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MULTIPLE RADIAL SOLUTIONS FOR A SEMILINEAR DIRICHLET PROBLEM IN A BALL

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ABSTRACT. We prove that a semilinear elliptic boundary value problem in a ball has $4j-1$ radially symmetric solutions when the nonlinearity has a positive zero and the range of the derivative of the nonlinearity includes at least the first j eigenvalues. We make extensive use of the global bifurcation theorem, bifurcation from infinity, and bifurcation from simple eigenvalues.

1. INTRODUCTION

The purpose of this paper is to prove the existence of radially symmetric solutions to the semilinear elliptic problem

$$\begin{cases} \Delta u + f(u) & = 0 \quad \text{in} \quad \Omega \\ u & = 0 \quad \text{on} \quad \partial\Omega \end{cases} \tag{1.1}$$

where $f\colon \mathbb{R} \to \mathbb{R}$ is a differentiable function such that $f(0) = 0$,

$$f'(\infty) = \lim_{|u|\to\infty} \frac{f(u)}{u} \in \mathbb{R}, \tag{1.2}$$

$\Omega \subset \mathbb{R}^N$ is the unit ball in $\mathbb{R}^N$ centered at the origin, and Δ is the Laplacian operator.

Existence of radial solutions to problem (1.1) has been extensively studied (see [8], [9], [11], [12]). M. Esteban in [8] obtained the existence of radial solutions based on a priori estimates for solutions of (1.1). For studies on the existence of positive solutions of (1.1) we refer the reader to [7] and [13]. For other results see also [1], [2], [3], [4], and [14].

Let $\lambda_1 < \lambda_2 < \cdots < \lambda_k < \ldots$ be the eigenvalues of $(-\Delta)$ acting on radial functions of H_0^1 and $\varphi_1, \varphi_2, \ldots, \varphi_k, \ldots$ be the corresponding eigenfunctions, with $\varphi_k(0) = 1$.

Here we prove:

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Theorem A. *If f has a positive zero, and $f'(0) = f'(\infty) > \lambda_j$, then (1.1) posseses at least $4j-1$ radially symmetric solutions.*

We also give a simpler proof of the following result due to M. Esteban [8].

Theorem B. *If $0 < f'(0) < \lambda_{j+1}$, and $\lambda_{j+k} < f'(\infty)$, then (1.1) posseses at least $2k+1$ radially symmetric solutions.*

We prove Theorems A and B by obtaining a description of the graph of the set of solutions to

$$\begin{cases} \Delta u + \lambda f(u) = 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.3}$$

where $\lambda \in \mathbb{R}$ is a parameter (see figures 1 and 2). The main ingredients in arriving to these figures are the global bifurcation theorem (see [15]), bifurcation from a simple eigenvalue (see [5]), bifurcation from infinity (see [15]), as well as unique continuation properties of radial solutions to (1.3).

2. Properties of branches of solutions.

Let E be the Banach space of radially symmetric functions in $C^1(\Omega)$ satisfying $u(x) = 0$ if $\|x\| = 1$. For $u \in E$ we will denote by $u(r)$ the value of u on the sphere of radius $r \in [0,1]$. Let J denote the closure in $\mathbb{R} \times E$ to the set of nontrivial solutions to (1.3). First we prove:

Lemma 2.1. *If (λ, U) is a solution to (1.3) with $U \neq 0$, then there exists $\epsilon > 0$ such that if (λ, u) is also a solution to (1.3) and $\|(\lambda, u) - (\Lambda, U)\|_{\mathbb{R}\times E} < \epsilon$ then u has as many zeroes as U in (0,1).*

Proof. First we recall that the radial solutions to (1.3) are the solutions to the ordinary differential equation

$$u'' + \frac{N-1}{r}u' + \lambda f(u) = 0 \qquad 0 < r \leq 1, \tag{2.1}$$

$$u'(0) = 0, \tag{2.2}$$

$$u(1) = 0. \tag{2.3}$$

By uniqueness of solutions to initial value problems we know that if u satisfies (2.1), and $u(r) = u'(r) = 0$ for some $r \in [0,1]$ then $u \equiv 0$. Thus, in particular, $(u(r))^2 + (u'(r))^2 > 0$ for all $r \in [0,1]$. Hence u has finitely many zeroes and they are nondegenerate. Without loss of generality we can assume that $U(0) > 0$. Let $0 = a_0 < a_1 < a_2 < \cdots < a_k < 1 = a_{k+1}$ be the zeroes of U in (0,1), $b_1 = \max_{[0,a_1]} U$, $b_2 = \min_{[a_1,a_2]} U, b_3 = \max_{[a_2,a_3]} U, \ldots$, and $b = \min_{1\leq i\leq k+1} b_i$.

Let $0 < \epsilon_1 < \frac{b}{4}$. If $\|(\lambda, u) - (\Lambda, U)\|_{\mathbb{R}\times E} < \epsilon_1$ then u and U have same sign on the set $\{c_1, c_2, \ldots, c_k\}$ of critical points of U. Therefore $u(0) > 0$, and

$u(c_j)U(c_j) > 0$ for all $j = 1, \ldots, k$. By the mean value theorem there exist numbers $\alpha_1, \alpha_2, \ldots, \alpha_k$ with $\alpha_1 \in (0, c_1), \alpha_2 \in (c_1, c_2), \ldots, \alpha_k \in (c_{k-1}, c_k)$ so that $u(\alpha_1) = u(\alpha_2) = \cdots = u(\alpha_k) = 0$. Thus, u has at least k zeroes in (0,1).

We prove now that u has exactly k zeroes in (0,1). Let $\delta > 0$ be sufficiently small such that for some $\gamma_1 > 0$ and $\gamma_2 > 0$

(i) $|U'(x)| \geq \gamma_1 \quad \forall x \in C = [a_1 - \delta, a_1 + \delta] \cup \cdots \cup [1 - \delta, 1]$

(ii) $|U(x)| \geq \gamma_2 \quad \forall x \in D := [0, 1] - C$.

Let $\epsilon < min\{\frac{\gamma_1}{4}, \frac{\gamma_2}{4}, \epsilon_1\}$. If $x \in D$

$$|u(x)| \geq |U(x)| - |u(x) - U(x)| \geq \gamma_2 - \frac{\gamma_2}{4} > 0.$$

Suppose there exist $t_1, t_2 \in [a_j - \delta, a_j + \delta]$, for some $1 \leq j \leq k$, so that $u(t_1) = u(t_2) = 0$. By the mean value theorem there exists $t \in [t_1, t_2] \subset C$ such that $u'(t) = 0$. Thus

$$|U'(t)| = |u'(t) - U'(t)| < \frac{\gamma_1}{4}$$

which is a contradiction. Thus Lemma 2.1 is proven. □

Lemma 2.2. *If* Γ *is a connected component of* $\{(\lambda, u) \in \mathbb{R} \times E;$ (λ, u)*satisfies (1.3) and* $u \neq 0\}$ *then there exists a positive integer* k *such that if*
$(\lambda, u) \in \Gamma$ *then* u *has exactly* k *zeroes in* $[0, 1]$.

Proof. Let $(\Lambda, U) \in \Gamma$. Since solutions to (2.1)-(2.3) can not have degenerate zeroes, U has finitely many zeroes in [0,1], say k. We let $\Sigma = \{(\lambda, u) \in \Gamma; u$ has k zeroes in [0,1]$\}$. From Lemma 2.1 it follows that Σ is an open subset of Γ. Let us see now that Σ is also closed.

Let $\{(\lambda_n, u_n)\}_{n=1}^{\infty}$ be a sequence in Σ such that $(\lambda_n, u_n) \to (\lambda, u)$ with $(\lambda, u) \in \Gamma$. We prove now that u has a finite number of zeroes in $(0, 1)$. In fact, assume, on the contrary, that u has infinitely many zeroes in $(0, 1)$. Then, there exists a sequence $\{t_n\}_{n=1}^{\infty} \subset [0, 1]$ such that $u(t_n) = 0$. Without loss of generality we can assume $t_n \to \bar{t}$ where $\bar{t} \in [0, 1]$. By continuity of u we have $u(\bar{t}) = 0$. The mean value theorem gives the existence of a sequence $\{s_n\}_{n=1}^{\infty}$ so that $s_n \in (t_n, \bar{t}) \cup (\bar{t}, t_n)$ and $u'(s_n) = 0$. Since u' is continuous we see that $u'(\bar{t}) = 0$. Because u is a radial solution of (1.3), u satisfies

$$\begin{cases} u'' + \dfrac{N-1}{r} u' + \lambda f(u) = 0 & 0 < r \leq 1 \\ u'(0) = u(1) = 0. & \end{cases}$$

Therefore by uniqueness of solutions to the initial value problem we have $u \equiv 0$; which is a contradiction. This proves u has only a finite number of zeroes in

(0,1). Since $(\lambda_n, u_n) \to (\lambda, u)$, by Lemma 2.1 it follows that u has exactly $(k-1)$ zeroes in (0,1). Thus $(\lambda, u) \in \Sigma$. This proves that Σ is closed in Γ, and the Lemma follows. □

For $d \in \mathbb{R}$ and $\lambda > 0$ we will denote by $u(\,\cdot\,, \lambda, d)$ the solution to (2.1) with $u(0) = d$, and $u'(0) = 0$. Standard arguments on dependence on parameters show that u is a differentiable function in the variable (r, λ, d).

An elementary calculation shows that if $\alpha > 0$ then $u(\frac{r}{\alpha}, \lambda, d) = u(r, \frac{\lambda}{\alpha^2}, d)$. Differentiating this relation with respect to α and taking $\alpha = 1$ we obtain

$$ru'(r, \lambda, d) = 2\lambda u_\lambda(r, \lambda, d). \tag{2.4}$$

Lemma 2.3. *If J is a connected component of $\{(\lambda, d); d \neq 0, u(1, \lambda, d) = 0\}$ then there exist an open interval $(a, b) \subset \mathbb{R} - \{0\}$ and a continuous function $h : (a, b) \to (0, \infty)$ such that $(\lambda, d) \in J$ if and only if $d \in (a, b)$ and $\lambda = h(d)$. Moreover, if $a \in \mathbb{R} - \{0\}$ then $\lim_{d\to a} h(d) = \infty$. Similarly, if $b \in \mathbb{R} - \{0\}$ then $\lim_{d\to b} h(d) = \infty$.*

Proof. Let $(\gamma, w) \in J$. Since solutions to (2.1)-(2.3) can not have degenerate zeroes, from (2.4) we see that $u_\lambda(1, \gamma, w) \neq 0$. Thus, by the implicit function theorem there exists $\sigma > 0$ such that $(\lambda, d) \in J$, with $|\lambda - \gamma| < \sigma$ and $|d - w| < \sigma$ if and only if $\lambda = h_1(d)$ with $h_1 : (w - \sigma, w + \sigma) \to \mathbb{R}$ continuous. By continuous dependence on parameters we see that if $w + \sigma \in \mathbb{R} - \{0\}$ and $\liminf_{d \to w+\sigma} h_1(d) < \infty$, then $(z, w + \sigma) \in J$, where z is any accumulation point of any sequence of the form $\{h_1(s_k); s_k \to w + \sigma\}$. Because $u(1, 0, w + \sigma) = w + \sigma$ whereas $u(1, z, w + \sigma) = 0$ we see that $z \neq 0$. Since (2.4) also applies to $(\,\cdot\,, z, w + \sigma)$, using again the implicit function theorem, it follows that h_1 can be extended to $(w - \sigma, w + \sigma + \delta)$ until either $b \in \{0, \infty\}$ or $\lim_{d\to b} h_1(d) = \infty$. Similarly it is proven that either $a \in \{0, \infty\}$ or $\lim_{d\to a} h_1(d) = \infty$. Letting h denote the maximal extension of h_1 and noting that the graph of h is open and closed in J we conclude that J coincides with the graph of h, which proves the lemma. □

Remark 2.4. We denote by β the positive zero of f. For future reference we observe that, by uniqueness of solutions to the initial value problem for equation (2.1), if $u(s, \lambda, d) = \beta$ and $u'(s, \lambda, d) = 0$ for some $s \in [0, 1]$ then $u(t, \lambda, d) = \beta$ for all $t \in [0, 1]$. Thus if J is as in Lemma 2.3 then the domain of h can not include β.

3. Proof of Theorems A and B

Because λ_n $(1 \leq n < \infty)$ is a simple eigenvalue of $-\delta$ with Dirichlet boundary condition we have that 0 is a simple eigenvalue of $\delta + (\frac{\lambda_n}{f'(0)})\, I$ and

$\delta + (\frac{\lambda_n}{f'(\infty)})\, I$ with Dirichlet boundary condition. From general properties of bifurcation from simple eigenvalues (see [5]) and bifurcation from infinity (see [15]) it follows that $(\frac{\lambda_n}{f'(0)}, 0)$ and $(\frac{\lambda_n}{f'(\infty)}, \infty)$ are points of bifurcation of (2.1). More precisely there exists $\sigma > 0$ such that for $|r| \in (0, \sigma)$ the problem (2.1) has solutions of the form $(\lambda, r\varphi_n + \psi(r))$ with $\int_0^1 r^{n-1}\varphi_n\, \psi\, dr = 0$, $\|\psi\| = o(r)$, and $|\lambda - \frac{\lambda_n}{f'(0)}| = o(r)$ as $r \to 0$. Similarly, if (λ, u) is a solution to (2.1) on the branch bifurcating from $(\frac{\lambda_n}{f'(\infty)}, \infty)$ then $u = r\varphi_n + \psi(r)$ with $\int_0^1 s^{n-1}\varphi_n\, \psi\, ds = 0$, $\|\psi\| = o(r)$, and $|\lambda - \frac{\lambda_n}{f'(\infty)}| = o(r)$ as $r \to \infty$. We will denote by Γ_n^+ the connected component of nontrivial solutions to (2.1)-(2.3) bifurcating from $(\frac{\lambda_n}{f'(0)}, 0)$ and containing elements of the form $(\lambda, r\varphi_n + \psi(r))$ with $r > 0$. Similarly we define Γ_n^-. Also we define G_n^+ as the connected component of nontrivial solutions to (2.1)-(2.3) bifurcating from $(\frac{\lambda_n}{f'(\infty)}, \infty)$ and containing elements of the form $(\lambda, r\varphi_n + \psi(r))$ with $r > 0$. Similarly we define G_n^-.

Lemma 3.1. *Let $J = \{(\lambda, u(0)); (\lambda, u) \in G_2^-\}$. If h, a, and b are as in Lemma 2.3, then $a = -\infty$ and $b < 0$.*

Proof. Since G_2^- contains elements of the form $(\lambda, s\varphi_2 + \psi(s))$ with $s < 0$ and large, and λ is near $\frac{\lambda_2}{f'(\infty)}$ we have $a = -\infty$ and $\lim_{d \to -\infty} h(d) = \frac{\lambda_2}{f'(\infty)}$. Since $\|\psi(s)\| = o(s)$ as $s \to -\infty$ we have

$$\lim_{s \to -\infty} s\varphi_2(\alpha) + \psi(s)(\alpha) = +\infty \tag{3.1}$$

where α is the critical point of φ_2 in (0,1].

Let us see now that $g(d) = \max\{u(r, h(d), d); r \in [0,1]\}$ is a continuous function. In fact, Let $d_0 \in (a, b)$ and $\{d_n\}$ a sequence converging to d_0. Of course $\{h(d_n)\}$ converges to $h(d_0)$. Let t_n be such that $g(d_n) = u(t_n, h(d_n), d_n)$. Let $\bar{t}$ be a limit point of $\{t_n\}$. Since $u(\bar{t}, h(d_0), d_0) = \lim_{n \to \infty} u(t_n, h(d_n), d_n) \geq 0$, we have $\bar{t} > 0$. By continuous dependence on parameters $u'(\bar{t}, h(d_0), d_0) = \lim_{n \to \infty} u'(t_n, h(d_n), d_n) = 0$. Thus $\bar{t}$ is a critical point of $u(\cdot, h(d_0), d_0)$. Since elements in G_2^- have only one zero in (0,1) they have only one critical point. Thus $\bar{t}$ is the maximum of $u(\cdot, h(d_0), d_0)$, which proves that g is continuous. By (3.1) we see that $\lim_{d \to -\infty} g(d) = +\infty$. If $b = 0$ then $\lim_{d \to 0} g(d) = 0$, Thus by the intermediate value theorem there exists $d_1 < 0$ such that $g(d_1) = \beta$, i.e., there exists $t_1 \in (0,1)$ with $u(t_1, h(d_1), d_1) = \beta$, and $u'(t_1, h(d_1), d_1) = 0$. By uniqueness of solutions to initial value problem we have $u(t, h(d_1), d_1) \equiv \beta$, which contradicts that $d_1 < 0$; thus $b < 0$, which proves the Theorem. □

Notation. We will denote by h_1 and $b_1 < 0$ the continuous function and the number such that $\{(\lambda, u(0)); (\lambda, u) \in G_2^-\} = \{(h_1(d), d); d \in (-\infty, b_1)\}$.

Corollary 3.2. *If $J = \{(\lambda, u(0)); (\lambda, u) \in \Gamma_n^-\}$ with $n \geq 2$, and a, b are as in Lemma 2.3 then $a \geq b_1$.*

Proof. Suppose $a = -\infty$. Let h be as in Lemma 2.3. For $d < b_1$ we see that $h(d) < h_1(d)$, otherwise $\Gamma_n^- \cap G_2^- \neq \emptyset$, which contradicts that $b_1 < 0$. Let $\{d_n\}$ be a sequence converging to $-\infty$ with $\{h(d_n)\}$ converging to Λ. Thus $(\Lambda, -\infty)$ is a bifurcation point. Hence $\Lambda = \dfrac{\lambda_1}{f'(\infty)}$ or $\Lambda = \dfrac{\lambda_2}{f'(\infty)}$. Since elements in J correspond to solutions with at least one interior zero $\Lambda \neq \dfrac{\lambda_1}{f'(\infty)}$. Also since $b_1 < 0$ it follows that $\Lambda \neq \dfrac{\lambda_2}{f'(\infty)}$. This contradiction shows that $a > -\infty$. If $a < b_1$ then $h - h_1 : (a, b_1) \to \mathbb{R}$ is such that $\lim_{d \to a}(h - h_1)(d) = +\infty$ and $\lim_{d \to b_1}(h - h_1)(d) = -\infty$. Hence by the intermediate value theorem there exists $d < b_1$ with $h(d) = h_1(d)$ which contradicts that $\Gamma_n^- \cap G_2^- = \emptyset$, this proves the Corollary. □

Imitating the proof of Corollary 3.2 we also have

Corollary 3.3. *If $J = \{(\lambda, u(0)); (\lambda, u) \in G_n^-\}$ with $n > 2$ and a, b are as in Lemma 2.3 then $b \leq b_1$.*

Proof of Theorem A. Our proof consists of showing that the sets $\Gamma_k^{\pm}$ and $G_k^{\pm}$, for $k = 2, \ldots, j$, as well as Γ_1^+ and G_1^+ contain each a solution to (1.1). These $4j - 2$ solutions together with 0 yield the result.

Let $k \in \{1, \ldots, j\}$ and $J = \{(\lambda, u(0)); (\lambda, u) \in \Gamma_k^+\}$. Let (a,b) and h be as in Lemma 2.3. By the definition of Γ_k^+ we see that $a = 0$, and $\lim_{d \to 0} h(d) = \dfrac{\lambda_k}{f'(0)} < 1$. By Remark 2.4 we also have $b < \beta$. Thus by Lemma 2.3 $\lim_{d \to b} h(d) = \infty$. Hence by the intermediate value theorem there exists $d_k \in (a, b)$ such that $h(d_k) = 1$. Thus $(1, u(\cdot, 1, d_k)) \in \Gamma_k^+$ is a solution to (1.1).

Let now $J = \{(\lambda, u(0)); (\lambda, u) \in \Gamma_k^-\}$ with $k \in \{2, \ldots, j\}$ and a, b and h be as in Lemma 2.3. Since by Corollary 3.2 and Lemma 2.3 $a > b_1$ and $\lim_{d \to a} h(d) = \infty$ we see that there exists $\delta_k \in (a, 0)$ such that $h(\delta_k) = 1$. Thus $(1, u(\cdot, 1, \delta_k)) \in \Gamma_k^-$ is a solution to (1.1) with k zeroes in (0,1].

Similar arguments show that G_k^- for $k = 2, \ldots, j$, and G_k^+ for $k = 1, \ldots, j$ contain a solution to (1.1). This proves Theorem A. □

We summarize the above result in the following bifurcation diagram

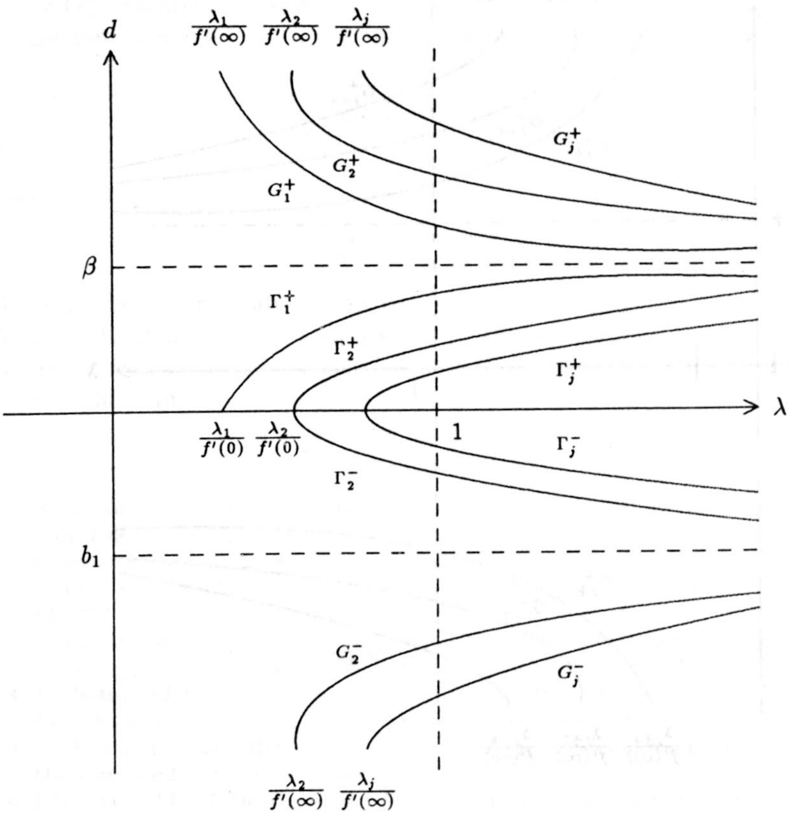

FIGURE 1. Bifurcation diagram for problem (1.3)

Proof of Theorem B. If there exists $\bar{x} \neq 0$ such that $f(\bar{x}) = 0$. Arguing as in the proof of Theorem A it follows that G_n^+ and G_n^- $(n \in \{j+1, \dots, j+k\})$ has a solution to (1.1) (see figure 2).

On the other hand if $f(x) \neq 0$ for $x \neq 0$, then there exists $m > 0$ such that $x\,f(x) - m\,x^2 > 0$ for $x \neq 0$. We claim now that $\Gamma_n^+ = G_n^+$ and $\Gamma_n^- = G_n^-$ for all $n = 1, 2, \dots$.

Let $J = \{(\lambda, u(0)); (\lambda, u) \in \Gamma_n^+\}$. Let a, b, and h be as in Lemma 2.3. By the definition of Γ_n^+ we have $a = 0$. We claim that h is a bounded function. If

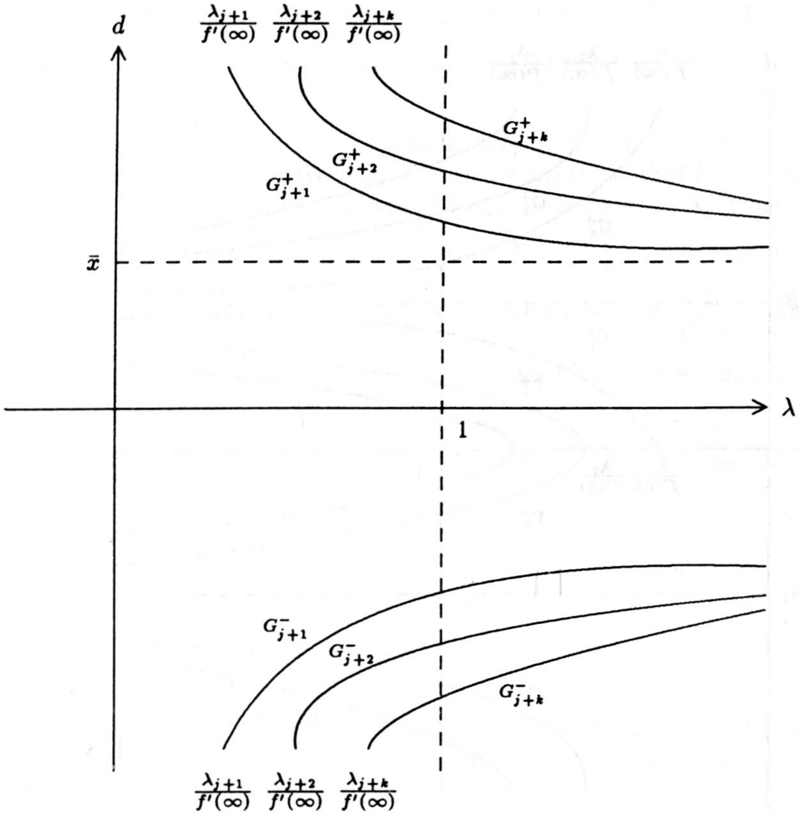

FIGURE 2. Bifurcation diagram for problem (1.3)

not there exists $d \in (a, b)$ with $h(d) > \dfrac{\lambda_n}{m}$. Thus comparing

$$u'' + \frac{N-1}{r} u' + \left(\frac{h(d)\, f(u)}{u} \right) u = 0$$

with

$$\varphi_n'' + \frac{N-1}{r} \varphi_n' + \lambda_n \varphi_n = 0.$$

We see that u has at least $(n+1)$ zeroes in [0,1] which contradicts that if $(\lambda, u) \in \Gamma_n^+$ then u has n zeroes in [0,1]. Thus, h is bounded and, in particular, $b = \infty$.

Let $\{d_n\}$ be a sequence tending to $+\infty$ such that $\{h(d_n)\}$ converges. Thus $(\lim_{n\to\infty} h(d_n), +\infty)$ is a point of bifurcation of (1.3) and $u(\cdot, h(d_n), d_n)$ has n zeroes in $[0,1]$. Hence $h(d_n) = \dfrac{\lambda_n}{f'(\infty)}$ and $(h(d_n), u(\cdot, h(d_n), d_n)) \in G_n^+$. Thus $\Gamma_n^+ = G_n^+$. Similar arguments show that $\Gamma_n^- = G_n^-$.

Let now $n = j+1, \ldots, j+k$. By the definition of Γ_n^+ and G_n^+ we have,

$$\lim_{d\to+\infty} h(d) = \frac{\lambda_n}{f'(\infty)} < 1$$

and

$$\lim_{d\to 0} h(d) = \frac{\lambda_n}{f'(0)} > 1.$$

Thus by the intermediate value theorem there exists $d_n \in (0,\infty)$ such that $h(d_n) = 1$. Hence $(1, u(\cdot, 1, d_n)) \in G_n^+ = \Gamma_n^+$ is a solution to (1.1). Similarly it can be seen that there exists $\delta_n < 0$ with $(1, u(\cdot, 1, \delta_n)) \in G_n^- = \Gamma_n^-$, which proves Theorem B. □

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