$$p(\xi_i = k) = x^k(1-x)\ ,\ x \in (0,1)\ ,\ i = 1,2,\dots\ ,$$

then

$$E(\xi_i) = \frac{x}{1-x}\ ,\ \rho_2 = E(\xi_i - E(\xi_i))^2 = \frac{x}{(1-x)^2}$$

and $\eta_n = \sum_{i=1}^n \xi_i$ is a random variable with distribution

$$P(\eta_n = k) = \binom{n+k-1}{k} x^k(1-x)^n. \tag{2.1}$$

Lemma 2.3. *If y is a positive valued random variable with a non-degenerate probability distribution then (by Schwarz's inequality)*

$$E(y^3) \le (E(y^2).E(y^4))^{1/2},$$

provided $E(y^2), E(y^3)$ and $E(y^4) < \infty$.

Lemma 2.4. *For $k \in N$ and $x \in (0,1)$,*

$$p_{n,k}(x) \le 19/(6\sqrt{n}\ x^{3/2}), \quad n = 1,2,\dots\ . \tag{2.2}$$

Proof. By (2.1) we have

$$\begin{aligned} p_{n,k}(x) &= P(\eta_n = k) = P(k-1 < \eta_n \le k) \\ &= P\left(\frac{k-1-nx/(1-x)}{\sqrt{nx}/(1-x)} < \frac{\eta_n - nx/(1-x)}{\sqrt{nx}/(1-x)} \le \frac{k-nx/(1-x)}{\sqrt{nx}/(1-x)}\right), \end{aligned}$$

and using Lemma 2.1 we obtain

$$\left| P(\eta_n = k) - \frac{1}{\sqrt{2\pi}} \int_{\frac{k-1-nx/(1-x)}{\sqrt{nx}/(1-x)}}^{\frac{k-nx/(1-x)}{\sqrt{nx}/(1-x)}} e^{-t^2/2} dt \right| < \frac{2(0.409)\rho_3}{\sqrt{n}\ \rho_2^{3/2}}. \tag{2.3}$$

Next, we estimate ρ_3. By an easy computation we have

$$\begin{cases} T_0(x) &= \sum_{k=0}^\infty x^k(1-x) = 1\ , \\ T_1(x) &= \sum_{k=0}^\infty kx^k(1-x) = \frac{x}{1-x}, \\ T_2(x) &= \sum_{k=0}^\infty k^2x^k(1-x) = \frac{x(1+x)}{(1-x)^2}, \\ T_3(x) &= \sum_{k=0}^\infty k^3x^k(1-x) = \frac{x^3+4x^2+x}{(1-x)^3}, \\ T_4(x) &= \sum_{k=0}^\infty k^4x^k(1-x) = \frac{x^4+11x^3+11x^2+x}{(1-x)^4}. \end{cases} \tag{2.4}$$

Also, if $M_r(x)(\equiv \sum_{k=0}^{\infty}(k-\frac{x}{1-x})^r x^k(1-x))$ stands for the central moment of order r about the mean $\frac{x}{1-x}$, it is easily checked, by using (2.4), that

$$M_2(x)=\sum_{j=0}^{2}\tbinom{2}{j}T_{2-j}(x)(T_1(x))^j(-l)^j=\frac{x}{(1-x)^2}$$

and that

$$M_4(x)=\sum_{j=0}^{4}\tbinom{4}{j}T_{4-j}(x)(T_1(x))^j(-l)^j=\frac{x^3+7x^2+x}{(1-x)^4}.$$

Thus, in view of Lemma 2.3, we have

$$\rho_3\le (M_2(x)\cdot M_4(x))^{1/2}=\left(\frac{x}{(1-x)^2}\cdot\frac{x^3+7x^2+x}{(1-x)^4}\right)^{1/2}$$
$$\le\frac{3}{(1-x)^3},\quad x\in(0,1).$$

So, the right hand side of (2.3) is less than $8/(3\sqrt{n}\,x^{3/2})$, and the absolute value of the second term on the left hand side of (2.3) is less than $(1-x)/\sqrt{2\pi n x}$, i.e., than $1/(2\sqrt{n}\,x^{3/2})$ $\big($as $1/\sqrt{2\pi}<1/2$ and $x\in(0,1)\big)$.

Hence

$$p_{n,k}(x)=P(\eta_n=k)\le\frac{19}{6\sqrt{n}\,x^{3/2}}.$$

Lemma 2.5 [5]. *For $k\ge 1, x\in(0,1)$ and $n>2$, we have*

$$\sum_{j=0}^{k-1}p_{n-1,j}(x)=\frac{(n+k-2)(n+k-3)}{n-2}\int_x^1 p_{n-2,k-1}(t)dt.$$

Lemma 2.6. *For $k\ge 1, x\in(0,1)$ and $n>2$, we have*

$$\left|\sum_{j=2}^{k+1}p_{n,j}(x)-\sum_{j=1}^{k-1}p_{n-1,j}(x)\right|\le 19/(6\sqrt{n}\,x^{3/2}).$$

Proof. Since

$$\sum_{j=0}^{k+1}p_{n,j}(x)=P(\eta_n\le k+1)\quad\text{and}\quad\sum_{j=0}^{k-1}p_{n-1,j}(x)=P(\eta_{n-1}\le k-1),$$

Lemma 2.1 gives

$$\left|\sum_{j=0}^{k+1} p_{n,j}(x) - \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\frac{k+1-nx/(1-x)}{\sqrt{nx/(1-x)}}} e^{-t^2/2} dt\right| < \frac{4}{3\sqrt{n}\ x^{3/2}}$$

$$\left|\sum_{j=0}^{k-1} p_{n-1,j}(x) - \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\frac{k-1-(n-1)x/(1-x)}{\sqrt{(n-1)x/(1-x)}}} e^{-t^2/2} dt\right| < \frac{4}{3\sqrt{(n-1)}x^{3/2}}.$$

On the other hand,

$$p_{n,0}(x) + p_{n,1}(x) = o\left(\frac{1}{n}\right).$$

Hence, for n sufficiently large we have

$$\left|\sum_{j=2}^{k+1} p_{n,j}(x) - \sum_{j=0}^{k-1} p_{n-1,j}(x)\right| < \frac{8}{3\sqrt{n}\ x^{3/2}} + \frac{1}{\sqrt{2\pi}} \int_{I_{n,k}} e^{-t^2/2} dt$$

$$\leq \frac{19}{6\sqrt{n}\ x^{3/2}}$$

where $I_{n,k} = \left[\frac{k-1-nx/(1-x)}{\sqrt{nx/(1-x)}}, \frac{k-nx/(1-x)}{\sqrt{nx/(1-x)}}\right]$, and the lemma is proved.

Lemma 2.7 [5]. *For all $x \in (0,1)$ and n sufficiently large we have*

$$\hat{M}_n((t-x)^2, x) = \frac{2x(1-x)^2}{n-2} + o\left(\frac{1}{n^2}\right),$$

$$\hat{M}_n((t-x)^4, x) \leq \frac{8}{n^2}.$$

Lemma 2.8. *Let $K_n(x,t) = \sum_{k=1}^{\infty} \frac{(n+k-2)(n+k-3)}{n-2} p_{n,k+1}(x) p_{n-2,k-1}(x)$. If n is sufficiently large, then:*

(i) *For $0 \leq y < x$,*

$$\int_0^y K_n(x,t)dt \leq \frac{3x(1-x)^2}{n(x-y)^2}.$$

(ii) *For $x < z \leq 1$,*

$$\int_y^1 K_n(x,t)dt \leq \frac{3x(1-x)^2}{n(y-x)^2}.$$

The proof of this lemma easily follows from Lemma 2.7.

3. Main result

In this section, we state and prove the following theorem:

Theorem 3.1. *Let f be a function of bounded variation on $[0,1], x \in (0,1)$. Then, for sufficiently large n, we have*

$$\left|\hat{M}_n(f,x) - \frac{1}{2}\{f(x+) + f(n-)\}\right| \le \frac{7}{nx}\sum_{k=1}^{n} V_{x-x/\sqrt{k}}^{x+x/\sqrt{k}}(g_x) + \frac{4.75}{\sqrt{n}\, x^{3/2}}|f(x+) - f(x-)|,$$

where $g_x(t)$ and $V_a^b(g_x)$ are defined as in Theorem 1.1.

Proof. First

$$\left|\hat{M}_n(f,x) - \frac{1}{2}\{f(x+) + f(x-)\}\right| \le \hat{M}_n(g_x, x) + \frac{1}{2}|f(x+) - f(x-)| \cdot |\hat{M}_n(\text{ Sign}(t-x), x)|$$

Thus, to estimate the left hand side we need estimates for $\hat{M}_n(g_x, x)$ and $\hat{M}_n(\text{ Sign}(t-x), x)$.

To estimate $\hat{M}_n(\text{ Sign}(t-x), x)$, we first decompose it into two parts as follows:

$$\hat{M}_n(\text{ Sign}(t-x), x) = \int_0^1 \text{Sign}(t-x) K_n(x,t)dt$$

$$= \int_x^1 K_n(x,t)dt - \int_0^x K_n(x,t)dt = A_n(x) - B_n(x), \text{ say}.$$

Making use of Lemma 2.5, we have

$$\begin{aligned} A_n(x) &= \int_x^1 K_n(x,t)dt \\ &= \sum_{k=1}^{\infty} p_{n,k+1}(x)\frac{(n+k-2)(n+k-3)}{n-2}\int_x^1 p_{n-2,k-1}(t)dt \\ &= \sum_{k=1}^{\infty}\left(p_{n,k+1}(x)\sum_{j=0}^{k-1} p_{n-1,j}(x)\right). \end{aligned}$$

Now, from Lemma 2.6, it follows that

$$\left|A_n(x) - \sum_{k=1}^{\infty}\left(p_{n,k+1}(x)\sum_{j=2}^{k+1} p_{n,j}(x)\right)\right| \le 19/(6\sqrt{n}\, x^{3/2}).$$

Let $S = \sum_{k=2}^{\infty} \left(p_{n,k}(x) \sum_{j=2}^{k} p_{n,j}(x)\right)$

$$= p_{n,2}p_{n,2} + p_{n,3}(p_{n,2} + p_{n,3}) + \cdots + p_{n,m}(p_{n,2} + p_{n,3} + \cdots + p_{n,m}) + \cdots$$

Since

$$\begin{aligned}(p_{n,2}+&p_{n,3} + \cdots + p_{n,m} + \cdots)(p_{n,2} + p_{n,3} + \cdots + p_{n,m} + \cdots)\\ &=p_{n,2}(p_{n,2} + \cdots + p_{n,m} + \cdots) + \cdots + p_{n,m}(p_{n,2} + \cdots + p_{n,m} + \cdots) + \cdots\\ &=1 + o\left(\frac{1}{n}\right),\end{aligned}$$

then

$$\begin{aligned}p_{n,2}(p_{n,3} + p_{n,4} + \cdots) &+ p_{n,3}(p_{n,4} + p_{n,5} + \cdots) + \cdots\\ &+ p_{n,m}(p_{n,m+1} + p_{n,m+2} + \cdots) + \cdots\\ &= p_{n,3}p_{n,2} + p_{n,4}(p_{n,2} + p_{n,3})\\ &\quad + p_{n,5}(p_{n,2} + p_{n,2} + p_{n,3} + p_{n,4}) + \cdots\\ &= 1 - S + o\left(\frac{1}{n}\right),\end{aligned}$$

Now, we can easily obtain that

$$p_{n,2}^2 + p_{n,3}^2 + \cdots + p_{n,m}^2 + \cdots = 2S - 1 + o\left(\frac{1}{n}\right),$$

and using Lemma 2.4, that

$$\left|S - \frac{1}{2}\right| \leq \frac{19}{12\sqrt{n}\, x^{3/2}} \sum_{k=2}^{\infty} p_{n,k}(x) + o\left(\frac{1}{n}\right) \leq \frac{19}{12\sqrt{n}\, x^{3/2}}.$$

Hence

$$\left|A_n(x) - \frac{1}{2}\right| \leq \frac{19}{4\sqrt{n}\, x^{3/2}}.$$

On the other hand,

$$A_n(x) + B_n(x) = \int_0^1 K_n(x,t)dt = 1 + o\left(\frac{1}{n}\right)$$

Therefore,

$$|A_n(x) - B_n(x)| = \left|2A_n(x) - 1 + o\left(\frac{1}{n}\right)\right| \leq \frac{19}{2\sqrt{n}\, x^{3/2}}. \tag{3.2}$$

Since the estimate of $\hat{M}_n(g_x, x)$ is exactly the same as given in [5], we thus have, by using Lemma 2.8, that

$$|\hat{M}_n(g_x, x)| \le \frac{7}{nx} \sum_{k=1}^{n} V_{x-x/\sqrt{k}}^{x+(1-x)/\sqrt{k}}(g_x). \tag{3.3}$$

Combining the estimates (3.1)-(3.3), the theorem follows.

Remark. It may be shown paralleling [5] that our Theorem 3.1 is essentially the best; i.e., it can not be asymptotically improved.

References

1. R. N. Bhattacharya and R. R. Rao, *Normal Approximation and Asymptotic Expansions*, Wiley, New York, 1976.
2. F. Cheng, *On the rate of convergence of Bernstein polynomials of functions of bounded variation*, J. Approx. Theory **39** (1983), 259–274.
3. R. A. DeVore, *The Approximation of Continuous Functions by Positive Linear Operators*, Lecture Notes in Math, Springer, New York, 1972.
4. S. Guo, *On the rate of convergence of the Durrmeyer operator for function of bounded variation*, J. Approx. Theory **51** (1987), 183–192.
5. S. Guo, *Degree of approximation to function of bounded variation by certain operators*, Approx. Theory and its Appl. **4:2** (1988), 9–18.
6. Z. Wang, *Fundamental Theory of Probability and its Applications*, (in chinese), Beijing, 1979.

(Recibido en febrero de 1995)

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