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INEQUALITIES FOR JACOBIANS: INTERPOLATION TECHNIQUES

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ABSTRACT. Recently new and surprising integrability properties were discovered for the Jacobians of orientation preserving maps. These results have interesting applications to variational problems in elasticity theory, compensated compactness and other areas. We review some of the main results and present new estimates that resolve some open problems in the area. Our methods are based on techniques from Interpolation theory.

§1. INTRODUCTION

Recently it has been discovered that the Jacobians of orientation preserving maps, and other related nonlinear quantities, enjoy better integrability properties than those known for the Jacobians of standard maps. The first results in this direction were obtained by Müller [22], and have been extended in many different directions by a number of authors including Coifman, Lions, Meyer, and Semmes [7], Iwaniec and Sbordone [9], Brezis, Fusco and Sbordone [5], Iwaniec and Lutoborski [11], Iwaniec and Greco [9], and many others. These developments have interesting applications in the study of the equations of non-linear elasticity, variational problems, compensated compactness, etc. At present time the area is growing at a very fast rate and we have not attempted to give complete bibliographical references.

The purpose of this paper is to give a brief account of some of these estimates and point out new complementary results. The new estimates confirm some conjectures proposed in [9]. In our presentation we shall emphasize the important rôle played by some new ideas of interpolation theory. The basic interpolation theorems are well known and commonly used tools in analysis. However, in order to tackle the applications at hand, one needs techniques that have been developed more recently, like the theory of commutator estimates initiated by Rochberg and Weiss [24], and developed further in, among other articles, [14], [6], [17]; and the theory of extrapolation spaces developed by Jawerth and Milman (cf. [15], [16], and [18]).

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In this paper we review some of the main results related to the integrability of Jacobians of orientation preserving maps and announce and give indications of the proofs of several new complementary estimates. We refer to [milman3] for complete details and further applications of the methods outlined here.

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§2. Estimates for Jacobians of orientation preserving maps

Let Ω be a bounded open set in R^n, $f : \Omega \to R^n$ be a *smooth* mapping, we say that f is *orientation preserving* if its Jacobian $Jf = \det Df$ is nonnegative a.e.. A typical assumption on the smoothness of f is that f is in the Sobolev class $W^1_n(\Omega, R^n)$; more generally one assumes that $f \in W^1_{\overrightarrow{p}}(\Omega, R^n)$ for some for some fixed n-uple of exponents $\overrightarrow{p} = (p_1, p_2, ..., p_n)$, $p_i \in (1, \infty)$, $i = 1, ..n$, with $\frac{1}{p_1} + ... \frac{1}{p_n} = 1$, this means that the coordinate functions of $f = (f_1, ...f_n)$, satisfy $f_i \in W^1_{p_i}(\Omega)$. (The class $W^1_n(\Omega, R^n)$ thus corresponds to the case $\overrightarrow{p} = (n, ..., n)$). Observe that by Hadamard's inequality $Jf \leq |\nabla f_1| \cdots |\nabla f_n|$, and therefore by Hölder's inequality we have that $f \in W^1_{\overrightarrow{p}}(\Omega, R^n)$ implies that $Jf \in L^1(\Omega)$. It was recently discovered by Müller [22] that if f is an *orientation preserving* map then one has a better result

Theorem 1. *Let Ω be a bounded open domain in $R^n, n \geq 2$, and let $f : \Omega \to R^n$ be an orientation preserving map in the Sobolev class $W^1_n(\Omega, R^n)$. Then $\forall K \subset \Omega$, compact we have that $Jf \in L(LogL)(K)$.*

This result has since been extended and applied in many different directions by a number of authors. In [7] Coifman, Lions, Meyer and Semmes point out the rôle of the Hardy space $H^1(R^n)$ proving, among other things, the following

Theorem 2. *Let $f : R^n \to R^n$, of class $W^1(R^n, R^n)$, then $Jf \in H^1(R^n)$.*

The relationship with Müller's result is given by a theorem of Stein stating that if $g \geq 0$, *then* $g \in H^1_{loc}(R^n) \Longleftrightarrow g \in L(LogL)_{loc}(R^n)$.

In a different direction Iwaniec and Sbordone give in their paper [12] sufficient conditions to guarantee the *local integrability* of the Jacobian of an orientation preserving map

Theorem 3. *Let $B \subset 3B$ be concentric balls in R^n, and let $f : 3B \to R^n$ be an orientation preserving map such that $f \in \Delta_{s\in 1,n)}(n - s)W^1_s(3B, R^n)$, then $Jf \in L^1(B)$.*

Here we denote by $\Delta_{s\in 1,n)}(n - s)W^1_s(3B, R^n)$ the space of maps such that

$$\sup_{s\in[1,n)} (n - s)\|f\|_{W^1_s(3B,R^n)} < \infty$$

It is shown in [12] that

$$W^1_{L^n(LogL)^{-1}}(3B, R^n) \subset \Delta_{s\in[1,n)}(n-s)W^1_s(3B, R^n) \qquad (1)$$

$$W^1_{L^{n,\infty}}(3B, R^n) \subset \Delta_{s\in[1,n)}(n-s)W^1_s(3B, R^n) \qquad (2)$$

where for a function space X, and a domain Ω, we denote by $W^1_X(\Omega, R^n)$ the class of maps $f = (f_1, ...f_n)$ with components such that $\nabla f_i \in X(3B)$, $i = 1, ...n$.

Corollary 1. *Let $B \subset 3B$ be concentric balls in R^n, and let $f : 3B \to R^n$ be an orientation preserving mapping, then if $f \in W^1_{L^n(LogL)^{-1}}(3B, R^n) \cup W^1_{L^{n,\infty}}(3B, R^n)$ (set theoretic union!), we have $Jf \in L^1(B)$.*

More recently, Brezis, Fusco and Sbordone [5], have shown the following interpolation between Theorem 1 and Corollary 1,

Theorem 4. *Let Ω be a bounded open domain in $R^n, n \geq 2$, and let $f : \Omega \to R^n$ be an orientation preserving map in the Sobolev class $W^1_{L^n(LogL)^\theta}(\Omega, R^n)$, $\theta \in [-1, 0)$. Then $\forall K \subset \Omega$, we have $Jf \in L(LogL)^{\theta+1}(K)$.*

Another related result by Greco and Iwaniec [9] is

Theorem 5. *Let Ω be a bounded open domain in $R^n, n \geq 2$, and let $f : \Omega \to R^n$, be an orientation preserving map in the Sobolev class $W^1_{L^n(LogL)}(\Omega, R^n)$. Then $\forall K \subset \Omega$, we have that $Jf \in L(LogL)^2(K)$.*

In view of this it was conjectured in [9] that Theorem 4 should hold $\forall \theta \in R$. We have in fact recently confirmed this conjecture and obtained the following

Theorem 6. *Let Ω be a bounded open domain of R^n, and let f be an orientation preserving map of class $W^1_{L^n(LogL)^\theta}(\Omega, R^n)$ then $Jf \in L^n(LogL)^{\theta+1}$ (K), $\forall K \subset \Omega$, $\forall \theta \in R$.*

In [11] and [9], Iwaniec and his collaborators have obtained more refined versions of Theorem 4 and Theorem 5, for example in [9] it is shown that

Theorem 7. *Let Ω be a bounded open domain in R^n, $n \geq 2$, $\overrightarrow{p} = (p_1, \cdots, p_n)$, with $p_i \in (1, \infty)$, $i = 1, \cdots, n$, $\frac{1}{p_1} + ... + \frac{1}{p_n} = 1$, and let $W^1_{L^{\overrightarrow{p}}(LogL)}(\Omega, R^n) = \{f/f : \Omega \to R^n, f = (f_1, ...f_n), \nabla f_i \in L^{p_i}(LogL), i = 1, \cdots, n\}$. Then, for an orientation preserving map $f \in W^1_{L^{\overrightarrow{p}}(LogL)}(\Omega, R^n)$ we have that $\forall K \subset \Omega$, $Jf \in L(LogL)^2(K)$. In fact, if Ω is a cube in $R^n, 0 < \sigma < 1$, we have*

$$\int_{\sigma\Omega} Jf(x) \log^2(e + \frac{Jf(x)}{Jf_{\sigma\Omega}})dx \leq \frac{c}{(1-\sigma)\sigma^n} [[\nabla f_1]]_{p_1} ... [[\nabla f_n]]_{p_n}$$

where $[[\nabla f_i]]_{p_i} = \left\{ \int_{\sigma\Omega} |\nabla f_i|^{p_i} \log\left(e + \frac{|\nabla f_i|}{|\nabla f_i|_\Omega}\right) dx \right\}^{1/p_i}$, *and* $H_{\sigma\Omega}$ *denotes the integral mean of* H *over the cube* $\sigma\Omega$.

We have also been able to extend this last result in the direction of Theorem 6 and thus confirm in part a conjecture of [9].

Theorem 8. *Let* Ω *be a cube in* R^n, $n \geq 2$, $\vec{p} = (p_1, \cdots, p_n)$, *with* $p_i \in (1, \infty)$, $\alpha \geq 0$ *and let* $f \in W_{L^{\vec{p}}(LogL)^\alpha}(\Omega, R^n)$ *be an orientation preserving map, then* $Jf \in L(LogL)^{\alpha+1}(\Omega, R^n)$.

The key to our proof of Theorem 8 is the following estimate by Iwaniec and Lutoborski [11]

Theorem 9. *Let* Ω *be a cube in* R^n, $n \geq 2$, $\vec{p} = (p_1, \cdots, p_n)$, *with* $p_i \in (1, \infty)$, $i = 1, \cdots, n$, $\frac{1}{p_1} + ... + \frac{1}{p_n} = 1$, *and let* $f \in W^1_{\vec{p}}(\Omega, R^n)$ *be an orientation preserving map, then*

$$\frac{1}{|\sigma\Omega|} \int_{\sigma\Omega} Jf(x)dx \leq c(\sigma) \prod_{i=1}^{n} \left\{ \frac{1}{|\Omega|} \int_\Omega |f_i(x)|^{p_i} dx \right\}^{1/p_i} \tag{3}$$

We shall not give a complete proof of Theorem 8 here, but below shall give the proof of Theorem 6 for $\alpha \geq 0$, and moreover outline the proof in the case $\alpha < 0$. Using (3), rearrangement inequalities for maximal operators as in the proof of Theorem 6 and Hölder's inequality it is not difficult to complete the details of the proof of Theorem 8. For complete details see [20].

Let us now close this section indicating an application, due to Müller [22], to the study of weak compactness for sequences of the type $\{J(f_j, x)\}_{j \in N}$. Recall the criteria of de La Vallée Poussin stating that for a set K of finite measure, a sequence $\{f_j\}_{j \in N}$ is relatively weakly sequentially compact in $L^1(K)$ if and only there exists a positive function γ defined on R_+ with $\lim_{x \to \infty} \gamma(x)/x = \infty$ such that

$$\sup_j \int_K \gamma(|f_j(x)|)dx < \infty .$$

Thus, Müller [22] proves that if $\{u_j\}_{j \in N}$ is a sequence of orientation preserving mappings, $u_j : \Omega \to R^n$, and $u_j \rightharpoonup u$ (weakly) in $W^1_n(\Omega, R^n)$, then $\forall K \subset \Omega$ compact we have

$$J(x, u_j) \rightharpoonup J(x, u) \text{ weakly in } L^1(K) .$$

In fact under these assumptions a result of Ball [1] asserts that $J(x, u_j) \rightharpoonup^* J(x, u)$ *weak* * in the sense of measures. Therefore by Theorem 1 and de La Vallée Pousin's criteria we conclude.

§3. Commutator methods

In this section we discuss applications of the theory of commutator estimates of interpolation theory to obtain estimates for Jacobians of orientation preserving mappings and their relevance in estimates for perturbed Hodge type decompositions of vector fields. The theory of commutator estimates in interpolation theory has been developed in, among other papers, [24], [14], [17], [6], [21]. This theory has been applied to study the integrability of Jacobians of orientation preserving maps in [13], [9]. It is also interesting to remark here that one of the concrete examples of the abstract theory of commutator estimates, i.e. the commutator theorem of Coifman, Rochberg and Weiss [8], which states that for all the Riesz transforms $R_j = \frac{\partial}{\partial x_j}(-\Delta)^{1/2}, j = 1, \cdots, n$, and for $b \in BMO(R^n)$, the commutators $[R_j, b]$ are bounded on $L^p(R^n), 1 < p < \infty$, is also fundamental in the approach of Coifman, P. Lions, Meyer and Semmes [7] to the theory of compensated compactness.

Remark. Observe that the interpolation theory approach to the theorem of Coifman, Rochberg and Weiss is based on the theory of weighted norm inequalities for singular integrals.

A detailed development of the relationship between the commutator methods of interpolation theory and compensated compactness will be given elsewhere.

In order to state and prove our next result let us recall Kalton's extension [17] of the commutator theorem of Rochberg and Weiss [24]. Let us say that an operator Ω, defined on some L^p space and values on the measurable functions is a *centralizer* , if there exists a function δ: $R_+ \to R_+$, such that $\forall u \in L^\infty$, the commutator $[\Omega, M_u]$, where M_u is the multiplication operator defined by $M_u f = uf$, satisfies $||[\Omega, M_u]f||_p \leq \delta(||f||_p)$. We then say, by abuse of language, that Ω and M_u *commute*. Similarly we say that a centralizer Ω is a *symmetric centralizer* if in addition it *commutes* with all the operators of the form $S_\sigma f = f \circ \sigma$, generated by measure preserving transformations σ.

Theorem 10. *Let $1 \leq p_1 < p < p_2 < \infty$, and let Ω be a symmetric centralizer on L^p, then every operator T of weak types (p_i, p_i), $i = 1, 2$, commutes with Ω.*

In fact Kalton proves a much more general result holds involving rearrangement invariant spaces and in particular the result is valid for $L(p, q)$ spaces.

We also need to review briefly some basic results concerning the Hodge decomposition. We are trying to decompose a vector field $F = \nabla u + H$, where H is a divergence free vector field, $divH = 0$. This is done as follows. Suppose first that $F \in L^p(R^n, R^n)$, then we select u to be such that $\Delta u = divF$, i.e. by letting

$$\nabla u = KF$$

where K is the matrix operator given by

$$K = -\begin{bmatrix} R_1 \otimes R_1 & R_1 \otimes R_2 & \dots & R_1 \otimes R_n \\ \vdots & \vdots & \ddots & \vdots \\ R_n \otimes R_1 & \dots & \dots & R_n \otimes R_n \end{bmatrix}$$

and the $R_j, j = 1, ...n$, are the Riesz transforms. Therefore the decomposition we seek is $F = KF + (I - K)F$, and we have the right control in the L^p norms. For vector fields defined on a smooth domain Ω a similar result holds, and again ∇u is given by a singular integral.

We now consider operators of the form $\Omega_\phi f = f\phi(f)$,where the (generally non-linear) operator ϕ is selected in such a way that Ω_ϕ is a symmetric centralizer, i.e. satisfies the conditions of Kalton's theorem. The methods of [9] yield then the following result

Theorem 11. *(cf. [9]) Let $f : R^n \to R^n$, be a mapping of class $C_0^\infty(R^n, R^n)$, and let Ω_ϕ be a commutator on L^n, and let δ be the function associated to Ω_ϕ by Kalton's theorem, then*

$$\int_{R^n} J(x, f)\phi(|Df|)dx \leq c\delta(\| \, |Df| \, \|_n)\| \, |Df| \, \|_n^{n-1}.$$

Proof. We use the method of [9] and the notation of our previous discussion. Using Hodge decomposition write

$$\Omega_\phi(Df)(x) = Dg(x) + H(x) \tag{4}$$

with $g \in W^{1,s}(R^n, R^n)$, and where $H = (I - K)\Omega_\phi(Df) \in L^s(R^n, GL(n))$ is a divergence free matrix-field, $1< s < \infty$, and since this decomposition is unique we have $(I - K)(Df) = 0$. $I - K$ is bounded on L^p for $1< p < \infty$, and therefore by Kalton's theorem we get

$$\|(I - K)\Omega_\phi(Df) - \Omega_\phi((I - K)Df)\|_n \leq c\delta(\| \, |Df| \, \|_n) \tag{5}$$

thus,

$$\|H\|_n \leq c\delta(\| \, |Df| \, \|_n).$$

Using the notation of differential forms, $J(x, f)dx = df_1 \wedge df_2 \wedge \cdots df_n$, and (4) takes the form

$$\phi(Df)df_k = dg_k + h_k, \ k = 1, \cdots, n \tag{6}$$

where the h_k are differential forms of degree one whose coefficients coincide with the entries of the k-th column of H. Computing using (6) for $k = 1$ we get

$$\int_{R^n} J(x, f)\phi(Df(x))dx = \int_{R^n} dg_1 \wedge df_2 \wedge \cdots \wedge df_n + \int_{R^n} h_1 \wedge df_2 \wedge \cdots \wedge df_n.$$

Given the assumption on the vector field f we see that $\int_{R^n} dg_1 \wedge df_2 \wedge \cdots \wedge df_n = 0$, by Stokes' theorem, and the second integral can be estimated, by Hadamard's inequality and (5), as follows

$$\int_{R^n} h_1 \wedge df_2 \wedge \cdots \wedge df_n \leq \int_{R^n} |H(x)||Df(x)|^{n-1} dx$$
$$\leq |||H|||_n |||Df|||_n^{n-1} \leq c\delta(|||Df|||_n)|||Df|||_n^{n-1}$$

and the desired result follows. □

Example 1. In the case $\phi(f)(x) = \log|f(x)|$, then we can take $\delta(x) = cx$. This is the result of Greco and Iwaniec [9]. We can also deal with

$$\phi(f) = (\log|f|)^{\alpha}, 0 \leq \alpha \leq 1,$$

$$\phi(f) = (\log|f|)^{\alpha} |\log\ r_f(|f(x)|)|^{\beta}, 0 \leq \alpha, \beta \leq \alpha + \beta \leq 1$$

where r_f is the rank function defined for $t \geq 0$, by

$$r_f(t) = |\{s : |f(s)| > |f(t)|, or\ s \leq t \text{ and } |f(s)| = |f(t)|\}|$$

For other examples of symmetric centralizers see [17].

Another commutator result relevant in this theory was obtained by Iwaniec and Sbordone [12] and applied to Hodge decompositions in [12] and in [9].

Theorem 12. *(cf. [12]) Let $1 < r_i < \infty$, $i = 1, 2$, $r \in [r_1, r_2]$ and suppose that $T : L^r(\Omega, E) \to L^r(\Omega, E)$, where E is a Hilbert space. Then $\forall \epsilon$ such that $\frac{r}{r_2} - 1 \leq \epsilon \leq \frac{r}{r_1} - 1$, we have*

$$\|TS_{\epsilon} - S_{\epsilon}T\|_{\frac{r}{1+\epsilon}} \leq c_r|\epsilon|\|f\|_r$$

where $S_{\epsilon}f = \left(\frac{|f|}{\|f\|_p}\right)^{\epsilon} f$, and c_r is independent of f.

The following application of Theorem 12 to Hodge decomposition is important in the study of the integrability properties of the Jacobian transformation as well as in other problems (cf. [12]).

Theorem 13. *Let $B = B(a, R)$ be a ball in R^n and let $f \in W_r^1(R^n)$, $r > 1$. Then, for each $\epsilon \in (1 - r, 1)$ the vector field $|\nabla f|^{-\epsilon}\nabla f \in L^{\frac{r}{1-\epsilon}}(R^n)$ can be decomposed as*

$$|\nabla f(x)|^{-\epsilon}\nabla f(x) = \nabla g(x) + H(x), \ a.e.\ x \in B$$

where $g \in W^1_{\frac{r}{1-\epsilon}}(R^n)$ and $H \in L^{\frac{r}{1-\epsilon}}(R^n, R^n)$ is divergence free and such that

$$\|\ |H|\ \|_{\frac{r}{1-\epsilon}} \leq c|\epsilon| \cdot \|\ |\nabla f|\ \|_r^{\frac{1-\epsilon}{r}}$$

We now present an extension of Theorem 12, obtained in [19], using the real method of interpolation. We start by giving a brief review of the relevant definitions.

We generally assume that the reader is familiar with the basic definitions of interpolation theory, and we refer to [4] for background information. Let $\bar{A} = (A_0, A_1)$ be a Banach pair, let $a \in \Sigma(\bar{A}) = A_0 + A_1$, and recall that the E functional of a is defined, for $t > 0$, by

$$E(t, a; \bar{A}) = \inf_{\|a_1\|_{A_1} \leq t} \{\|a_0\|_{A_0} : a = a_0 + a_1\}$$

The corresponding interpolation spaces $\bar{A}_{\theta,q;E}$, $0 < \theta < \infty$, $0 < q \leq \infty$, are defined using the quasi-norms

$$\|f\|_{\bar{A}_{\theta,q;E}} = \{\int_0^\infty [t^\theta E(t, f, \bar{A})]^q \frac{dt}{t}\}^{1/q} . \tag{7}$$

Let us write $D_E(t; \bar{A}) = D_E(t)a = a_0(t)$, for an almost optimal decomposition, that is a decomposition such that

$$E(t, a; \bar{A}) \approx \|D_E(t)a\|_{A_0} . \tag{8}$$

Then, we define the corresponding operators Ω_E associated with this method, by

$$\Omega_E a = \int_1^\infty D_E(t)a\frac{dt}{t} - \int_0^1 (I - D_E(t))a\frac{dt}{t} . \tag{9}$$

One can define, similarly, operators associated with other methods of interpolation like the K and J methods, or the complex method. The main results of [24] and [14] state that if T is a bounded operator $T : \bar{A} \to \bar{B}$, and F denotes any of these methods of interpolation, then there exists a constant c(F) such that if we let $[\Omega_F, T] = \Omega_{F(\bar{B})}T - T\Omega_{F(\bar{A})}$, then

$$\|[\Omega_F, T]f\|_{F(\bar{B})} \leq c\|f\|_{F(\bar{A})}.$$

In order to state the corresponding analogs of the result of [12] we consider variants of the Ω operators. Let $\alpha \in (-1, 1)$, $\alpha \neq 0$, and define

$$\Omega_{E,\alpha} a = \Omega_\alpha a = \alpha \left(\int_1^\infty D_E(t)a\, t^\alpha \frac{dt}{t} - \int_0^1 (I - D_E(t))a\, t^\alpha \frac{dt}{t}\right) . \tag{10}$$

Then we have the following

Theorem 14. *(cf. [19]) Let $\bar{A}$ and $\bar{B}$ be a Banach pairs, $T : \bar{A} \to \bar{B}$ be a bounded operator, then there exists a constant $c \geq 0$ such that if $\theta + \alpha > 0$,*

$$\|[\Omega_\alpha, T]f\|_{(B_0,B_1)_{\theta/(\alpha+1),q;E}} \leq \frac{c}{\theta}|\alpha|(2c_\alpha)^{\theta/(\alpha+1)}(\alpha+1)^{1/q}\|f\|_{(A_0,A_1)_{\theta+\alpha,q;E}} .$$

In the case of the $L^p(E)$ spaces it is not hard to check by computation that we recover Theorem 12. As an application we now give an extension of Theorem 13 to the setting of $L(p,q)$ spaces. Observe that the proof of Theorem 13 depends only on two steps:

(i) the boundedness of singular integrals on L^p spaces (see the discussion above on Hodge decompositions), and

(ii) Theorem 12

Both of these steps can be extended to the setting of $L(p,q)$ spaces. In fact, the singular integral operators are bounded on $L(p,q)$ spaces, by interpolation, while a special case of Theorem 14 implies that Theorem 14 holds in the setting of $L(p,q)$ spaces. We have thus obtained the following.

Theorem 15. *Let $B = B(a,R)$ be a ball in R^n and let $f \in W^1_{p,q}(R^n)$, $p > 1.1 \leq q \leq \infty$. Then, for each $\epsilon \in (1-p,1)$ the vector field $|\nabla f|^{-\epsilon}\nabla f \in L^{\frac{p}{1-\epsilon},q}(R^n)$ can be decomposed as*

$$|\nabla f(x)|^{-\epsilon}\nabla f(x) = \nabla g(x) + H(x), \ a.e.\ x \in B$$

where $g \in W^1_{\frac{p}{1+\epsilon},q}(R^n)$ and $H \in L^{\frac{p}{1-\epsilon},q}(R^n,R^n)$ is divergence free and such that

$$\|\,|H|\,\|_{\frac{p}{1-\epsilon},q} \leq c|\epsilon| \cdot \|\,|\nabla f|\,\|^{1-\epsilon}_{p,q}$$

§4. An Application of Extrapolation Spaces

In the last few years a theory of extrapolation spaces has been emerging through the work of Jawerth and Milman (cf. [15], [16], and [19]). One of the main applications of the theory is to provide a framework to study the limiting spaces and inequalities of classical analysis. It is not possible here to go into details for which I must refer the reader to the quoted papers. I simply wish to point out that a simple application of extrapolation allows one to compute the spaces that appear in Theorem 3. Corollary 1 follows readily from this characterization.

Theorem 16. *$u \in \Delta_{\in[1,n)}(n-s)W^1_s(3B,R^n)$ if an only if the distributional derivatives of its components, $\frac{\partial u_j}{\partial x_i}$, are such that*

$$sup_{t\in(0,1)}\rho(t)\left[\sum_{i,j}\int_0^{t^{\frac{n}{n-1}}}(\frac{\partial u_j}{\partial x_i})^*(s)ds + t\{\int_{t^{\frac{n}{n-1}}}^1[(\frac{\partial u_j}{\partial x_i})^*(s)]^n ds\}^{1/n}\right] < \infty$$

where $\rho(t) = t^{-1}(1 + \log\ 1/t)^{-1}$.

In closing this brief section we just wish to remark that extrapolation methods can be very useful in dealing with the type of spaces considered in this article. For a reader who may be interested in the abstract methods per se we mention that the papers quoted above also include a number of open problems in the theory.

§5. Rearrangement Inequalities: Müller's Theorem

The main tool in Müller's work is an estimate of the Maximal function of Hardy-Littlewood of the Jacobian of an orientation preserving map. We now rewrite this estimate in a form suitable for our needs here.

Let us start by recalling that for $u \in L^1_{loc}(R^n)$ the maximal function of Hardy-Littlewood M is defined by

$$Mu(x) = \sup_{Q \ni x} \frac{1}{|Q|} \int_Q |u(y)|dy, x \in R^n$$

where the supremum is taken over all the cubes Q with sides parallel to the coordinate axes. As it is well known the maximal operator of Hardy-Littlewood is bounded on $L^p(R^n), 1 < p \leq \infty$, and is of weak type (1,1). Let B be a fixed ball and consider the maximal operator M acting on functions supported on B, then we also have the following well known result of Stein (cf. [25])

$$M : LLogL(B) \to L^1(B)$$

In fact Stein's theorem is actually a characterization of the space $LLogL(B)$: $u \in L(LogL)(B)$ *iff* $Mu \in L^1(B)$. It will be important for our purposes to give a quantitative form to these statements using the following estimate by Herz (cf. [23], and [3])

$$(Mu)^*(t) \approx u^{**}(t) = \frac{1}{t}\int_0^t u^*(s)ds,\ t > 0\,, \tag{11}$$

where we have used * to denote the non-increasing rearrangement of a function, $\approx$ to denote equivalence within absolute multiplicative constants (i.e. independent of u, or t). In fact, integrating (11) we obtain

$$\int_0^{|B|} (Mu)^*(t)dt \approx \int_0^{|B|} (\frac{1}{t}\int_0^t u^*(s)ds)dt = \int_0^{|B|} u^*(t)\log\frac{|B|}{t}dt\,. \tag{12}$$

Observe that the left hand side of (12) is exactly $\int_B Mu(x)dx$, while the right hand side is an equivalent (Lorentz type) norm for the Orlicz space $LLogL(B)$ (cf. [26]). More generally we have the following relationships between the usual

Orlicz and Lorentz norms for the $L^p(LogL)^\alpha$ spaces. For $\alpha \in R, 1 \leq p < \infty$ we consider the following functionals on the $L^p(LogL)^\alpha(B)$ spaces

$$\|f\|_{L^p(LogL)^\alpha} = \left\{ \int_0^{|B|} f^{**}(s)^p \left(1 + \log \frac{|B|}{s}\right)^\alpha ds \right\}^{1/p} \tag{13}$$

$$\|f\|^*_{L^p(LogL)^\alpha} = \left\{ \int_0^{|B|} f^*(s)^p \left(1 + \log \frac{|B|}{s}\right)^\alpha ds \right\}^{1/p} \tag{14}$$

In fact the expression (14) defines for $\alpha \geq 0$, $1 \leq p < \infty$, an equivalent norm on $L^p(LogL)^\alpha(B)$, while for $\alpha < 0$, it is a quasi-norm defining a topology that coincides with the usual one provided by the corresponding Orlicz norms. The expression (13) defines a norm for $\alpha \in R, 1 \leq p < \infty$. Moreover, for $1 < p < \infty, \alpha \in R$, we have

$$\|f\|^*_{L^p(LogL)^\alpha} \approx \|f\|_{L^p(LogL)^\alpha} \tag{15}$$

(cf. [2] for an extensive treatment).

The case $p = 1$, as can it be seen from (12) must be dealt with separately. Thus, a slight modification of the argument leading to (12) allows us to obtain further integrability results. In fact multiplying both sides of (11) by $\phi_\alpha(t) = \left(1 + \log \frac{|B|}{t}\right)^\alpha$, $\alpha \geq 0$ (in the previous argument we used $\alpha = 0$!), and integrating leads to

$$\|Mu\|_{L(LogL)^\alpha(B)} \approx \|u\|_{L(LogL)^{\alpha+1}(B)}, \alpha \geq 0 \tag{16}$$

For $\alpha < 0$ we have moreover,

$$\|Mu\|_{L(LogL)^\alpha(B)} \approx \begin{cases} \|u\|^*_{L(LogL)^{\alpha+1}(B)} & \alpha \neq -1 \\ \|u\|^*_{L(LogLogL)(B)} & \alpha = -1 \end{cases} \tag{17}$$

We also point out that these results admit *local* versions. Indeed, let us state a local version suitable for our purposes here. Let Q be a fixed cube R^n, with sides parallel to the coordinate axes, and consider the localized maximal operator of Hardy-Littlewood defined for $f \in L^1(Q)$ as follows

$$M_Q f(x) = \sup_{Q' \ni x, Q' \subset Q} \frac{1}{|Q'|} \int_{Q'} |f(y)| dy, x \in Q$$

Then, we can repeat the analysis above for the localized maximal operators of Hardy-Littlewood, since the following analog of (11) holds,

$$(M_Q f)^*(t) \approx (f\chi_Q)^{**}(t), 0 < t \leq |Q|, \tag{18}$$

with constant of equivalence independent of f. In fact the proof of (18) is the same as the one for (11). Since this is rather easy we indicate the proof for completeness sake. Let then $f \in L^1(Q)$, extend it to be zero on Q^c and call this new function $\hat{f}$. It is clear that $M_\Omega f \leq M\hat{f}$, and we deduce that $(M_Q f)^*(t) \leq c(f\chi_Q)^{**}(t)$. On the other hand for $t > 0$ let $B = \{x \in Q : M_Q f(x) > (M_Q f)^*(t)\}$. Suppose that B is not empty, otherwise there is nothing to prove, then B is an open set relative to Q and we can get a disjoint cover of $B \subset \bigcup_{j=1}^\infty Q_j$, where the Q_j are pair wise disjoint, $Q_j \subset Q, |B| \approx \sum_{j=1}^\infty |Q_j|$, and moreover $Q_j \cap (Q\backslash B) \neq \phi$,$\forall j = 1, \cdots$ (cf. [3]). Define $b = f\chi_B = \sum_{j=1}^\infty f\chi_{Q_j}$, $g = f\chi_{Q\backslash B}$, so that $f = b + g$. Consider the K functional for the pair $(L^1(Q), L^\infty(Q))$, then, as it is well known, we have (cf. [4])

$$\begin{aligned} t\,(f\chi_Q)^{**}(t) &= K(t, f, L^1(Q), L^\infty(Q)) \\ &= \inf\{\|h_1\|_{L^1(\Omega)} + t\|h_\infty\|_{L^\infty(\Omega)} : f = h_1 + h_\infty\} \end{aligned}$$

thus, comparing with the previously constructed decomposition we must have

$$(f\chi_Q)^{**}(t) \leq t^{-1}\|b\|_{L^1(\Omega)} + \|g\|_{L^\infty(\Omega)}$$

Now, by construction

$$\|g\|_{L^\infty(\Omega)} \leq (M_Q f)^*(t)$$

To compute the $\|b\|_{L^1(Q)}$ let us select elements x_j from the non-empty sets $Q_j \cap (Q\backslash B), j = 1, ..,$then

$$\frac{1}{|Q_j|}\int_{Q_j} |f(y)|dy \leq M_Q f(x_j) \leq (M_Q f)^*(t)$$

consequently, using that the sum of the measures of the cubes $\{Q_j\}_{j=1}^\infty$ add up to the measure of B (up to a fixed factor), and the fact that $|B| \leq t$, we obtain

$$t^{-1}\|b\|_{L^1(\Omega)} \leq t^{-1}\sum_{j=1}^\infty \int_{Q_j} |f(y)|dy = t^{-1}\sum_{j=1}^\infty \frac{|Q_j|}{|Q_j|}\int_{Q_j} |f(y)|dy \leq c\,(M_Q f)^*(t)$$

and the result follows.

After all this preliminary work, let us now state

Theorem 17. *(cf. [22]) Let Ω be an open bounded set in R^n and let f be an orientation preserving map in the class $W_n^1(\Omega, R^n)$, and let $Q \subset \Omega$, a cube with sides parallel to the coordinate axes, and let $\tilde{Q} = Q/2$ be the cube concentric with Q and with sidelenght equal to half the sidelenght of Q. Then,*

$$M_{\tilde{Q}}(J(f,.))(x) \leq (M_Q(|adjDf|)(x))^{\frac{n}{n-1}} , \tag{19}$$

where for a matrix H, $adjH$ is defined so that $H\ adjH = (\det H)\ I$.

We are now ready to prove our main result in this section

Theorem 18. *Let Ω be an open bounded set in R^n and let f be an orientation preserving map in the class $W^1_{L^n(LogL)^\alpha}(\Omega, R^n)$, $\alpha \geq 0$, then $\forall K \subset \Omega$, $Jf \in L(LogL)^{\alpha+1}(K)$.*

Proof. Let Q be any cube with sides parallel to the coordinate axes contained in Ω, and let $\tilde{Q}$ be defined as in the statement of Theorem 17. Observe that taking decreasing rearrangements in (19) preserves the inequality therefore, combining with (18), gives for a suitable absolute constant c,

$$\left(J(f,.)\chi_{\tilde{Q}}\right)^{**}(t) \leq c\left\{(\chi_Q|adjDf|)^{**}(t)\right\}^{\frac{n}{n-1}}$$

Combining this estimate with Hadamard's inequality, we readily get

$$\left(J(f,.)\chi_{\tilde{Q}}\right)^{**}(t) \leq c\left\{\left(\chi_Q|Df|^{n-1}\right)^{**}(t)\right\}^{\frac{n}{n-1}}$$

This estimate is valid for all orientation preserving maps f in the class $W^1_n(\Omega, R^n)$. Note that for $\alpha \geq 0$, we have $W^1_{L^n(LogL)^\alpha}(\Omega, R^n) \subset W^1_n(\Omega, R^n)$. Consequently, using the local versions of (16) and (17), we see that for $\alpha \geq 0$,

$$||Jf||_{L(LogL)^{\alpha+1}(\tilde{Q})} \leq c\left\{||\,|Df|\,||_{L^n(LogL)^\alpha(Q)}\right\}^n$$

and the result follows. □

The previous result encompasses Müller's Theorem 1 which corresponds to the case $\alpha = 0$, while if $\alpha = 1$, we obtain the Greco-Iwaniec Theorem 5. The remaining cases seem to be new.

§6. Rearrangement Inequalities and the Iwaniec-Sbordone Theorem

In order to complete the proof of Theorem 6 we briefly indicate how to deal with the case $\alpha < 0$. Instead of using Müller's Theorem 17 we use the maximal inequalities of Iwaniec and Sbordone [12] incorporating the relevant parts of the analysis of [5] where the case $\alpha \in [-1, 0)$ is treated. Assume then that $\alpha < -1$. In [12] it is shown that if Ω is a bounded open set in R^n, $\exists\, c > 0$, such that for all orientation preserving maps $f \in W^1_{n-\epsilon}(\Omega, R^n)$,$\forall Q \subset \Omega$, $\tilde{Q}$ defined as before, we have

$$M_{\tilde{Q}}f(x) \leq c\left\{M_Q(|Df|^{\frac{n^2}{n+1}})\right\}^{\frac{n+1}{n}}, \tag{20}$$

as long as we can guarantee that

$$\lim_{\epsilon\to 0}|\epsilon|\int_Q |Df|^{n-\epsilon}dx = 0\,. \tag{21}$$

It is easy to check, as in [5] that indeed (21) holds for $f \in W^1_{L^n((LogL)^{-\alpha}} \subset W^1_{n-\epsilon}(\Omega, R^n), \forall \epsilon \in (0, n-1)$. Taking rearrangements in (20) we arrive to the estimate

$$\left(f\chi_{\tilde{Q}}\right)^{**}(t) \leq c\left\{\left(\chi_Q|Df|^{\frac{n^2}{n+1}}\right)^{**}(t)\right\}^{\frac{n}{n+1}};$$

consequently

$$\int_0^1 \left(f\chi_{\tilde{Q}}\right)^{**}(t)(1+\log\frac{1}{t})^\alpha dt \leq \int_0^1 c\left\{\left(\chi_Q|Df|^{\frac{n^2}{n+1}}\right)^{**}(t)\right\}^{\frac{n}{n+1}}(1+\log\frac{1}{t})^\alpha dt$$

$$\leq c\int_0^1 \left(\chi_Q|Df|^{\frac{n^2}{n+1}}\right)^{*\frac{n}{n+1}}(t)(1+\log\frac{1}{t})^\alpha dt \text{ (since } \frac{n}{n+1} > 1)$$

$$\leq c\int_0^1 \left(\chi_Q|Df|^n\right)^*(t)(1+\log\frac{1}{t})^\alpha dt$$

Finally we conclude with an appeal to (17). □

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