THE NEED OF IMPATIENCE FOR GENERAL EXISTENCE THEOREMS FOR EQUILIBRIA AND PARETO OPTIMA IN MATHEMATICAL ECONOMIC SYSTEMS

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ABSTRACT. We consider a Mathematical Economic System (M.E.S.) of Arrow-Debreu type,

\[ E = \{X^i, Y^i, \alpha_{ij}, X^j, Y^j \subset X = l^\infty(E), \] 

where \( E = E(\tau_0) \) is a locally convex \( \mathbb{R} \)-vector space. We assume \( E \) barreled, later on we assume \( E \) a reflexive Banach Latice.

Example. \( L^p, L^p(\mu), 1 \leq p \leq \infty \) Hilbert spaces.

Question: Which topologies \( \tau \) in \( X \) are suitable for Economic Models?

§1. MYOPIC ECONOMIC AGENTS

Let:

\[ l^\infty(E) = \{z = (z_0, z_1, \ldots, z_n, \ldots), \{z_n\} \text{ bounded in } E(\tau_0)\}. \]

\[ z = (z_0, \ldots, z_n, 0, \ldots) + (0, \ldots, 0, z_{n+1}, z_{n+2}, \ldots) = uz^{(n)} + z^{(n)} \]

\[ z^{(n)} = n - \text{tail of } z. \]

Definition 1.1. \( \succ \) is myopic iff \( x \succ z \) implies \( x \succ z + w^{(n)} \) for \( n > n_0(x, z, w). \)

Definition 1.2. A topology \( \tau \) in \( X = l^\infty(E) \) is myopic iff all \( \tau \)-continuous preferences are myopic.
Example 1.3. Let $E$ be a Banach space,
\[
\tau_\infty : \|z\| := \sup_n \|z_n\|, \text{ topology of uniform convergences},
\]
\[
\beta : \|z\|_\lambda := \sup_n \|\lambda z_n\|, \lambda \in c_0, \text{ strict topology},
\]
\[
\tau_\pi : \|z\|_n := \sup(\|z_0\|, \ldots, \|z_n\|), \text{ product topology},
\]
then $\beta, \tau_\pi$ are myopic, $\tau_\infty$ is not myopic.

Definition 1.4. A topology $\tau$ in $X$ is regular, if $i_n : E \rightarrow X(\tau) = l^\infty(E), x \mapsto (0, \ldots, 0, x, 0, \ldots)$ is continuous.

$\tau_\infty, \beta, \tau_\pi$ are regular.

Proposition 1.5.

a) $\tau$ is myopic $\iff z^{(n)} \to 0$ for any $z \in X$.
b) There exist a strongest regular and myopic topology $\tau_{SM}$ on $X = l^\infty(E)$, $E$ a locally convex space.

Theorem 1.6.

a) $X(\tau_{SM})' =: X' = l_0^\infty(E') = \{u = (u_n) \mid \|u\|_B = \sup_{x \in B} \|u_n(x)\| < \infty \text{ for each bounded } B \subset X(\beta') = X'\}$ for all Banach and Fréchet spaces $E$.
b) $\tau_\infty \supset \tau_{MA} \supset \tau_{SM} \supset \beta \supset \sigma = \sigma(X, X')$, $\tau_{MA} \supset \tau_{MA}(X, X') = \text{ Mackey-topology}$. All these topologies have the same class of bounded sets.
c) $\tau_1'$ defined by $|u|_B, B$ bounded in $E$, is the strong topology $\tau_b$ on $X'$.
d) $X'' = (X \tau_0 = l_0^\infty(E'')) = \{x = (x_n), x_n \in E'', \text{ equicontinuous}\}$.
e) $X'' = X$ if $E$ is a reflexive, barreled space, $\sigma(X, X') = \sigma(X'', X')$.

Theorem 1.7. If $E$ is a reflexive Banach lattice, then $\tau_{MA} = \tau_{SM} = \beta, \tau_{MA}$ the Mackey topology $\tau_{MA}(X, X')$.

Theorem 1.8 (N. Flügel, 1989. Diplomarbeit, Münster). $\tau_{MA} = \tau_{SM}$ for all barreled spaces $E$.

Corollary 1.9. $\tau_\infty \supset \tau \supset \sigma$ implies $\tau$ is myopic iff $\tau$ is admissible for the duality $(X, X') \rightarrow X(\tau)' = X' = l_0^\infty(E')$.

If $E = \{X^i, i, a^i, Y^j, \alpha_{ij}\}$ is a Private Ownership Economic System $X^i, Y^j \subset X = l^\infty(E)$ we use the

Definition 1.10.

a) $(x^i, y^j)$ is a state of $E$ if $x^i \in X^i, y^j \in Y^j$ for all $i, j$.
b) A state $(x^i, y^j)$ is attainable iff $\sum x^i = \sum y^j + \sum \alpha^i$ (demand = supply on all markets).
c) A state $(x^i, y^j)$ is individual rational iff $x^i \gtrless_i a^i$ for all $i$.
d) A state $(x^i, y^j)$ is Pareto-optimal if it is attainable and for all attainable states $(\hat{x}^i, \hat{y}^j)$ it holds: $\forall \hat{x}^i \gtrless_i x^i$ implies $\hat{x}^i \sim_i x^i$. 
e) A state \((x^i, y^i)\) is an equilibrium at prices \(p_*\), if (1) \(y^i\) maximizes \(p_* \cdot y\) in \(Y^j\). (2) \(x^i\) maximizes \(\succeq_i\) in the budget set \(B^i(p_*): B^i(p_*) := \{x \in X^i : p \cdot x \leq w^i(p) \equiv p \cdot a^i + \sum \alpha_{ij} \sup p \cdot y, y \in Y^j\}\) (3) \((x^i, y^i)\) is attainable: \(\sum x^i = \sum y^i + \sum a^i\) (4) \(p_* \neq 0\).

f) A state \((x^i, y^i)\) is a quasiequilibrium at \(p_*\) if (1), (3), (4) are true and (2') \(x^i\) maximizes \(\succeq_i\) on the budget set \(B^i(p_*)\) or

\[ w^i(p_*) := \min(p \cdot x, x \in X^i) \text{ and } x^i \in B^i(p_*). \]

g) If \(w^i(p_*) = \min(p \cdot x, x \in X^i)\) we say: consumer \(i\) is in the minimum – wealth situation.

§2. EXISTENCE THEOREMS (ARAUJO (1985) FOR \(E = \mathbb{R}\))

For the Mathematical Economic System \(\mathcal{E}\) we assume:

A0 \(\mathcal{E}\) is a reflexive Banach Lattice, \(X^i = X_+ = l^\infty(E), a^i \in X_+.\)

A1 \(\succeq_i\) is total and transitive (= preference relation)

A2 \((\tau): \succ_i\text{ is }\tau\text{ continuous: } M^i_+(x) = \{x \in X^i : z \succ_i x\} \text{ and } M^i_-(x) = \{x \in X^i : x \succ_i z\}\) are \(\tau\)-closed sets.

A5 \(\succeq_i\) is convex: \(M^i_+(x)\) is a convex set for all \(x, i\).

\(B(\tau):\)

(i) \(0 \in Y^j, Y^j\) is \(\tau\)-closed, convex (or \(\overline{Y} = \sum Y^j\) is a \(\tau\)-closed, convex set).

(ii) \(Y^j \cap \{X_+ - \sum_{v \neq j} Y^v - a\}\) is \(\tau\)-bounded, \(a = \sum a^i\).

Theorem 2.1. If \(\tau_{MA} \supset \tau_{SM} \supset \tau_1, \tau_2 \supset \sigma\), every M.E.S. that satisfies A0, A1, A2(\(\tau_1\)), A5, \(B(\tau_2)\) has individual, rational, Pareto-optimal states \((x^i, y^i)\).

The existence of equilibria needs some more assumptions:

A4* : \(\forall i \exists z^i_0 \in X^i : z^i_0 \leq a^i, \forall x \in X^i : x + \lambda z^i_0 \succ_j xj = 1, 2, \ldots, k.\) (\(z^i_0\) is universally desired, includes “non-station”).

P : \(\exists y^i_0 \in Y^j : s_0 = \sum y^i_0 + \sum a^i = y_0 + a \gg 0 \text{ and } \exists \lambda_0 \geq 1 : \sum x^i = \sum y^i + a \leq \lambda_0 s_0 \text{ for all attainable states } (x^i, y^i).\) (\(z \gg 0\) means \(p \cdot z \gg 0\) for all positive linear functionals \(0 \neq p \in X')\)

M : \(x \geq z \rightarrow x \succeq_i z, i = 1, 2, \ldots, k.\)

Theorem 2.2. If \(\tau_{MA} \supset \tau_{SM} \supset \tau_1, \tau_2 \supset \sigma\), every M.E.S. that satisfies A0, A1, A2(\(\tau_1\)), A4*, A5, \(B(\tau_2)\), P, M has an equilibrium \((x^i, y^i, p_*)\) where \(p_*\) is a continuous, positive linear functional on a subspace \(L \subset X\) which contains all attainable \(x^i, y^i\) and \(L\) is a Banach sublattice of \(X\).

Idea of Proof. Approximation by finite dimensional subsystems

\[ \mathcal{E}_F = \{F \cap X^i, \succeq_i, a^i, F \cap Y^j, \alpha_{ij}\}, F \ni a^i, z^i_0, y^i_0 \]
with a basis of positive elements. The Debreu’s Existence Theorem of 1962 can be applied so that $E_F$ has a quasiequilibrium $(x_F^*, y_F^*, p_F)$. $M$ implies $p_F \geq 0$.

$A4^*$ implies $(x_F^*, y_F^*)$ is an equilibrium at $p_F$. $B(\tau_2)$ implies: $(y_F^*)_F$ is $\tau_2$-bounded; hence, $\sigma$-bounded and $\sigma$-relative compact.

$$\{x_F^i\}_F$$ is $\tau_2$-bounded; therefore, $\beta$-bounded, and so, implies $(x_F^i)_F$ is $\beta$-bounded since $0 \leq x_F^i \leq \sum x_F^i$ and $\beta$ is a solide topology. $(\sum x_F^i)_F$ is bounded implies $\sigma$-bounded and $\sigma$-relative compact. There exist accumulation points $(x^i_*, y^j_*)$ of $(x_F^i, y_F^j)$. $X^i$ is convex, $\tau_1$-closed, therefore $\sigma$-closed and $x^i_* \in X^i$; $Y^j$ is $\tau_2$-closed, convex, therefore $\sigma$-closed and $y^j_* \in Y^j$; $(x^i_*, y^j_*)$ is an attainable state.

There exists a $\sigma$-bounded and $\sigma$-closed, solide, absolutely convex set $B$, that contains all attainable $x^i_*$, $y^j_*$ and supplies $\sum y^j + a = s$. $L = X_B = \bigcup B$ is a Banach lattice with $B$ as unit ball.

Normalization of the price vector $p_F : p_F(s_0) = 1, ||p_F|| \leq \lambda_0$. By Alaoglu: $p_F \rightharpoonup p_*$ pointwise on $L$; $p_*(s_0) = \lim p_F(s_0) = 1, ||p_*|| \leq \lambda_0$. Maximal of $x^i_*$ and $y^j_*$ with respect to $p_*$ can be proved as in Araujo’s case $E = \mathbb{R}$.

§3. Inverse Results

We assume $\tau_\infty \supset \tau_1, \tau_2 \supset \sigma = \sigma(X, X')$, but $\tau_1, \tau_2$ not necessarily myopic. If $\tau_1$ or $\tau_2$ is not myopic we can construct counterexamples of M. E. S.’s, such that all conditions of Theorem 2.1 resp 2.2 are satisfied, but individual rational, Pareto-Optimal states do not exist. We get the following:

**Theorem 3.1.** If $E$ is a reflexive $B$-lattice with separable dual $E'$, $\tau_1$ and $\tau_2$ are locally convex topologies on $X = L^\infty(E)$, $\tau_\infty \supset \tau_1, \tau_2 \supset \sigma$, such that every M. E. S. that satisfies $A0, A1, A2(\tau_1), A5, B(\tau_2)$ has an individual rational, Pareto-optimal state, then $\tau_{MA} = \tau_{SM} \supset \tau_1, \tau_2; \tau_1, \tau_2$ are myopic.

**Theorem 3.2.** If $E$ is a reflexive separable $B$-lattice with separable dual $E'$, $\tau_1$ and $\tau_2$ locally convex topologies on $X = L^\infty(E)$, such that every M. E. S. that satisfies $A0, A1, A2(\tau_1)$, $A4^*$, $A5$, $B(\tau_2)$, $P, M$, has an equilibrium, then $\tau_{MA} = \tau_{SM} \supset \tau_1, \tau_2; \tau_1, \tau_2$ are myopic topologies.

“Impatience”, as formalized by myopicity, of the economic agents is necessary for general existence results.

**References**


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