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THE NEED OF IMPATIENCE FOR GENERAL EXISTENCE THEOREMS FOR EQUILIBRIA AND PARETO OPTIMA IN MATHEMATICAL ECONOMIC SYSTEMS

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ABSTRACT. We consider a Mathematical Economic System (M.E.S.) of Arrow-Debreu type,

$$\mathcal{E} = \{X^i, \succsim i, a^i, \ , Y^j, \alpha_{ij}\}, X^i, Y^j \subset X = l^\infty(E),$$

where $E = E(\tau_0)$ is a locally convex $\mathbb{R}-$ vector space. We assume E barreled, later on we assume E a reflexive Banach Latice.

Example. l^p, $L^p(\mu), 1 \le p \le \infty$ Hilbert spaces.

Question: Which topologies τ in X are suitable for Economic Models ?

§1. MYOPIC ECONOMIC AGENTS

Let :

$$l^\infty(E) = \{z = (z_0, z_1, \ldots, z_n \ldots), \{z_n\} \text{ bounded in } E(\tau_0)\}.$$
$$z = (z_0, \ldots, z_n, 0 \ldots) + (0, \ldots, 0, z_{n+1}, z_{n+2}, \ldots) = uz^{(n)} + z^{(n)}$$
$$z^{(n)} = n - \text{tail of } z.$$

Definition 1.1. $\succsim$ is *myopic* iff $x \succ z$ implies $x \succ z + w^{(n)}$ for $n > n_0(x, z, w)$.

Definition 1.2. A topology τ in $X = l^\infty(E)$ is *myopic* iff all τ-continuous preferences are *myopic*.

Example 1.3. Let E be a Banach space,

$$\tau_\infty : \|z\| := \sup_n \|z_n\|, \text{ topology of uniform convergences,}$$

$$\beta : \|z\|_\lambda := \sup_n \|\lambda_n z_n\|, \ \lambda \in c_0, \text{ strict topology,}$$

$$\tau_\pi : \|z\|_n := \sup(\|z_0\|, \dots, \|z_n\|), \text{ product topology,}$$

then β, τ_π are *myopic* , τ_∞ is not *myopic*.

Definition 1.4. A topology τ in X is *regular*, if $i_n : E \to X(\tau) = l^\infty(E)$, $x \mapsto (0, \dots, 0, x, 0, \dots)$ is continuous.

τ_∞, β, τ_π are regular.

Proposition 1.5.

a) τ is myopic $\iff z^{(n)} \xrightarrow{\tau} 0$ for any $z \in X$.

b) There exist a strongest regular and myopic topology τ_{SM} on $X = l^\infty(E)$, E a locally convex space.

Theorem 1.6.

a) $X(\tau_{SM})' =: X' = l_b^1(E') = \{u = (u_n) \mid \|u\|_B = \sum_0^\infty \sup_{x \in B} |u_n(x)| < \infty$ for each bounded $B \subset X(\beta)' = X'\}$ *for all Banach and Fréchet spaces E.*

b) $\tau_\infty \supset \tau_{MA} \supset \tau_{SM} \supset \beta \supset \sigma = \sigma(X, X')$, $\tau_{MA} \supset \tau_{MA}(X, X')$ = Makey-topology. All these topologies have the same class of bounded sets.

c) τ_1' defined by $|u|_B$, B bounded in E, is the strong topology τ_b on X'.

d) $X'' = (X\tau_b = l_e^i nfty(E'') = \{x = (x_n), x_n \in E''$, equicontinuous$\}$.

e) $X'' = X$ if E is a reflexive, barreled space, $\sigma(X, X') = \sigma(X'', X')$.

Theorem 1.7. *If E is a reflexive Banach lattice, then $\tau_{MA} = \tau_{SM} = \beta$, τ_{MA} the Mackey topology $\tau_{MA}(X, X')$.*

Theorem 1.8 (N. Flügel, 1989. Diplomarbeit, Münster). *$\tau_{MA} = \tau_{SM}$ for all barreled spaces E.*

Corollary 1.9. *$\tau_\infty \supset \tau \supset \sigma$ implies τ is myopic iff τ is admissible for the duality $(X, X') \leftrightarrow X(\tau)' = X' = l_b^1(E')$.*

If $\mathcal{E} = \{X^i, \succsim_i, a^i, Y^j, \alpha_{ij}\}$ is a *Private Ownership Economic System* $X^i, Y^j \subset X = l^\infty(E)$ we use the

Definition 1.10.

a) (x^i, y^j) is a *state of* $\mathcal{E}$ if $x^i \in X^i, y^j \in Y^j$ for all i, j.

b) A state (x^i, y^j) is *attainable* iff $\sum x^i = \sum y^j + \sum a^i$ (demand = supply on all markets).

c) A state (x^i, y^j) is *individual rational* iff $x^i \succsim_i a^i$ for all i.

d) A state (x^i, y^j) is *Pareto−optimal* if it is attainable and for all attainable states $(\tilde{x}^i, \tilde{y}^j)$ it holds: $\forall \tilde{x}^i \succsim_i x^i$ implies $\tilde{x}^i \sim_i x^i$.

e) A state (x_*^i, y_*^j) is an *equilibrium* at prices p_*, if (1) y_*^j maximizes $p_* \cdot y$ in Y^j. (2) x_*^i maximizes $\succsim_i$ in the budget set $B^i(p_*)$: $B^i(p_*) := \{x \in X^i : p \cdot x \leq w^i(p) \equiv p \cdot a^i + \sum \alpha_{ij} \sup p \cdot y, y \in Y^j\}$ (3) (x_*^i, y_*^j) is attainable : $\sum x_*^i = \sum y_*^j + \sum a^i)$ (4) $p_* \neq 0$.

f) A state (x_*^i, y_*^j) is a *quasiequilibrium* at p_* if (1), (3), (4) are true and (2') x_*^i maximizes $\succsim_i$ on the budget set $B^i(p_*)$ or

$$w^i(p_*) := min(p_* \cdot x, x \in X^i) \text{ and } x_*^i \in B^i(p_*).$$

g) If $w^i(p_*) = min(p \cdot x, x \in X^i)$ we say: consumer i is in the *minimum – wealth situation*.

§2. Existence theorems (Araujo (1985) for $E = \mathbb{R}$)

For the Mathematical Economic System $\mathcal{E}$ we assume :

A0 E is a reflexive Banach Lattice, $X^i = X_+ = l^\infty(E)$, $a^i \in X_+$.

A1 $\succsim_i$ is total and transitive (= preference relation)

A2 (τ) : $\succsim_i$ is τ continuous: $M_+^i(x) = \{x \in X^i : z \succsim_i x\}$ and $M_-^i(x) = \{x \in X^i : x \succsim_i z\}$ are τ-closed sets.

A5 $\succsim_i$ is convex: $M_+^i(x)$ is a convex set for all x, i.

$B(\tau)$:

(i) $0 \in Y^j, Y^j$ is τ-closed, convex (or $\overline{Y} = \sum Y^j$ is a τ-closed, convex set).

(ii) $Y^j \cap \{X_+ - \sum_{v \neq j} Y^v - a\}$ is τ-bounded, $a = \sum a^i$.

Theorem 2.1. *If $\tau_{MA} \supset \tau_{SM} \supset \tau_1, \tau_2 \supset \sigma$, every M.E.S. that satisfies $A0, A1, A2(\tau_1), A5, B(\tau_2)$ has individual, rational, Pareto-optimal states (x_*^i, y_*^j).*

The existence of equilibria needs some more assumptions:

$A4^*$: $\forall i \exists z_0^i \in X^i : z_0^i \leq a^i$, $\forall x \in \hat{X}^j : x + \lambda z_0^i \succ_j x j = 1, 2, \ldots, k$. ($z_0^i$ is universally desired, includes "non-station").

P : $\exists y_0^j \in Y^j : s_0 = \sum y_0^j + \sum a^i = y_0 + a >> 0$ and $\exists \lambda_0 \geq 1 : \sum x^i = \sum y^j + a \leq \lambda_0 s_0$ for all attainable states (x^i, y^j). ($z >> 0$ means $p \cdot z > 0$ for all positive linear functionals $0 \neq p \in X'$)

M : $x \geq z \rightarrow x \succsim_i z, i = 1, 2, \ldots, k$.

Theorem 2.2. *If $\tau_{MA} \supset \tau_{SM} \supset \tau_1, \tau_2 \supset \sigma$, every M.E.S. that satisfies $A0, A1, A2(\tau_1), A4^*, A5, B(\tau_2), P, M$ has an equilibrium (x_*^i, y_*^j, p_*) where p_* is a continuous, positive linear functional on a subspace $L \subset X$ which contains all attainable x^i, y^j and L is a Banach sublattice of X.*

Idea of Proof. Aproximation by finite dimensional subsystems

$$\mathcal{E}_F = \{F \cap X^i, \succsim_i, a^i, F \cap Y^j, \alpha_{ij}\}, F \ni a^i, z_0^i, y_0^j$$

with a basis of positive elements. The Debreu's Existence Theorem of 1962 can be applied so that $\mathcal{E}_F$ has a quasiequilibrium (x_F^i, y_F^j, p_F). M implies $p_F \geq 0$.

$A4^*$ implies (x_F^i, y_F^j) is an equilibrium at p_F. $B(\tau_2)$ implies: $\{y_F^j\}_F$ is τ_2-bounded; hence, σ-bounded and σ-relative compact.

$\{\sum x_F^i\}_F$ is τ_2-bounded; therefore, β-bounded, and so, implies $\{x_F^i\}_F$ is β-bounded since $0 \leq x_F^i \leq \sum x_F^i$ and β is a solide topology. $\{\sum x_F^i\}$ is bounded implies σ-bounded and σ-relative compact. There exist accumulation points (x_*^i, y_*^j) of (x_F^i, y_F^j). X^i is convex, τ_1- closed, therefore σ-closed and $x_*^i \in X^i$; Y^j is τ_2-closed, convex, therefore σ-closed and $y_*^j \in Y^j$; (x_*^i, y_*^j) is an attainable state.

There exists a σ-bounded and σ-closed, solide, absolutely convex set B, that contains all attainable x_*^i, y_*^j and supplies $\sum y^j + a = s$. $L = X_B = \cup_n B$ is a Banach lattice with B as unit ball.

Normalization of the price vector $p_F : p_F(s_0) = 1, \|p_F\| \leq \lambda_0$. By Alaoglu: $p_F \mapsto p_*$ pointwise on L; $p_*(s_0) = \lim p_F(s_0) = 1$, $\|p_*\| \leq \lambda_0$. Maximality of x_*^i and y_*^j with respect to p_* can be proved as in Araujo's case $E = \mathbb{R}$.

§3. Inverse results

We assume $\tau_\infty \supset \tau_1, \tau_2 \supset \sigma = \sigma(X, X')$, but τ_1, τ_2 not necessarily *myopic*. If τ_1 or τ_2 is not *myopic* we can construct counterexamples of M. E. S.'s, such that all conditions of Theorem 2.1 resp 2.2 are satisfied, but individual rational, Pareto-Optimal states do not exist. We get the following:

Theorem 3.1. *If E is a reflexive B-lattice with separable dual E', τ_1 and τ_2 are locally convex topologies on $X = l^\infty(E)$, $\tau_\infty \supset \tau_1$, $\tau_2 \supset \sigma$, such that every M. E. S. that satisfies $A0, A1, A2(\tau_1), A5, B(\tau_2)$ has an individual rational, Pareto-optimal state, then $\tau_{MA} = \tau_{SM} \supset \tau_1, \tau_2; \tau_1, \tau_2$ are myopic .*

Theorem 3.2. *If E is a reflexive separable B-lattice with separable dual E', τ_1 and τ_2 locally convex topologies on $X = l^\infty(E)$, such that every M. E. S. that satisfies $A0, A1, A2(\tau_1), A4^*, A5, B(\tau_2), P, M$, has an equilibrium, then $\tau_{MA} = \tau_{SM} \supset \tau_1, \tau_2; \tau_1, \tau_2$ are myopic topologies.*

"Impatience", as formalized by *myopicity*, of the economic agents is necessary for general existence results.

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