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A NONSTANDARD FIXED POINT RESULT IN $L^1[0,1]$

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ABSTRACT. We give a survey on Maurey's contribution to metric fixed point theory

§1. INTRODUCTION

The most frequently cited fixed point theorem in analysis is the *Banach contraction principle*, which states that if (M,d) is a complete metric space and $T: M \to M$ is a contraction mapping (i.e. there exists $k \in (0,1)$ such that $d(Tx,Ty) \le kd(x,y)$ for all $x,y \in M$), then T has a unique fixed point in M. Moreover, for each $x_0 \in M$, the Picard iterates $(T^n(x_0))$ converges to a fixed point. This theorem has its origins in Euler and Cauchy's work [8] on the existence and uniqueness of a solution of the differential equation

$$\frac{dy}{dt} = f(x,t), \quad y(0) = y_0 \ .$$

The Lipschitz condition $k < 1$ is crucial even for the existence part of the result. Mappings which satisfy the Lipschitz condition for $k = 1$ are known as *nonexpansive mappings*, and the theory of such mappings is fundamentally different from that of contraction mappings. For example, even if a nonexpansive mapping T has a nonempty set of fixed points $Fix(T)$, the Picard iterates may fall to converge. Also, $Fix(T)$ need not contain just one point.

In the linear case, the most important result is the theorem of Brouwer [5], which states that any continuous map $T: B(\mathbf{R}^n) \to B(\mathbf{R}^n)$ has a fixed point, where $B(\mathbf{R}^n)$ is the unit ball in $\mathbf{R}^n$.

The underlying causes behind Brouwer's theorem are the compactness and convexity of the unit ball of $\mathbf{R}^n$. Thus Schauder [29] extended this result to obtain the same conclusion for any compact convex set in any linear topological space which is locally convex. Let us add that this result was known to Poincaré [27] in an equivalent form.

Combining the two fundamental results, we come to the following basic problem.

Fixed point problem in Banach spaces. *Given a Banach space X and a nonempty, closed, bounded convex subset K of X, what types of conditions on K (or X) will guarantee the existence of fixed points for every nonexpansive mapping of K onto itself.*

Definitions.

1. A bounded, closed, convex subset K of a Banach space X is said to have the fixed point property (f.p.p.) if every nonexpansive mapping $T : K \to K$ has a nonempty fixed point set $(Fix(T)) \neq \emptyset)$.
2. The Banach space X is said to have the f.p.p. if every weakly compact convex subset of X has f.p.p.

As the compactness assumption in Brouwer's theorem is fundamental, one might naturally conjecture that any Banach space has the fixed point property. This question was closed in 1978 (published in 1980), when Alspach gave an example of a weakly compact convex subset K of $L^1[0,1]$ and an isometry T defined on K, which fails to have a fixed point. Let us add that the problem as stated before, originated in four papers which appeared in 1965. In the first of these, Browder [6] proved that the problem has a positive answer when X is a Hilbert space. Almost immediately thereafter, both Browder [7] and Göhde [12] extended this conclusion to Banach spaces which are uniformly convex. At the same time, Kirk [21] observed that the presence of a geometric property called *normal structure* guarantees the fixed point property. More on fixed point property could be found in [1, 11].

§2. Basic definitions and results

In this survey we will consider only the weak topology. For other topologies we recommend the books [1, 11]. Let X be a Banach space and C a nonempty, weakly compact, convex subset of X. Assume that C does not have f.p.p. Therefore there exists a nonexpansive mapping $T : C \to C$ with an empty fixed point set. Define

$$\mathcal{F} = \{K \subset C \ ; \ K \text{ nonempty, closed convex and } TK \subset K\} .$$

Since C is weakly compact, the $\mathcal{F}$ is downward directed, i.e. every decreasing chain of elements of $\mathcal{F}$ has a nonempty intersection in $\mathcal{F}$. Using Zorn's lemma, there exists minimal elements in $\mathcal{F}$.

Definition. A convex set K is said to be minimal for T if K is a minimal element of $\mathcal{F}$.

Since T does not have a fixed point, then any minimal set consists of more than one point. Kirk was the first to investigate the structure of minimal sets.

Properties of minimal sets. *Let K be a minimal set for T.*

1. $\overline{conv}\,(TK) = K$.

2. *([25]) Let $\alpha : K \to [0,\infty)$ be a lower semi-continuous convex function. Assume that*
$$\alpha((Tx) \leq \alpha(x) \ ,$$
for every $x \in K$. Then α is a constant function.
3. *([21]) $r(x) = \sup\{\|x-y\| \ ; \ y \in K\} = \text{diam}\,(K)$, for any $x \in K$.*
4. *([10, 16]) Let (x_n) be in K so that*
$$\lim_{x\to\infty} \|x_n - TX_n| = 0 \ .$$
Then for any $x \in K$, we have
$$\lim_{n\to\infty} \|x_n - x\| = \text{diam}\,(K) \ .$$

A point $x \in K$ is said to be *diametral* if
$$r(x) = \sup\{\|x-y\| \ ; \ y \in K\} = \text{diam}\,(K) \ .$$
A set consisting only of diametral points is called *diametral*.

Definition. A Banach space has normal structure if it does not contain a diametral weakly compact convex subset.

This property was introduced in 1948 by Brodskii and Milman [4]. For a while it was thought that any reflexive Banach space has normal structure property. James disproved this statement by renorming the Hilbert space ℓ_2 with the new norm
$$\|x\|_\beta = \max\{\|x\|_{\ell_2}, \|x\|_{\ell_\infty}\} \ ,$$
for $\beta > 0$. Write $X_\beta = (\ell_2, \| \ \|_\beta)$. James proved that $X_{\sqrt{2}}$ fails to have normal structure. In fact, $X_{\sqrt{2}}$ has normal structure if and only if $\beta < \sqrt{2}$. Once this question was resolved, it was natural to ask wether $X_{\sqrt{2}}$ has f.p.p. This question was answered positively in [16]. The main idea is based on property 4 of minimal sets. Since a nonexpansive map could have a nonempty fixed point set, it is natural to ask if sequences defined in property 4 exist. The answer is yes. Indeed let C be a bounded closed convex set and $T : C \to C$ be a nonexpansive mapping. Then for every $\epsilon \in (0,1)$, the map
$$T_\epsilon(x) = (1-\epsilon)T(X) + \epsilon z \ ,$$
where z is a given point in C, is a contraction and therefore has a unique fixed point. Denote by x_n the fixed point of $T_{1/n}$. It is easy to see that
$$\|x_n - T(x_n)\| \leq \frac{1}{n} \text{ diam }(C) \ .$$
Therefore,
$$\lim_{n\to\infty} \|x_n - t(x_n)\| = 0 \ .$$
Such sequences are called *approximated fixed point sequences* (a.f.p.s.).

§3. Use of ultraproducts

Since approximate fixed point sequences always exist, it is somehow natural to think about an extension of the Banach space X generated by sequences. One of these extensions is the ultrapower extension.

Let $\mathcal{U}$ be a nontrivial ultrafilter over $\mathbf{N}$, the set of natural numbers. The ultrapower space $\tilde{X}$ of X is the quotient space of

$$\ell_\infty(X) = \{(x_n) \ ; \ x_n \text{ such that } \sup_n \|x_n\| < \infty\}$$

by the closed subspace

$$\mathcal{N} = \{(x_n) \in \ell_\infty(X) \ ; \ x_n \in X \text{ such that } \lim_{\mathcal{U}} \|x_n\| = 0 \ .$$

We will not distinguish between $(x_n) \in \ell_\infty(X)$ and $\widetilde{(x_n)} = x_n + \mathcal{N} \in \tilde{X}$. It is not hard to show that the quotient norm is given by

$$\|\widetilde{(x_n)}\|_{\tilde{x}} = \lim_{\mathcal{U}} \|x_n\|_X \ .$$

It is also clear that X is isometric to a subspace of $\tilde{X}$ through the mapping

$$x \to (x, x, \cdots) \ .$$

We will write $\tilde{x}$, $\tilde{y}$, $\cdots$ for the general elements of $\tilde{X}$ and by $x, y, \cdots$ for the general elements of X viewed as a subspace of $\tilde{X}$. For more on this construction we recommend [1, 30].

Remark. Another definition of the ultrapower space $\tilde{X}$ of X not using filters is as follows. Consider the Banach space

$$\ell_\infty(X) = \{(x_n) \ ; \ x_n \in X \text{ such that } \sup_n \|bx_n\| < \infty\}$$

and its closed subspace

$$c_0(X) = \{(x_n) \in \ell_\infty(X) \ ; \ x_n \in X \text{ such that } \lim_{n \to infty} \|x_n\| = 0\} \ .$$

The ultrapower space $\tilde{X}$ of X is the quotient space of $\ell_\infty(X)$ by $c_0(X)$. It is not hard to see that the quotient norm is given by

$$|\widetilde{(x_n)}\|_{\tilde{X}} = \limsup_{n \to \infty} \|x_n\|_X$$

Among the most interesting properties satisfied by $\tilde{X}$ is a canonical extension defined as follows. Let D be a subset of X and define

$$\tilde{D} = \{\widetilde{(x_n)} \in \tilde{X} \ ; \ x_n \in D\}$$

Then $\tilde{D}$ will be a bounded, closed, convex subset provided D is a bounded, closed, convex subset of X. Moreover, if $T : D \to D$ is a map, then

$$\tilde{T} : \tilde{D} \to \tilde{D}$$

defined by

$$\tilde{T}(\widetilde{x_n}) = (\widetilde{T(x_n)}) ,$$

is a map provided

$$\lim_{\mathcal{U}} \|Tx_n - ty_n\| = 0, \text{ whenever } \lim_{\mathcal{U}} \|x_n - y_n\| = 0 .$$

If T is nonexpansive, then $\tilde{T}$ exists and is also nonexpansive. It is not hard to see that $Fix(\tilde{T})$ consists of approximate fixed point sequences of T. Therefore, $Fix(\tilde{T})$ is never empty.

Maurey was the first to investigate the structure of $\tilde{K}$, where K is a minimal set associated to a nonexpansive map T.

Properties of $\tilde{K}$. *Let K be a minimal set for T, where T is a nonexpansive map. Consider $\tilde{T}$ and $\tilde{K}$. Then the following hold.*

1. *diam* $(\tilde{K}) =$ *diam* $(K) =$ *diam* $(Fix(\tilde{T}))$
2. *([25]) $Fix(\tilde{T})$ is metrically convex, i.e. for any $\tilde{x}$ and $\tilde{y}$ in $Fix(\tilde{T})$ and $\alpha \in [0,1]$, there exists $\tilde{z} \in Fix(\tilde{T})$ so that*

$$\|\tilde{x} - \tilde{z}\| = (1-\alpha)\|\tilde{z} - \tilde{y}\|, \text{ and } \|\tilde{y} - \tilde{z}\| = \alpha\|\tilde{x} - \tilde{y}\| .$$

3. *([23]) Let $(\tilde{w}_n)$ be an a.f.p.s. for $\tilde{T}$ in $\tilde{K}$, then for any $x \in K$ we have*

$$\lim_{n\to\infty} \|\tilde{w}_n - x\| = \textit{diam}\,(K) .$$

4. *([23]) Let $\tilde{W}$ be any nonempty closed convex subset of $\tilde{K}$ which is inaviant under $\tilde{T}$. Then for every $x \in K$ we have*

$$\sup\{\|\tilde{w} - x\| \ ; \ \tilde{w} \in \tilde{W}\} = \textit{diam}\,(K) .$$

§4. Some fixed point theorems

Before we give Maurey's theorem, we need some preliminary results on the ultapower of $L^1[0,1]$. Let $\mathcal{U}$ be an ultrafilter on the set of natural numbers $\mathbf{N}$ and (X_n) be abstract sets. Consider the cartesian product ΠX_n and define the relation $\mathcal{R}$ as follows

$$(x_N)\ \mathcal{R}\ (y_n) \text{ if and only if } \{n \ ; \ x_n = y_n\} \in \mathcal{U} .$$

The *ultraproduct* of (X_n), denoted $\widetilde{(X_n)}$, is defined as the quotient of ΠX_n over $\mathcal{R}$. When $X_n = X$ for all n, $\widetilde{(X_n)} = \tilde{X}$ is called the ultrapower of X.

Let (Ω, Σ, μ) be a measure space. Without any loss of generality, we will restrict our attention to the case when μ is a probability measure. Consider the following collection of subsets of Ω:

$$\tilde{\Sigma}_0 = \{\widetilde{(A_n)}\ ;\ A_n \in \Sigma\}\ .$$

It is easy to verify that $\tilde{\Sigma}_0$ is a Boolean algebra on $\tilde{\Omega}$. We define a measure $\tilde{\mu}_0$ on $\tilde{\Sigma}_0$ by setting

$$\tilde{\mu}_0(\widetilde{(A_n)}) = \lim_{\mathcal{U}} \mu(A_n)\ .$$

It is not hard to see that the measure $\tilde{\mu}_0$ is σ-additive on $\tilde{\Sigma}_0$. Consequently, $\tilde{\mu}_0$ can be extended to a σ-additive measure $\tilde{\mu}$ on $\tilde{\Sigma}$, the least σ-algebra containing $\tilde{\Sigma}_0$.

Heinrich [13] proved that the ultrapower of $L^1[0,1]$ is canonically isometric to an L^1-sum of $L^1[\widetilde{[0,1]}]$ and $L^1(\nu)$, for some measure ν. Let K be a weakly compact convex subset of $L^1[0,1]$. Using the fact that K is equi-integrable, one can show that the convex subset $\tilde{K}$ of the ultrapower of $L^1[0,1]$ is in fact a subset of $L^1[\widetilde{[0,1]}]$. Therefore the elements of $\tilde{K}$ can be seen as functions in a L^1-space. Maurey [25] proved the following fundamental lemma.

Lemma. *Let K be a weakly compact convex subset of $L^1[0,1]$ that fails f.p.p. and let $T : K \to K$ be nonexpansive with empty fixed point set. Assume that K is minimal for T. Consider $\tilde{T} : \tilde{K} \to \tilde{K}$ in the ultrapower of $L^1[0,1]$. Let $\tilde{x}_0, \tilde{x}_1, \cdots, \tilde{x}_n$ be fixed points of $\tilde{T}$, such that*

$$\|\tilde{x}_n - \tilde{x}_0\| = \sum_{i=1}^{n} \|\tilde{x}_i - \tilde{x}_{i-1}\|\ .$$

Then the functions $(|\tilde{x}_i - \tilde{x}_{i-1}|)$ have disjoint support.

Let K and T as above. Let $\tilde{x}$ and $\tilde{y}$ be in $Fix(\tilde{T})$, with $\|\tilde{x} - \tilde{y}\| = \operatorname{diam}(K) > 0$. Since $Fix(\tilde{T})$ is metrically convex, then there exists $\tilde{z} \in Fix(\tilde{T})$ such that

$$\|\tilde{x} - \tilde{z}\| = \|\tilde{y} - \tilde{z}\| = \frac{1}{2}\|\tilde{x} - \tilde{y}\|\ .$$

Iterating this process, we can find for every $n \in \mathbf{N}$, $\tilde{x}_0, \tilde{x}_1, \cdots, \tilde{x}_n \in Fix(\tilde{T})$ such that

$$\|\tilde{x}_n - \tilde{x}_0\| = \sum_{i=1}^{n} \|\tilde{x}_i - \tilde{x}_{i-1}\|\ .$$

Using the fundamental lemma, we get that the functions $(|\tilde{x}_i - \tilde{x}_{i-1}|)$ have disjoint support. Set

$$\tilde{z}_i = \frac{\tilde{x}_i - \tilde{x}_{i-1}}{\|\tilde{x}_i - \tilde{x}_{i-1}\|}\ .$$

Hence

$$\|\sum_{i=1}^{n} \alpha_i \tilde{z}_i\| = \sum_{i=1}^{n} |\alpha_i| .$$

This implies that the closed subspace generated by $\tilde{K}$ contains ℓ_1^n isometrically for every n. If K was supposed to be a subset of a reflexive subspace $\mathcal{R}$ of $L^1[0,1]$, we get a contradiction with the fact that any ultrapower of $\mathcal{R}$ is reflexive [28] (see also [24]). This completes the proof of the following result.

Theorem [25]. *Let $\mathcal{R}$ be a reflexive subspace of $L^1[0,1]$. Then $\mathcal{R}$ has the fixed point property.*

The second interesting result proved by Maurey concerns the Banach space c_0. Let us recall that before Maurey's result it was unknown wether c_0 has the the fixed point property. However some partial results were known [26]. Let us mention that the proof given there is highly technical and uses a careful study of the structure of some weakly compact convex subsets of c_0.

Theorem [25]. *The Banach space c_0 has the fixed point property.*

Proof. Assume the contrary, and let $K, T, \tilde{k}$ and $\tilde{K}$ be defined as usual. Let (x_n) be an a.f.p.s. of T in K. We may suppose that (x_n) is weakly convergent to 0 and that $\mathrm{diam}(K) = 1$. Using the canonical basis of c_0, one can find a subsequence $(x_n{}')$ of (x_n) such that

$$\| \, |x_n{}'| \wedge |x_n| \, \| \leq dfrac1n$$

and

$$\lim_{ntoo\infty} \|x_n - x_n'\| = 1 .$$

Put $\tilde{x} = \widetilde{(x_n)}$ and $\tilde{y} = \widetilde{(x_n{}')}$. Then $\tilde{x}$ and $\tilde{y}$ are in $Fix(\tilde{T})$. Let $\tilde{z} = \widetilde{(z_n)}$ be a quasi-middle fixed point of $\tilde{x}$ and $\tilde{y}$. Then by use of the lattice inequalities

$$|z_n| \leq |z_n - x_n| \wedge |z_n - x_n'| + |x_n| \wedge |x_n'|$$

one can obtain

$$\|\tilde{z}\| = \lim_{\mathcal{U}} \|z_n\| \leq \frac{1}{2} .$$

This contradicts the fact that $\tilde{z} \in Fix(\tilde{T})$ and $\|\tilde{z} - 0| = 1$.

Looking into the proof carefully, one can see that Maurey used the basis of c_0 and the lattice structure of c_0. Clearly any Banach space that has an unconditional basis "will somehow" share c_0's result. In particular, the authors in [3] used the above ideas to get a more general result in a large class of Banach lattices. But maybe the most important use of Maurey's proof can be found in [9]. Indeed the authors proved that $X_\beta = (\ell_2, \| \, \|_\beta)$ has the fixed point property, which settles up a long standing problem. Lin's result [22] on Banach spaces with unconditional basis may explain how Maurey's ideas were deep and profound. First let us recall some basic definitions on Schauder basis.

Recall that a sequence $\{e_n\}$ in X is called a Scahuder basis for X if for each $x \in X$ there exists a unique sequence of scalars (x_n) such that

$$x = \sum_{n=1}^{\infty} x_n e_n .$$

$\{e_n\}$ is said to be unconditional if there exists a constant $\lambda \geq 1$ such that for every convergent series $\sum_{n=1}^{\infty} x_n e_n$ and every sequence of signs (ϵ_n) ($\epsilon_n = \pm 1$), the series $\sum_{n=1}^{\infty} \epsilon_n x_n e_n$ converges and satisfies

$$\|\sum_{n=1}^{\infty} \epsilon_n x_n e_n\| \leq \lambda \|\sum_{n=1}^{\infty} x_n e_n\|.$$

The smallest constant λ is called the unconditional constant of $\{e_n\}$.

Theorem [22]. *Let X be a Banach space with an unconditional basis $\{e_n\}$. Assume that the constant of unconditionality λ staisfies*

$$\lambda < \frac{\sqrt{33}-3}{2}$$

Then X has the fixed point property.

As a corollary we get the following result.

Theorem [22]. *Let X be Banach space with a montone unconditional Schauder basis (i.e. $\lambda = 1$). Then X has the fixed point property.*

In [18] it was noticed that unconditionality is not the key behind this positive result. Specially that there are Banach spaces which fails to have an unconditional basis or even be embeded ina Banach space with an unconditional basis. The most important one is the quasi-reflexive space James' space [14]. Before we state the main result of [18], we need the following.

Let $\{e_n\}$ be a Schauder basis for X. Define the natural projections associated to $\{e_n\}$ as

$$P_F(\sum_{n=1}^{\infty} x_n e_n) = \sum_{n \in F} x_n e_n ,$$

where F is a subset of the set of natural numbers $\mathbf{N}$. Usually we denote by P_n the natural projection of the segment $[1, n]$. Set

$$c = \sup\{\|P_n\| \ ; \ n \geq 1\} ,$$

$$c_1 = \sup\{\|I - P_n\| \ ; \ n \geq 1\} ,$$

$$c_2 = \sup\{\|P_F\| \ ; \ F \text{ is a segment of } \mathbf{N}\} ,$$

$$\mu = \sup\{\|u - v\| \ ; \ u \text{ and } v \text{ are disjoint blocks with } \|u + v\| \leq 1\}$$

where by a block of $\{e_n\}$ we mean any element $u = \sum_{n=p}^{n=k} x_n e_n$. Note that these constants are finite.

Theorem [17]. *Let X be a Banach space with Scahuder basis $\{e_n\}$. Assume that*

$$c_1\mu + c + c_2 < 4 \ .$$

Then X has the fixed point property.

A corollary to this result concerns James quasi-reflexive space J. First let us give the definition of J. Let $(x_n) \in \mathbf{R}^{(\mathbf{N})}$ and define $\|(x_n)\|_J$ by

$$\|(x_n)\|_J = \sup N(p_1, p_2, \cdots, p_n) \ ,$$

with

$$N(p_i) = \frac{1}{\sqrt{2}}\left[(x_{p_1} - x_{p_2})^2 + \cdots, x_{p_{n-1}} - x_{p_n})^2 + (x_{p_n} - x_{p_1})^2\right]^{1/2} ,$$

where the supremum is taken over all positive integers n and all increasing sequences of positive integers $(p_1, p_2, \cdots, p_n)$.

Definition. *The James space J is the completion of $\mathbf{R}^{(\mathbf{N})}$ with respect to the norm $\|\cdot\|_J$.*

We have

$$c = c_1 = 1 \ , \quad \text{and} \quad c_2 = \mu = \sqrt{2} \ .$$

Since $\sqrt{2} + \sqrt{2} + 1 < 4$, we get the following corollary.

Corollary [18]. *The quasi-reflexive space J has the fixed point property.*

Let us add that more applications of Maurey's theorem can be found in [2, 15, 17, 19, 20].

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