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TOTAL ABSOLUTE CURVATURE OF CURVES IN LORENTZ MANIFOLDS

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§1. INTRODUCTION

The total absolute curvature of manifolds has been discussed several times and some papers about it already belong to the classical literature on that topics. All of them deal with manifolds inmersed in Riemannian spaces. Even [7], is related to spaces of constant curvature with positive definite metric. The total absolute curvature has not been studied on spaces with indefinite metrics or more simply, on Lorentz manifolds or Minkowski n-spaces. Every semi-riemannian submanifold has a normal neighborhood in the manifold where it is contained, but that is not true, in general, for immersed submanifolds, [5], p. 200. In this note we want to show the total absolute curvature of a curve imbedded into a Lorentz manifold $\mathbb{R}_1^{N+1}$.

§2. THE TOTAL ABSOLUTE CURVATURE

Let $f\ :\longrightarrow \mathbb{R}_1^{N+1}$ be an imbedding of a curve into the $(n+1)$-Minkowski space with semi-riemannian metric g and inner product denoted by $\langle,\rangle$, of signature $(1, N)$. From the imbedding, bundles are induced according to the following diagram

$$\begin{array}{ccccc} B & \xrightarrow{\ i\ } & F_1(X) & \xrightarrow{\ j\ } & F(\mathbb{R}_1^{N+1}) \\ & & \Big\downarrow{\scriptstyle \pi_1} & & \Big\downarrow{\scriptstyle \pi} \\ & & X & \xrightarrow{\ f\ } & \mathbb{R}_1^{N+1} \end{array} \tag{1}$$

where $F_1(X)$ is the bundle of unit normals, $F(\mathbb{R}_1^{N+1})$ is the bundle of orthonormal frames, B is the sub-bundle of $(p, qe_1, \ldots, e_{N+1})$ for $p \in X$, $q = f(p)$, e_1 tangent vector and $e_2, \ldots, e_{N+1}$ orthogonal vectors to $F(X)$ at q, or the indefinite sphere bundle.

It seems appropiate to express explicitly that there are three classes of orthonormal frames attached to a curve in $\mathbb{R}_1^{N+1}$. For an orthonormal frame $e_1, \ldots, e_{N+1}$, they are:

1) *Null frame*: (for null curves) if e_1 and E_2 are nulls ($\langle e_1, e_1\rangle = \langle e_2, e_2\rangle = 0$), $\langle e_1, e_2\rangle = -1$ and $\langle e_i, e_i\rangle = 1$ for $i = 3, \ldots, n+1$.
2) *Timelike frame*: (for timelike curves) if e_1 is a timelike vector ($\langle e_1, e_1\rangle = -1$) tangent to the curve and the others are spacelike vectors, i.e., $\langle e_i, e_i\rangle = 1$ for $i = 2, \ldots, N+1$.
3) *Spacelike frame*: (for spacelike curves) if e_1 is tangent to the curve, $\langle e_i e_i\rangle = 1$ for $i = 1, \ldots, N$ and $\langle e_{N+1}, e_{n+1}\rangle = -1$, i.e., timelike vector.

The pullback of g by f, $f^*(g)$, will be the semi-riemannian metric on X induced by f from g.

Let w'_a and w'_{ab} be the forms on $f(\mathbb{R}_1^{N+1})$ associated to the semi-riemannian connection of g, $1 \le a, b \le N+1$. These forms satisfy the structure equstions

$$\begin{cases} dw'_a = \Sigma_b w'_b \wedge w'_{ab} \,, \\ dw'_{ab} = \Sigma_c w'_{ac} \wedge w'_{cb} + \Omega'_{ab} \,. \end{cases}$$

From the diagram (1) we have that

$$w_a = w'_a \circ d_j \circ d_i \qquad \text{and} \qquad w_{ab} \circ d_j \circ d_i$$

are forms on B which satisfy $W_r = 0$ for $2 \le r \le N+1$.

Applying the structure equations and $w_r = 0$ we obtain

$$w_{ir} = A_{rij} w_j \qquad \text{where} \qquad A_{rij} = A_{rji} \,.$$

The function $G(p, e_r) = (-1)\det(A_{rij})$ is called the Lipschitz-Killing curvature of f at g in the direction of the unitary vector e_r at q. This function is differentiable and bounded in $F_1(X)$, so we can consider the following integral:

$$K^*(p) = \int |G(p_1, e_r)| d\sigma \,,$$

where $d\sigma$ is the volume element of the fiber over p, and according to [8] the total absolute curvature of f is

$$\text{tac } (f) = \frac{\int_x K^*(p) dV}{V^{-1} \int_x c(N) dV} \,,$$

where V is the volume of X and $c(N)$ is the area of the unitary N-sphere.

As an example we consider the case of an imbedding of the circle S^1 into $\mathbb{R}_1^3$ by f. We will distinguish three cases: a) $f(S^1)$ is a pure spacelike curve,

b) $f(S^1)$ is a pure timelike curve, c) $f(S^1)$ is timelike or spacelike by parts (piece-pure). In [3], a pure curve was defined as a curve whose tangent vector at every point is timelike or spacelik, and a piece-pure curve as one that has only a finite number of null points.

For a), $X(s) = f(S^1)$ in $\mathbb{R}^3_1$ is a spacelike curve; we have the Frenet frame (t, n, b) that satisfies $\langle t, t\rangle = 1$, $\langle b, b\rangle = -1$. Any other frame will satisfy

$$\begin{aligned} e_1 &= t(s) \\ e_2 &= n(s) \text{ ch } \theta + b(s) \text{ sh } \theta \qquad \text{for} \qquad \theta \in \mathbb{R} \\ e_3 &= n(s) \text{ sh } \theta + b(s) \text{ ch } \theta \ . \end{aligned}$$

We get

$$\begin{aligned} w_1 &= \langle dX, e_1\rangle \\ w_{12} &= \langle de_1, e_2\rangle = \langle t'(s), (n(s) \text{ ch } \theta + b(s) \text{ sh } \theta)\rangle \ , \end{aligned}$$

and

$$A_{211} = \frac{\langle t', n\rangle \text{ ch } \theta + \langle t', b\rangle \text{ sh } \theta}{\langle t, t\rangle} = k(s) \text{ ch } \theta \ ,$$

where $k(s)$ is the curvature of $X(s)$ at s. The Lipschitz-Killing curvature is

$$k^*(p) = |k(s)| \int \text{sh } \theta d\theta \ ,$$

where the integral is over the unitary sphere in the osculator plane $T, n)$, that means that it is the euiclidean sphere S^1. We obtain

$$k^*(p) = 4|k(s)| \cdot \text{sh } \frac{\pi}{2} \ .$$

For b), $f(S^1)$ is a timelike curve and we take a timelike frame. The function curvature is

$$\int_{S^1_1} |k(s)| \text{ ch } \theta d\theta \ ,$$

but this integral is divergent. Also coming back to the expression tac(f) it is not possible to compute $c(N)$ because it corresponds to the area of non-compact spheres. Remember that the only compact connected 2-dimensional manifold which admits a Lorentz metric is the torus T.

For all above, the case c) disappears.

The diagram (1) can be shown with more detail and extented to

$$\begin{array}{ccccc} P & \xrightarrow{\psi} & F_1(X) & \longrightarrow & X \\ {\scriptstyle i}\downarrow & \searrow & {\scriptstyle \phi}\downarrow & & \downarrow \\ G(1, N) & \xleftarrow[\rho]{} & S^N_1 \cong F(\mathbb{R}^{N+1}_1) & \longrightarrow & \mathbb{R}^{N+1}_1 \ , \end{array}$$

where $F(\mathbb{R}_1^{N+1})$ is the Stiefel manifold of frames with one timelike vector and the rest are spacelike vectors. Therefore, we can identify it with the sphere S_1^N. Thus, the application ϕ i, in fact, the generalized Gauss map, [5] p. 196.

Let P be the bundle given by $\phi^*(T(S_1^N))$ over $F(X)$ with projection ψ and fiber $P_x = \{y \in T_{\phi(x)}(S_1^N)\ ;\ g(y,y) = 1\}$ which could be immersed by t in a sphere of $\mathbb{R}_1^{N+1}$. Following [7] step by step, it is not difficult to verify that

$$d\phi = |dG| d\alpha$$

and

$$di = |g(t, f \circ \pi \circ \psi)|(\psi^*(d\phi \wedge \theta) ,$$

θ being the form on P_x defined by

$$\theta(w_1, \cdots, w_n) = \det\ (\phi(\psi(x), t(x), dt(w_1), \cdots, dt(w_N)) ,$$

for $x \in P$, and

$$di(w_1, \cdots, w_N) = |dG(i_*(w_1, \cdots, i_*(w_N))| ,$$

where dG is the volume element of the indefinite Grassman manifold, [2].

Taking into account the last results we can transfer the integration to the Grassman manifold. Thus, together with [6], p. 254, we find the following expression for the total absolute cuervature of f:

$$\text{tac } (f) = \frac{1}{O_N} \int_{G(1,N)} \nu_N dL_1 ,$$

where ν_N is the number of N-hyperplanes of $\mathbb{R}_1^{N+1}$ that are orthogonal to L_1 and contain the tangent vector of X, and O_N is the volume element of the unitary N-sphere (euclidean).

In particular, [1] and [4] give us the density of lines for $N = 1$ and 2, respectively.

Finally, we want to remark that the integration is over the whole indefinite Grasssman manifold.

In [2] we showed the measure of indefinite Grassman manifolds classified according to its signature as well as its causal condition.

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