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THE DISTRIBUTIONAL MULTIPLICATIVE PRODUCT $P_{\pm}^{-\frac{s}{2}}.\delta(x)$

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ABSTRACT. In this note we obtain the following distributional multiplicative product of distributions:

$$P_{+}^{-\frac{s}{2}}.\delta(x)=\begin{cases}0 & \text{if } s=1,3,5,\dots,\\ \dfrac{1}{2.2^{\frac{s}{2}}(\frac{s}{2})!n(n+1)\cdots(n+s-2)}\,L^{\frac{s}{2}}\delta & \text{if } s=2,4,6,\dots.\end{cases}$$

and

$$P_{-}^{-\frac{s}{2}}.\delta(x)=\begin{cases}0 & \text{if } s=1,3,5,\dots,\\ \dfrac{1}{2.2^{\frac{s}{2}}(\frac{s}{2})!n(n+1)\cdots(n+s-2)}\,L^{\frac{s}{2}}\delta & \text{if } s=2,4,6,\dots.\end{cases}$$

(cf. formulae (II,17) and (II,18)).

Here $P_{\pm}^{\lambda}$ are defined by (II,5) and (II,6), $\delta(x)$ is the Dirac measure and L^k is the ultrahyperbolic operator iterated k-times (k integer ≥ 1) defined by (I,4).

Our result generalizes the product due to Cheng Lin Zhi and Li Chen Kuan (cf. [7], p. 348, Theorem 4).

§1. INTRODUCTION

Let $x=(x_1,x_2,\dots,x_n)$ be a point of the n-dimensional Euclidean space R^n. Consider a non-degenerate quadratic form in n variables of the form

$$P=P(x)=x_1^2+\cdots+x_\mu^2-x_{\mu+1}^2-\cdots-x_{\mu+\nu},\tag{I,1}$$

where $n=\mu+\nu$. The distribution $(P\pm i_0)^\lambda$ is defined by

$$(P\pm i_0)^\lambda=\lim_{\varepsilon\to 0}\{P\pm i\varepsilon|x|^2\}^\lambda,\tag{I,2}$$

where $\varepsilon>0, |x|^2=x_1^2+\cdots+x_n^2,\quad \lambda\in C$ (cf. [4], p. 274).

The distributions $(P\pm i_0)^\lambda$ are analytic in λ everywhere except at $\lambda=-\frac{n}{2}-k,\quad k=0,1,\dots,$ where they have simple poles (cf. [4], p. 275), with residues given by (cf. [4], p.276)

$$Res_{\lambda=-\frac{n}{2}-k}(P \pm io)^{\lambda} = a_{k,n} L^k\{\delta(x)\}, \tag{I,3}$$

where

$$L^k = \left\{\frac{\partial^2}{\partial x_1^2} + \cdots + \frac{\partial^2}{\partial x_\mu^2} - \frac{\partial^2}{\partial x_{\mu+1}^2} - \cdots - \frac{\partial^2}{\partial x_{\mu+\nu}^2}\right\}^k \tag{I,4}$$

is the ultrahyperbolic operator, iterated k-times (k integer ≥ 1), $\delta(x)$ is the Dirac measure,

$$a_{k,n} = \frac{e^{\mp\nu\frac{\pi i}{2}}}{2^{2k}k!\Gamma(\frac{n}{2}+k)}, \tag{I,5}$$

and

$$\Gamma(\alpha) = \int_0^\infty e^{-x}x^{\alpha-1}dx. \tag{I,6}$$

The particular case of (I,2) when λ is a negative integer, which is not a pole, has been studied by Bresters and De Jager (cf. [3] and [5], respectively).

From [3], p. 577, formula 4.1, we have

$$(P \pm io)^{-m} = P^{-m} \mp \frac{(-1)^{m-1}}{(m-1)!}\pi i \delta^{(m-1)}(P), \tag{I,7}$$

$m \neq \frac{n}{2} + k, \quad k = 0, 1, 2 \ldots .$

On the other hand, from [6], page 192, formulas (9.1) and (9.2) we have,

$$(P \pm io)^{-m} . L^k\delta(x) = C(m,n,k)L^{k+m}\delta(x), \tag{I,8}$$

$m < \frac{n}{2}$, m is a positive integer, k is a non-negative integer and

$$C(m,n,k) = [4^m(k+1)\ldots(k+m)(\frac{n}{2}+k)\ldots(\frac{n}{2}+k+m-1)]^{-1}. \tag{I.9}$$

From (I,9) and considering the well-known formula

$$\Gamma(z+t) = z(z+1)\ldots(z+t-1)\Gamma(z) \tag{I,10}$$

$t = 1, 2, \ldots$ and z is a complex number, we have

$$C(m,n,k) = \frac{\Gamma(k+1)\Gamma(\frac{n}{2}+k)}{2^{2m}\Gamma(k+m+1)\Gamma(m+\frac{n}{2}+k)} \,. \tag{I,11}$$

In the following, we shall need the following results,

$$(P \pm io)^{\lambda}.(P \pm io)^{\mu} = (P \pm io)^{\lambda+\mu} \tag{I,12}$$

([8], page 33, formula (I,3;1)) and

$$(P \pm io)^{-k}.(P \pm io)^{-l} = (P \pm io)^{-(k+l)} \tag{I,13}$$

([6], page 191, formula (8,3)), where λ and μ are complex numbers such that λ, μ and $\lambda + \mu \neq -\frac{n}{2} - s$, $s = 0, 1, 2, \ldots, k$ and l are positive integers such that $k + l - 1 < \frac{n-2}{2}$.

From [7], p. 348, we know that the product $r^{-p}.\delta(x)$ exists and

$$r^{-p}.\delta(x) = \begin{cases} Z(m,p,\delta) & \text{if } p = 2,4,6,\ldots, \\ 0 & \text{elsewhere;} \end{cases} \tag{I,14}$$

where $x \in R^m$ and $Z(m,p,\delta)$ is defined by

$$Z(m,p,\delta) = \frac{\Delta^{\frac{p}{2}}\delta(x)}{2 2^{\frac{p}{2}}(\frac{p}{2})! m(m+2)\ldots(m+p-2)}, \tag{I,15}$$

m dimension of the space, $r^2 = x_1^2 + \cdots + x_m^2$ and Δ is the Laplacian operator.

In this paper we give sense to the distributional multiplicative products $P_{+}^{-\frac{s}{2}}.\delta(x)$ and $P_{-}^{-\frac{s}{2}}.\delta(x)$, here $P_{\pm}^{\lambda}$ are defined by (II,5) and (II,6) respectively; $\delta(x)$ is, as usual, the Dirac measure.

Our final formula (II,17) results a generalization of the equation (I,14) due to Cheng Lin Zhi and Li Chen Kuan (cf. [7], page 348, Th.4).

§II. THE DISTRIBUTIONAL MULTIPLICATIVE PRODUCTS $P_{+}^{-\frac{s}{2}}.\delta(x)$ AND $P_{-}^{-\frac{s}{2}}.\delta(x)$

From (I,7) and taking into account (I,3) the following formula is valid,

$$P^{-m}.L^k\delta(x) = \frac{1}{2}\{(P+io)^{-m} + (P-io)^{-m}\}.L^{k+m}\delta(x) \tag{II,1}$$

$m \neq \frac{n}{2} + k$, $k = 0, 1, 2, \ldots$.

In fact, from (I,3) and (I,7) we have,

$$
\begin{aligned}
P^{-m}.L^k\delta(x) &= \frac{1}{2}(a_{k,n})^{-1}\{(P+io)^{-m}+(P-io)^{-m}\}.\\
&\quad .\lim_{\gamma\to 0}[\gamma(P\pm io)^{\gamma-\frac{n}{2}-k}] =\\
&= \frac{1}{2}(a_{k,n})^{-1}.\lim_{\gamma\to 0}[\gamma(P\pm io)^{\gamma-\frac{n}{2}-k}].\\
&\quad .\{(P+io)^{-m}+(P-io)^{-m}\} =\\
&= \frac{1}{2}(a_{k,n})^{-1}\{\lim_{\gamma\to 0}[\gamma(P\pm io)^{\gamma-\frac{n}{2}-k}].(P+io)^{-m}+\\
&\quad +\lim_{\gamma\to 0}[\gamma(P\pm io)^{\gamma-\frac{n}{2}-k}].(P-io)^{-m}\}. \qquad \text{(II,2)}
\end{aligned}
$$

From (II,2) and remembering (I,12), (I,13) and (I,3), we have

$$
\begin{aligned}
P^{-m}.L^k\delta(x) &= \frac{1}{2}(a_{k,n})^{-1}\{Res_{\gamma=-\frac{n}{2}-(k+m)}(P\pm io)^{\gamma}+\\
&\quad + Res_{\gamma=-\frac{n}{2}-(k+m)}(P\pm io)^{\gamma}\} = \frac{1}{2}(a_{k,n})^{-1}2a_{k+m,n}.\\
&\quad .L^{k+m}\delta(x) = \frac{e^{\pm\frac{\nu\pi i}{2}}2^{2k}k!}{2^{(k+m)}(k+m)!\Gamma(\frac{n}{2}+k+m)}.\frac{\Gamma(\frac{n}{2}+k)L^{k+m}\delta(x)}{e^{\mp\frac{\nu\pi i}{2}}} =\\
&= \frac{k!}{2^{2m}(k+m)!}.\frac{\Gamma(\frac{n}{2}+k)}{\Gamma(\frac{n}{2}+k+m)}L^{k+m}\delta(x)\\
&= C(m,n,k)L^{k+m}\delta(x).
\end{aligned}
$$

Therefore,

$$
P^{-m}.L^k\delta(x) = C(m,n,k)L^{k+m}\delta(x) \quad \text{if } m<\frac{n}{2}. \qquad \text{(II,3)}
$$

On the other hand, from [6], p. 189, formula 6.9, we have,

$$
P^{-m} = P_+^{-m} + (-1)^m P_-^{-m} \qquad \text{(II,4)}
$$

where

$$
P_+^{\lambda} = \begin{cases} P^{\lambda} & \text{if } P>0,\\ 0 & \text{if } P\le 0,\end{cases} \qquad \text{(II,5)}
$$

and

$$
P_-^{\lambda} = \begin{cases} (-P)^{\lambda} & \text{if } P<0,\\ 0 & \text{if } P\ge 0.\end{cases} \qquad \text{(II,6)}
$$

From (II,3) and (II,4) we obtain,

$$P_{+}^{-m}.L^{k}\delta(x)=\frac{1}{2}C(m,n,k)L^{k+m}\delta(x) \tag{II,7}$$

and

$$P_{-}^{-m}.L^{k}\delta(x)=\frac{1}{2}(-1)^{m}C(m,n,k)L^{k+m}\delta(x), \tag{II,8}$$

where $C(m,n,k)$ is given by (I,11).

Putting $k=0$ in (II,7) and (II,8), we have,

$$P_{+}^{-m}.\delta(x)=\frac{1}{2}C(m,n,0)L^{m}\delta(x) \tag{II,9}$$

$m=1,2,\ldots$ and

$$P_{-}^{-m}.\delta(x)=\frac{1}{2}(-1)^{m}C(m,n,0)L^{m}\delta(x) \tag{II,10}$$

$m=1,2,\ldots$.

Here,

$$C(m,n,0)=\frac{4^{-m}\Gamma(\frac{n}{2})}{m!\Gamma(\frac{n}{2}+m)}=\frac{4^{-m}}{m!(\frac{n}{2})(\frac{n}{2}+1)\ldots(\frac{n}{2}+m-1)}. \tag{II,11}$$

If $s=2m$, from (II,9), (II,10) and (II,11) we have,

$$P_{+}^{-\frac{s}{2}}.\delta(x)=\frac{1}{2}C(\frac{s}{2},n,0)L^{\frac{s}{2}}\delta(x), \tag{II,12}$$

$s=2,4,6,\ldots$ and

$$P_{-}^{-\frac{s}{2}}.\delta(x)=\frac{1}{2}(-1)^{\frac{s}{2}}C(\frac{s}{2},n,0)L^{\frac{s}{2}}\delta(x), \tag{II,13}$$

$s=2,4,6,\ldots$, where

$$C(\frac{s}{2},n,0)=\frac{4^{-\frac{s}{2}}}{(\frac{s}{2})!(\frac{n}{2})(\frac{n}{2}+1)\ldots(\frac{n}{2}+\frac{s}{2}-1)}. \tag{II,14}$$

On the other hand, it follows from (I,3) that

$$L^{\frac{s}{2}}\delta(x)=a_{\frac{s}{2},n}\cdot\lim_{\alpha\to 0}\alpha(P\pm io)^{\alpha-\frac{n}{2}-\frac{s}{2}} \tag{II,15}$$

where $a_{\frac{s}{2},n}=\frac{e^{\mp\nu\pi i/2}}{2^{s}\Gamma(\frac{s}{2}+1)\Gamma(\frac{n}{2}+\frac{s}{2})}$.

We know that $(P\pm io)^{\lambda}$ have simple poles at $\lambda=-\frac{n}{2}-l,\ l=0,1,2,\ldots$.

If $s=1,3,5,\ldots$ and $\beta=-\frac{n}{2}-\frac{s}{2}$, then β is not a pole of (I,2), therefore from (II,15) and considering (I,12) we have,

$$L^{s/2}\delta(x) = a_{s/2,n} \lim_{\alpha\to 0} \alpha(P\pm io)^{\alpha}.(P\pm io)^{\beta} = a_{s/2,n}.0.(P\pm io)^{\beta} = 0 \qquad \text{(II,16)}$$

if $s = 1,3,5,\dots$. Therefore from (II,12), (II,13) and considering (II,14) and (II,16), we conclude that the following formulas are valid

$$P_+^{-s/2}.\delta(x) = \begin{cases} 0 \text{if } s = 1,3,5,\cdots, \\ \frac{1}{2.(\frac{s}{2})!2^{s/2}n(n+1)\cdots(n+s-2)} L^{s/2}\delta \text{if } s = 2,4,6,\cdots; \end{cases} \qquad \text{(II,17)}$$

and

$$P_-^{-s/2}.\delta(x) = \begin{cases} 0 & \text{if } s = 1,3,5,\dots, \\ \frac{1}{2(\frac{s}{2})!2^{s/2}n(n+1)\dots(n+s-2)} L^{s/2}\delta & \text{if } s = 2,4,6,\dots. \end{cases} \qquad \text{(II,18)}$$

The formula (II,17) generalizes the product $r^{-p}.\delta(x)$ given by Cheng Lin Zhi and Li Chen Kuan ([7], page 348).

In fact, putting $\nu = 0$ in (II,17) and considering (II,1) and (II,5) we have,

$$r^{-s}.\delta(x) = \begin{cases} 0 & \text{if } s = 1,3,5,\dots \\ \frac{1}{2(\frac{s}{2})!2^{s/2}n(n+1)\dots(n+s-2)} \Delta^{s/2}\delta & \text{if } s = 2,4,6,\dots \end{cases} \qquad \text{(II,19)}$$

where Δ es the Laplacian operator. The formula (II,19) coincides with the formula (I,14).

Also, putting $\mu = 1$ in (II,17) and (II,18) and considering (I,1), (I,4) and (II,5) we have,

$$\sigma_+^{-s}.\delta(x) = \begin{cases} 0 & \text{if } s = 1,3,5,\dots, \\ \frac{1}{2(\frac{s}{2})!2^{s/2}n(n+1)\dots(n+s-2)} \Box^{s/2}\delta & \text{if } s = 2,4,6,\dots; \end{cases} \qquad \text{(II,20)}$$

and

$$\sigma_-^{-s}.\delta(x) = \begin{cases} 0 & \text{if } s = 1,3,5,\dots, \\ \frac{1}{2(\frac{s}{2})!2^{s/2}n(n+1)\dots(n+s-2)} \Box^{s/2}\delta & \text{if } s = 2,4,6,\dots. \end{cases} \qquad \text{(II,21)}$$

where

$$\sigma_+^{\lambda} = \begin{cases} \sigma^{\lambda} & \text{if } \sigma > 0, \\ 0 & \text{if } \sigma \le 0; \end{cases} \qquad \text{(II,22)}$$

and

$$\sigma_-^\lambda = \begin{cases} (-\sigma)^\lambda & \text{if } \sigma < 0, \\ 0 & \text{if } \sigma \geq 0. \end{cases} \tag{II,23}$$

Here

$$\sigma^2 = x_1^2 - x_2^2 - \ldots - x_n^2 \tag{II,24}$$

and

$$\Box = \frac{\partial^2}{\partial x_1^2} - \frac{\partial^2}{\partial x_2^2} - \ldots - \frac{\partial^2}{\partial x_n^2}. \tag{II,25}$$

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