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THE RIESZ TRANSFORMS FOR GAUSSIAN MEASURES

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§1. INTRODUCTION

This note concerns with the boundedness in L^p-spaces of the Riesz transforms associated with a class of second order elliptic differential operators. The proof of the results described here can be found in **[G]** and **[F-G-Sc]**.

It is a fundamental fact that the Hardy-Littlewood maximal function M, and the classical M. Riesz transforms are bounded in L^p for $p > 1$, and of weak-type 1-1. E. M. Stein discovered in 1983, **[S2]** that the strong type constant of such operators can be bounded independently of the dimension n, and shortly after, Stein and Stromberg **[S-St]** proved that the weak-type constant of M can be bounded by a universal constant times the dimension n. This last result is because it is possible to control M by a universal constant times n times the Ergodic maximal operator E of the heat semigroup. Then by the powerful theorem of Hopf-Dunford-Schwartz about the weak-type 1-1 of E the result follows. It is a very interesting open problem to determine if the weak-type 1-1 constants of the Hardy-Litlewood and of the Riesz transforms can be bounded independently of n. By having dimensionless inequalitites we can extend them to the infinite-dimensional space R^N. However, the trouble is that those inequalitites would be proved with respect to the Lebesgue measure and unfortunately the Lebesgue measure does not extend to infinite dimensions. The most typical infinite-dimensional measure is the Gaussian measure, and the singular integrals naturally associated with this measure are the Riesz transforms associated with the Ornstein-Uhlenbeck semigroup whose infinitesimal generator is the "Laplacian" $L_B = \frac{1}{2}\Delta_x - Bx \cdot \text{grad}_x$. The probability measure $\gamma_B(x)\,dx = C\,e^{-Bx\cdot x}\,dx$ makes L_B self-adjoint. The interest of the infinite dimensional formulation comes from the Malliavin calculus.

In order to introduce and clarify this notion of Riesz's transforms, we shall first construct the classical M. Riesz transforms by defining their action on the eigenvectors of a boundary value problem for the Laplacian.

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Let Δ be the Laplacian in R^n and the eigenvalue problem

$$\Delta u = \lambda u,$$

with boundary conditions

$$u(x) = O(1), \qquad \text{as } \|x\| \to \infty.$$

The set of eigenvalues of this problem consists of all non-positive numbers, and given $\lambda \leq 0$ the eigenvectors corresponding to λ are

$$e^{iy\cdot x}, \qquad \text{where } \|y\|^2 = -\lambda.$$

We define the j-th Riesz transform by

$$R_j(e^{iy\cdot})(x) = -\frac{1}{\|y\|}\frac{\partial e^{iy\cdot x}}{\partial x_j} = -i\frac{y_j}{\|y\|}\, e^{iy\cdot x}.$$

Given a function f and its Fourier transform $\hat{f}$ we write

$$f(x) = C_n \int_{R^n} \hat{f}(y) e^{iy\cdot x}\, dy,$$

and by "applying" R_j and "interchanging" the integral and R_j we get

$$R_j f(x) = \int_{R^n} R_j(e^{iy\cdot})(x)\, \hat{f}(y)\, dy = -i \int_{R^n} \frac{y_j}{\|y\|}\, e^{iy\cdot x} \hat{f}(y)\, dy,$$

which is the classical definition of the j-th Riesz transform.

Let B be an $n \times n$ positive definite symmetric matrix, and let L_B be the differential operator in R^n defined by

$$L_B = \frac{1}{2}\Delta - Bx \cdot \text{grad},$$

our purpose is to define a notion of Riesz's transforms naturally associated with L_B. The operator L_B is the infinite simal generator of an Ornstein-Uhlenbeck process, which is obtained by subjecting the particles of a Brownian motion to an elastic force. This force is reflected in the presence of the drift term $Bx \cdot$ grad, see **[Fe]**, p. 324. Also, the operator L_B received attention from the point of view of hypoellipticty, see **[Hör]**.

We look at the eigenvalue problem

$$L_B u = \lambda u,$$

with boundary conditions

$$u(x) = O(\|x\|^k), \qquad \text{for some } k \geq 0 \text{ as } \|x\| \to \infty.$$

These eigenvalues form a discrete set and are related to the Hermite polynomials.

The one-dimensional Hermite polynomials are defined by

$$H_0(x) = 1, \qquad H_n(x) = e^{x^2} \frac{d^n}{dx^n} e^{-x^2}, \qquad n \geq 1,$$

and some of their basic properties are

$$\int_{-\infty}^{+\infty} H_n(x)^2 e^{-x^2}\, dx = 2^n n! \sqrt{\pi}, \qquad n = 0, 1, \ldots$$

$$\int_{-\infty}^{+\infty} H_0(x) e^{-x^2}\, dx = \int_{-\infty}^{+\infty} e^{-x^2}\, dx = \sqrt{\pi},$$

and

$$\int_{-\infty}^{+\infty} H_n(x) e^{-x^2}\, dx = 0, \qquad \text{for } \quad n \geq 1.$$

Also

$$H'_{n+1}(x) = -2(n+1) H_n(x)$$

$$H_{n+1}(x) + 2x H_n(x) + 2n H_{n-1}(x) = 0, \qquad n \geq 0, \qquad H_{-1}(x) = 0$$

and

$$H''_n(x) - 2x H'_n(x) + 2n H_n(x) = 0.$$

If we assume that the matrix B is diagonal, i.e.,

$$B = D = \begin{pmatrix} d_1 & 0 & \cdots & 0 \\ 0 & d_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & d_n \end{pmatrix} \tag{1-1}$$

then by using the properties of the Hermite polynomials mentioned above is not difficult to see that the eigenvalues of L_D are of the form $\lambda = -(\alpha \cdot d)$, where $\alpha = (\alpha_1, \ldots, \alpha_n)$, α_i are non-negative integers, and $d = (d_1, \ldots, d_n)$. The corresponding eigenfunctions are the multidimensional Hermite polynomials defined by

$$H_\alpha^D(x) = H_{\alpha_1}(\sqrt{d_1} x_1) \ldots H_{\alpha_n}(\sqrt{d_n} x_n),$$

where $H_{\alpha_i}(.)$ are one-dimensional Hermite polynomials of degree α_i.

In the general case B is symmetric and positive definite, then there exists an orthogonal matrix A such that ABA^t is diagonal, i.e.,

$$ABA^t = D = \begin{pmatrix} d_1 & 0 & \dots & 0 \\ 0 & d_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & d_n \end{pmatrix}$$

In such case, the eigenvalues of L_B are again of the form $-(\alpha \cdot d)$ and the corresponding eigenfunctions now are $H_\alpha^B(x) = H_\alpha^D(Ax)$.

The Riesz transforms associated with L_B are defined as follows. Given $j, 1 \le j \le n$, and $H_\alpha^B(x)$ a multidimensional Hermite polynomial, the j-th Riesz transform $\mathcal{R}_j^B$ of H_α^B is defined by

$$\mathcal{R}_j^B(H_\alpha^B)(x) = -\frac{1}{\sqrt{\alpha \cdot d}} \frac{\partial}{\partial x_j} H_\alpha^B(x),$$

and by linearity the definition of $\mathcal{R}_j$ extends to any polynomial in R^n

The measure $e^{-Bx \cdot x}\, dx$ makes the operator L_B selfadjoint and therefore it is the natural measure to study the boundedness properties of the operators associated to L_B. Note that if $A = (a_{ij})$ then we have

$$\mathcal{R}_j^B(H_\alpha^B)(x) = \sum_{i=1}^n a_{ij}\, \mathcal{R}_j^D(H_\alpha^D)(Ax).$$

Therefore, if we define the vector

$$\mathcal{R}^B = (\mathcal{R}_1^B, \dots, \mathcal{R}_n^B) \tag{1-2}$$

then we have

$$\mathcal{R}^B(H_\alpha^B)(x) = A^t \mathcal{R}^D(H_\alpha^D)(Ax). \tag{1-3}$$

In view of the last formula it will be enough to consider the case when B is diagonal and to prove the inequalities for the vector defined by (1-2).

As in the case of the Laplacian there is a notion of Poisson semigroup associated to L_B. We assume for simplicity that $B = D$ as in (1-1). In fact, let T_t be the n-dimensional realization of the Ornstein-Uhlenbeck semigroup associated with D defined by

$$T_t f(x) = \int_{R^n} k_D(t, x, y) f(y)\, dy,$$

where

$$k_D(t, x, y) = \prod_{i=1}^n \frac{\sqrt{d_i}}{\sqrt{\pi}(1 - e^{-2d_i t})^{1/2}} \exp\left(-\frac{(e^{-d_i t} x_i - y_i)^2 d_i}{1 - e^{-2d_i t}} \right)$$

$t > 0, x \in R^n$. If we set $u(x,t) = T_t f(x)$ then u is a solution of the equation

$$u_t = \frac{1}{2}\Delta_x u - Dx \cdot \text{grad}_x u,$$

and therefore the infinitesimal generator of the semigroup T_t is L_D.

By using the principle of subordination we define the Poisson semigroup P_t associated with T_t by

$$P_t f(x) = \frac{1}{\sqrt{\pi}} \int_0^\infty \frac{e^{-u}}{\sqrt{u}} T_{t^2/4u} f(x)\, du.$$

We now introduce the following semigroups, which we shall see play an important role in the study of $\mathcal{R}^D$:

$$T_t^i = e^{-d_i t} T_t, \qquad i = 1, ..., n,$$

and let P_t^i be the Poisson semigroup associated with T_t^i.

Given a general semigroup T_t bounded in L^p_μ, the Poisson semigroup P_t associated with T_t is given by the subordinati formula above. The maximal operator associated with the semigroup P_t is defined by

$$P^* f(x) = f^*(x) = \sup_{t>0} \|P_t f(x)\| ,$$

and from the general theory of Poisson semigroups, this operator is of strong-type $(p,p), p > 1$ with respect to the measure μ, and the strong-type constant is bounded independently of the dimension n, see **[S1]**, p. 48. In our case, these results apply to the semigroups P_t and P_t^i defined above with the measure $e^{Dx \cdot x}\, dx$.

By using the properties of the Hermite polynomials mentioned above it is easy to see that

$$L_D H_\alpha^D(x) = -(\alpha \cdot d) H_\alpha^D(x)$$

$$T_t H_\alpha^D(x) = e^{-(\alpha \cdot d)t} H_\alpha^D(x)$$

$$P_t H_\alpha^D(x) = e^{-\sqrt{\alpha \cdot d}\, t} H_\alpha^D(x)$$

$$T_t^i H_\alpha^D(x) = e^{-(\alpha \cdot d + d_i)t} H_\alpha^D(x)$$

and

$$P_t^i H_\alpha^D(x) = e^{-\sqrt{\alpha \cdot d + d_i}\, t} H_\alpha^D(x),$$

$i = 1, ..., n$.

Given a polynomial f in R^n and $\mathcal{R}_1^D f, ..., \mathcal{R}_n^D f$ its Riesz's transforms we set

$$u_0(x,t) = P_t f(x)$$

$$u_j(x,t) = P_t^j(\mathcal{R}_j^D f)(x), \qquad j = 1, \ldots, n,$$

and we have

$$\frac{\partial u_0}{\partial x_i} = \frac{\partial u_i}{\partial t}, \qquad i = 1, \ldots, n \tag{1-4}$$

In fact, it is enough to check (1-4) in the case that $f = H_\alpha^D$. In such case we have

$$u_0(x,t) = e^{-\sqrt{\alpha \cdot d}\, t} H_\alpha^D(x)$$

$$u_j(x,t) = -e^{-\sqrt{\alpha \cdot d}\, t} \frac{1}{\sqrt{\alpha \cdot d}} \frac{\partial}{\partial x_j} H_\alpha^D(x), \qquad j = 1, \ldots, n,$$

and (1-4) inmmediatly follows. Note that $u(x,t) = P_t f(x)$ satisfies the equation

$$u_{tt} + \frac{1}{2}\Delta_x u - Dx \cdot \operatorname{grad}_x u = 0$$

and $v_i(x,t) = P_t^i f(x)$ satisfies

$$v_{tt} + \frac{1}{2}\Delta_x v - Dx \cdot \operatorname{grad}_x v - d_i v = 0,$$

$i = 1, \ldots, n$.

§2. The results in L^p, $p > 1$

We define $\gamma_n^B(x) = \dfrac{(\det B)^{1/2}}{\pi^{n/2}} e^{-Bx \cdot x}$, and note that $\int_{R^n} \gamma_n^B(x)\, dx = 1$. By $\|f\|_{p,\gamma_n^B}$ we denote the L^p-norm of the real-valued function f with respect to the measure $\gamma_n^B(x)\, dx$. Given a vector-valued function $h(x) = (h_1(x), \ldots, h_d(x))$, we set

$$\|h(x)\|_2 = \left(\sum_{i=1}^{d} \|h_i(x)\|^2 \right)^{1/2}$$

The main result concerning the boundedness of $\mathcal{R}_j$ in L^p for $p > 1$ is the following.

Theorem 1. *Let B be an $n \times n$ positive definite symmetric matrix with k different eigenvalues, and let $1 < p < \infty$. There exists a constant C_p depending only on p such that*

$$\|\mathcal{R}^B f_2\|_{p,\gamma_n^B} \le C_p\, k\, \|f\|_{p,\gamma_n^B},$$

for all polynomials f in R^n.

This theorem is proved in **[G]**. The proof is analytic and uses the Littlewood-Paley-Stein theory in **[S1]**. The idea is to construct appropriate g-functions defined in terms of the semigroups P_t and P_t^i that relate f and its Riesz's transforms by means of the equation (1-4), and to prove that these g-functions are bounded in L^p. Theorem 1 contains as a particular case the inequalities proved by P. A. Meyer using probabilistic methods for the case when $B = I$, the identity matrix, see **[Me]**, **[Gn]** and **[Pi]**. We mention that the study of the boundedness properties of $\mathcal{R}_j$ began with the work of B. Muckenhoupt, **[Mu]**, when the dimension $n = 1$. By using methods which seem to be applicable in one dimension only, he proved the boundedness of this transformation (in this case is only one operator) when $p > 1$ and the weak-type (1,1), the case he considered was when $B = 1$ and consequently the underlying measure is e^{-x^2}. By changing variables these imply the results in the case when $B = \rho > 0$ with the measure $e^{-\rho x^2}$.

For a diagonal matrix D as is (1-1) we introduce the following Littlewood-Paley-Stein functions

$$g(f)(x) = \left(\int_0^\infty t\|\nabla(P_t f(x))\|^2 \, dt\right)^{1/2}$$

$$g^i(f)(x) = \left(\int_0^\infty t\left(\|\nabla(P_t^i f(x))\|^2 + d_i(P_t^i f(x))^2\right) dt\right)^{1/2}$$

$$g_1^i(f)(x) = \left(\int_0^\infty t\|\frac{\partial}{\partial t}(P_t^i f)(x)\|^2 \, dt\right)^{1/2},$$

with the notation

$$\nabla u = (u_t, \frac{1}{\sqrt{2}}\mathrm{grad}_x u).$$

If $f = (f_1, ..., f_n)$, we define the vector-valued Littlewood-Paley function

$$g_1 f(x) = (g_1^1(f_1)(x), ..., g_1^n(f_n)(x)).$$

Theorem 1 is a consequence of the following.

Theorem 2. *Let $1 < p < \infty$, and let D be a positive definite diagonal matrix with k different eigenvalues. Then there exists a constant C_p only depending on p such that for every polynomial f in R^n we have*

$$\|g(f)\|_{p,\gamma_n^D} \le C_p \|f\|_{p,\gamma_n^D},$$

and

$$\|g^i(f)\|_{p,\gamma_n^D} \le C_p \|f\|_{p,\gamma_n^D}, \qquad i = 1, ..., n.$$

Also, if $f = (f_1, ..., f_n)$ then

$$\| \, \|f\|_2 \|_{p,\gamma_n^D} \le C_p \, k \, \| \, \|g_1(f)\|_2 \|_{p,\gamma_n^D}, \tag{1-7}$$

and

$$\|f_i\|_{p,\gamma_n^D} \le C_p \|g_1^i(f_i)\|_{p,\gamma_n^D}, \qquad i = 1, ..., n. \tag{1-8}$$

§3. The results in IN L^1

It has been an open problem, to determine if the Riesz transforms $\mathcal{R}_j$ are of weak-type 1-1. In **[F-G-Sc]** we solved this problem in the affirmative when $B = I$, the identity matrix, by showing a stronger result: the maximal singular operator(supremum over all ϵ-truncations) is of weak-type 1-1 with respect to Gaussian measure. The proof of this result is technically difficult, and we need to show pointwise estimates of the kernel of the operator in appropriate regions. This involves careful estimations of integrals involving the Gaussian measure and the study of maximal and singular integral operators appropriatly truncated. The techniques from spaces of homogeneous type, make it possible to extend the Calderón-Zygmund theory of singular integrals to that setting. In our case, however, they do not work due to the fact that the Gaussian measure is not doubling. The ideas developed in **[G]** and **[F-G-Sc]** can be very useful to deal non-doubling weights.

In order to prove the weak-type 1-1 result, the operator $\mathcal{R}_j$ is defined as the principal value of an integral operator. This definition is equivalent to the one previously given.

Let $T_t f$ be the semigroup associated with L_I, i.e., $u(x,t) = T_t f(x)$ is a solution of $u_t = L_I u$, and $u(x,0) = f(x)$. By using the principle of subordination mentioned in §2, let P_t be the Poisson semigroup associated with T_t. We now define the "fractional integral" of order 1

$$\mathcal{J}_1 f(x) = \int_0^\infty P_t f(x)\,dt,$$

and then the vector "Riesz transform" is

$$\mathcal{R} f(x) = \operatorname{grad}_x \mathcal{J}_1 f(x).$$

By using the explicit formula for the kernel of T_t and by performing the integrations it follows that

$$\mathcal{R}_j f(x) = \lim_{\epsilon \to 0} \int_{\|x-z\|>\epsilon} k_j(x,z) f(z)\,dz,$$

where

$$k_j(x,z) = \int_0^1 \left(\frac{1-r^2}{-\log r}\right)^{1/2} \frac{z_j - r x_j}{(1-r^2)^{(n+3)/2}} e^{-\frac{\|z-rx\|^2}{1-r^2}}\,dr, \quad j = 1, \dots, n.$$

We consider the maximal singular operator defined by

$$R_j^* f(y) = \sup_{\epsilon>0} \Big\| \int_{\|y-z\|>\epsilon} k_j(y,z) f(z)\,dz \Big\|.$$

The main result of **[F-G-Sc]** is the following.

Theorem 3. *There exists a constant $C = C(n)$ such that if $f \in L^1_\gamma(R^n)$ then*

$$\gamma\{x \in R^n : \|\mathcal{R}^*_j f(x)\| > \lambda\} \leq \frac{C}{\lambda}\|f\|_{L^1_\gamma},$$

for all $\lambda > 0$, $j = 1, ..., n$.

The proof of this theorem is based in careful estimates of integrals involving the Gaussian. It also uses some ideas developed by P. Sjögren, **[Sj]**, in particular, the following neighborhood of the diagonal. Given $R > 0$, let

$$N_R = \{(y,z) \in R^n \times R^n : \|y\| \leq R \text{ and } \|z\| \leq R, \\ \text{or } \|z\| \geq R/2 \text{ and } \|y - z\| \leq R/\|z\|\},$$

and $N^y_R = \{z : (y,z) \in N_R\}$. The operator $\mathcal{R}^*_j$ is majorized by the sum of

$$R^*_{j1} f(y) = \sup_{\epsilon > 0} \| \int_{N^y_R \cap \|y-z\| > \epsilon} k_j(y,z) f(z)\, dz\|, \quad \text{and}$$

$$R^*_{j2} f(y) = \int_{R^n \setminus N^y_R} \|k_j(y,z)\|\, \|f(z)\|\, dz,$$

and we proved that each of these operators are of weak-type 1-1 with respect to $\gamma_I(x)\, dx$. This proof contains many ideas that can be carried out to other situations, in particular to the case of a general matrix B.

The weak-type 1-1 constant we obtain in Theorem 3 grows exponentially with the dimension, and it is an open problem to determine if it is possible to obtain a weak-type 1-1 constant that can be bounded independently of the dimension.

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