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THE PLATEAU PROBLEM FOR THE PRESCRIBED MEAN CURVATURE EQUATION

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§1. INTRODUCTION

Given a Jordan curve Γ in $\mathbb{R}^3$ and $H : \mathbb{R}^3 \to \mathbb{R}^3$ a continuous and bounded function, we consider the Plateau problem in the unit disc $B = \{(u,v) \in \mathbb{R}^3 \ ; \ u^2 + v2 < 1\}$, .i.e we look for a vector function $X : \bar{B} \to \mathbb{R}^3$ which satisfies the following system of nonlinear differential equations

$$\begin{cases} (1)\ \Delta X = 2H(X) X_u \wedge X_v & \text{in } B \\ (2)\ |X_u|^2 - |X_v|^2 = 0 = X_u \cdot X_v & \text{in } B \\ (3)\ X\ |_{\partial B} : \partial B \to \Gamma & \text{is a parametrization of } \Gamma \end{cases} \tag{P}$$

where $X_u = \dfrac{\partial X}{\partial u}$, $X_v = \dfrac{\partial X}{\partial v}$ and $\wedge$ denotes the exterior product in $\mathbb{R}^3$.

As in [5] we define the following subset of the Sobolev space $H^1(B, \mathbb{R}^3)$:

$$C(\Gamma) = \{X \in H^1(B, \mathbb{R}^3) \ : \ X\ |_{\partial B} \in C^0(\partial B, \mathbb{R}^3)$$

$$\text{is a weakly monotone parametrization of } \ \Gamma\} \ .$$

We call $X \in C(\Gamma)$ a weak solution of (P) if for every $\varphi \in C_0^1(B, \mathbb{R}^3)$

$$\int_B \nabla X \cdot \nabla \varphi + 2H(X) X_u \wedge X_v \cdot \varphi = 0 \tag{4}$$

and if X satisfies (2) in B.

We will consider H prescribed and obtain weak solutions of (P) as critical points of the functional

$$D_H(X) = D(X) + 2V(X) \ ,$$

with

$$D(X) = \frac{1}{2} \int_B |\nabla X|^2$$

Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$-TEX being the Dirichlet integral and

$$V(X) = \frac{1}{3}\int_B Q(X)\cdot X_u \wedge X_v$$

the Hildebrandt volume, where for $\xi = (\xi_1, \xi_2, \xi_3)$ in $\mathbb{R}^3$, the associated function Q to H is

$$Q(\xi) = \left(\int_0^{\xi_1} H(s,\xi_2,\xi_3)ds, \int_0^{\xi_2} H(\xi_1,s,\xi_3)ds, \int_0^{\xi_3} H(\xi_1,\xi_2,s)ds\right),$$

which satisfies div $Q = 3H$ [3].

Notations. We denote by $W^{m,p}(B,\mathbb{R}^3)$ the usual Sobolev spaces [1] and $H^m(B,\mathbb{R}^3) = W^{m,2}(B,\mathbb{R}^3)$. For $X \in H^1(B,\mathbb{R}^3)$,

$$\|X\|_{L^2(\partial B,\mathbb{R}^3)} = \left(\int_{\partial B} |\operatorname{Tr} X|^2\right)^{1/2},$$

where Tr $: H^1(B,\mathbb{R}^3) \to L^2(\partial B,\mathbb{R}^3)$ is the usual trace operator [1] and for $Y \in L^\infty(U,\mathbb{R}^n)$, we denote

$$\|Y\|_\infty = \sup_{w\in U} |Y(w)|.$$

For example,

$$\|H\|_\infty = \sup_{\xi\in\mathbb{R}^3} |H(\xi)|,$$

$$\|Q\|_\infty = \sup_{\xi\in\mathbb{R}^3} |Q(\xi)|,$$

and

$$\|H(X)\|_\infty = \sup_{\xi\in B} |H(X(w))|,$$

$$\|Q(X)\|_\infty = \sup_{\xi\in B} |Q(X(w))|.$$

Concerning D_H (resp. V) we set

$$dD_H(X)(\varphi) = \lim_{t\to 0}\left[\frac{D_H(X+t\varphi) - D_H(X)}{t}\right],$$

whenever this limit exists (resp. $dV(X)(\varphi)$).

If $X \in H^1(B,\mathbb{R}^3)$, $\varphi \in C_0^1(B,\mathbb{R}^3)$ then $dD_H(X)(\varphi)$ exists and is equal to the first member of (4) [cf. 4, Lemma 2].

Finally, (n,σ) denotes polar coordinates in $\mathbb{R}^3$, in particular $\frac{\partial}{\partial n}$ is the normal derivative and $\frac{\partial}{\partial \sigma}$ the tangencial derivative on ∂B.

§2. Minima in subsets of $H^1(B,\mathbb{R}^3)$

We find weak solutions of (P). A first solution is a local minimum of D_H in a convenient subset of $H^1(B,\mathbb{R}^3)$ when Q is in a specific convex subset of L^∞ and Γ is a rectifiable Jordan curve in $\mathbb{R}^3$. Other solutions are either a local minimum of D_H with respect to the $W^{2,\infty}(B,\mathbb{R}^3)$-topology or a sequence of minima of D_H in convenient closed convex subsets of $H^m(B,\mathbb{R}^3)$ associated to H and Γ.

As in [5] we define

$$C^*(\Gamma) = \{X \in C(\Gamma) : X(P_j) = Q_j \ , \ j = 1,2,3\} \ ,$$

where $P_j = e^{i\varphi_j}$, $0 \le \varphi_1 < \varphi_2 < \varphi_3 < 2\pi$, and Q_j, $j = 1,2,3$ an oriented triple on Γ.

It is known that $C^*(\Gamma)$ is a weakly closed subset of $H^1(B,\mathbb{R}^3)$ but $C(\Gamma)$ is not and that for any rectifiable Jordan curve $\Gamma \subset \mathbb{R}^3$, $C(\Gamma) \neq \emptyset$ [5].

Finally, we know that if $X \in H^1(B,\mathbb{R}^3)$ is a critical point of D in the following sense: for any differentiable family of diffeomorphisms $g_\epsilon : B \to B_\epsilon$, with $g_0 =$ id there holds

$$\frac{d}{d\epsilon} D(X \circ g_\epsilon^{-1}, B_\epsilon)\bigg|_{\epsilon=0} = 0 \ ,$$

then X satisfies (2) in (P), i.e. the coordinates (u,v) are isothermal, and any minmizer X of D_H in $C(\Gamma)$ will be a critical point of D in this sense [5].

We give proofs based on the technical lemmas in section 3.

Theorem 1. *Let $H : \mathbb{R}^3 \to \mathbb{R}$ be continuous and bounded. If the function $Q \in C^1(\mathbb{R}^3,\mathbb{R}^3)$ associated to H satisfies $\|Q\|_\infty < \frac{3}{2}$ and $\frac{\partial Q_i}{\partial \xi_i} \in L^\infty(\mathbb{R}^3)$ for $i \neq j$, then given a Jordan curve Γ in $\mathbb{R}^3$ such that $C(\Gamma) \neq \emptyset$, the functional D_H has a minimum $\underline{X}$ in $C^*(\Gamma)$ and $\underline{X}$ is a weak solution of (P).*

Proof. From

$$|\zeta \cdot \eta \wedge \xi| \le \frac{1}{2}|\zeta|(|\eta|^2 + |\xi|^2) \ ,$$

for vectors in $\mathbb{R}^3$, we have for $X \in H^1(B,\mathbb{R}^3)$ that

$$\begin{aligned} |D_H(X)| &\le D(X) + \frac{2}{3}\int_B |Q(X)\cdot X_u \wedge X_v| \\ &\le D(X) + \frac{2}{3}\|Q\|_\infty D(X) \\ &< 2D(X) \ , \end{aligned}$$

so D_H is finite in $H^1(B,\mathbb{R}^3)$.

From Lemma 2 [4], D_H is weakly lower semicontinuous in $H^1(B,\mathbb{R}^3)$, and coercive in $C(\Gamma)$, beacuse for $X \in C(\Gamma)$, we have the Sobolev inequality, which is valid for $X \in H^1(B,\mathbb{R}^3)$ with $\operatorname{Tr} X \in L^\infty(\partial B,\mathbb{R}^3)$:

$$\begin{aligned}\|X\|_2^2 &\le k_1(\|\nabla X\|_2^2 + \|X\|^2_{L^2(\partial B,\mathbb{R}^3)}) \\ &\le k_2(D(X)) + \|X\|^2_{L^2(\partial B,\mathbb{R}^3)}) \le kD(X) + k(\Gamma)\ ,\end{aligned}$$

with k_1, k_2, k and $k(\Gamma)$ positive constants.

Then D_H is weakly lower semicontinuous and coercive in $C^*(\Gamma)$, and $C^*(\Gamma)$ is a weakly closed subset of $H^1(B,\mathbb{R}^3)$, so there is a minimum $\underline{X}$ of D_H in $C^*(\Gamma)$.

By conformal invariance of D_H ([5] and Lemma 1 below):

$$D_H(\underline{X}) = \inf_{C^*(\Gamma)} D_H(X) = \int_{C(\Gamma})D_H(X)\ .$$

But $\underline{X} + \epsilon\varphi \in C(\Gamma)$ for $\varphi \in C_0^1(B,\mathbb{R}^3)$, then we have that $dD_H(X)(\varphi) = 0$ and from Lemma 1 [4]:

$$\int_B \nabla\underline{X}\cdot\nabla\varphi + 2H(\underline{X}))\underline{X}_u \wedge \underline{X}_v \cdot \varphi = 0 \text{ for } \varphi \in C_0^1(B,\mathbb{R}^3)\ .$$

Finally, as in [5] for the case $H = H_0 \in \mathbb{R}$, from Lemma 1 we have that

$$\frac{d}{d\epsilon} D_H(X\circ g_\epsilon^{-1}, B_\epsilon)\Big|_{\epsilon=0} = \frac{d}{d\epsilon} D(X_0 g_\epsilon^{-1}, B_\epsilon)\Big|_{\epsilon=0} = 0$$

for any family of diffeomorphisms $g_\epsilon : B \to B_\epsilon$, with $g_0 =$ id and det $(dg) = g_{\epsilon 1u} g_{\epsilon 2v} - g_{\epsilon 2u} g_{\epsilon 1u} g_{\epsilon 1v} > 0$; hence the coordinates (u, v) are isothermal.

Let us recall that a function $g \in H^1(B,\mathbb{R}^3)$ is a solution to the classical Plateau problem for a curve Γ in $\mathbb{R}^3$ if g is harmonic in B and satisfies (2) and (3) in (P).

Theorem 2. *Let $\Gamma \subset \mathbb{R}^3$ be a rectifiable curve such that the solution to the classical Plateau problem is a function $g \in W^{2,\infty}(B,\mathbb{R}^3)$ and suppose that $H : \mathbb{R}^3 \to \mathbb{R}$ is a function satisfying the following properties:*

i) *$H \in C^1(\mathbb{R}^3) \cap W^{1,\infty}(\mathbb{R}^3)$ and $Q \in L^\infty(\mathbb{R}^3,\mathbb{R}^3)$*

ii) $0 < \|H\|_\infty \|g\|_\infty < \dfrac{3}{2}$

iii) *There exists a positive number $c > \|\nabla g\|_\infty$ such that*

$$|H(\xi)| \le \left(\frac{\lambda_1}{c}\right)^2 (|\xi| - \|g\|_\infty)_+ \text{ for } \xi \in \mathbb{R}^3,$$

where $\lambda_1 > 0$ is the first eigenvalue of $-\Delta$ in $H_0^1(B)$.

Then g is a weak solution of (P) and either g is a local minimum of D_H in

$$W^{2,\infty}(B,\mathbb{R}^3)\cap\Big\{X\in H^2(B,\mathbb{R}^3);\ \mathrm{Tr}\,X=\ \mathrm{Tr}\,g,$$

$$\mathrm{Tr}\,\frac{\partial X}{\partial r}=\ \mathrm{Tr}\,\frac{\partial g}{\partial r},\ \ \mathrm{Tr}\,\frac{\partial X}{\partial \sigma}=\ \mathrm{Tr}\,\frac{\partial g}{\partial \sigma}\Big\}\ .$$

or there exists a sequence X_n) in $W^{2,\infty}(B,\mathbb{R}^3)$ of distinct weak solutions of (P) with $X_n\to g$ in $W^{2,\infty}(B,\mathbb{R}^3)$.

Proof. From iii), $H(g)=0$ on B, so (P) holds trivially. Now we choose a positive number δ_1 such that

$$\delta_1<\min\{c-\|\nabla g\|_\infty,\frac{3}{2\|H\|_\infty}-\|g\|_\infty\}$$

and define

$$M_1=\{g+\varphi\ ;\varphi\in H^2(B,\mathbb{R}^3),\ \mathrm{Tr}\,\frac{\partial\varphi}{\partial r}=\ \mathrm{Tr}\,\frac{\partial\varphi}{\partial\sigma}=\ \mathrm{Tr}\,\varphi=0\ ,\|\varphi\|_{W^{2,\infty}}<\delta_1\}.$$

We have that M_1 is a nonempty convex, closed and bounded subset of $H^2(B,\mathbb{R}^3)$ and by Lemma 2 [4], D_H is weakly lower semicontinuous in M_1, because $\|Q(X)\|_\infty\le\|H\|_\infty\|X\|_\infty\le\|H\|_\infty(\|X-g\|_\infty+\|g\|_\infty)<3/2$ for $X\in M_1$. Hence there exists $X_1\in M_1$ such that

$$D_H(X_1)=\inf_{X\in M_1}D_H(X)\ .$$

Suppose that g is not a local minimum of D_H in

$$W^{2,\infty}(B,\mathbb{R}^3)\cap\Big\{X\in H^2(B,\mathbb{R}^3);\ \mathrm{Tr}\,X=\ \mathrm{Tr}\,g,$$

$$\mathrm{Tr}\,\frac{\partial X}{\partial r}=\ \mathrm{Tr}\,\frac{\partial g}{\partial r},\ \ \mathrm{Tr}\,\frac{\partial X}{\partial \sigma}=\ \mathrm{Tr}\,\frac{\partial g}{\partial \sigma}\Big\}\ .$$

As in the proof of theorem 2 in [4], we have that $dD_H(X_1)(\varphi)=0$ for all $\varphi\in C_0^1(B,\mathbb{R}^3)$. From this and from $X_1\in H^2(B,\mathbb{R}^3)$, it follows that (1) in (P) is fullfilled. $g\in H^2(B,\mathbb{R}^3)$ is a solution to the Plateau problem, in particular

$$\left|\frac{\partial g}{\partial\sigma}\right|-\left|\frac{\partial g}{\partial\eta}\right|=\frac{\partial}{\partial\sigma}\cdot\frac{\partial g}{\partial\eta}=0$$

on $\bar{B}$. From $X_1\in M_1$ we obtain that (η,σ) are isothermal on ∂B for X_1, so (u,v) are also isothermal on ∂B and in B (Lemma 2 below). Finally, $\mathrm{Tr}\ g=\mathrm{Tr}\ X_1$ on ∂B with $g\in C(\Gamma)$ gives (3) in (P). Now we choose

$$\delta_2=\min\{\delta_1,\|X_1-g\|_{W^{2,\infty}}\}$$

and define

$$M_2 = \{g+\varphi;\ \varphi \in W^{2,\infty}(B,\mathbb{R}^3),\ \mathrm{Tr}\,\frac{\partial\varphi}{\partial r} = \mathrm{Tr}\,\frac{\partial\varphi}{\partial\sigma} = \mathrm{Tr}\,\varphi = 0\ \|\varphi\|_{W^{2,\infty}} \le \delta_2\}\,.$$

Then there exists $X_2 \in M_2$ such that

$$d_H(X_2) = \inf_{X\in M_2} D_H(X)\,.$$

X_2 is a weak solution of (P) and $X_2 \ne g$, X_1, because $X_1 \notin M_2$. Hence we can define a sequence $(X_n) \subset W^{2,\infty}(B,\mathbb{R}^3)$ of weak solutions of (P) such that $X_n \to g$ in $W^{2,\infty}(B,\mathbb{R}^3)$.

Remark. If $\|Q\|_\infty < 3/2$, the condition $\|H\|_\infty\|g\|_\infty < 3/2$ is not necessary. In this case, we can define the sequence of convex subsets of $W^{2,\infty}(B,\mathbb{R}^3)$ as follows:

$$M_1 = \{g+\varphi;\varphi \in W^{2,\infty}(B,\mathbb{R}^3),\ \mathrm{Tr}\,\frac{\partial\varphi}{\partial r} = \mathrm{Tr}\,\frac{\partial\varphi}{\partial\sigma} = \mathrm{Tr}\,\varphi = 0 \text{ and } \|\varphi\|_{W^{2,\infty}} \le c - \|\nabla g\|_{W^{2,\infty}}\}\,,$$

$$\delta_2 = \min\{c - \|\nabla g\|_\infty\,,\ \frac{1}{2}\|X_1 - g\|_{W^{2,\infty}}\}$$

and

$$M_2 = \{g+\varphi;\ \varphi \in W^{2,\infty}(B,\mathbb{R}^3),\ \mathrm{Tr}\,\frac{\partial\varphi}{\partial r} = \mathrm{Tr}\,\frac{\partial\varphi}{\partial\sigma} = \mathrm{Tr}\,\varphi = 0\ \|\varphi\|_{W^{2,\infty}} \le \delta_2\}\,.$$

A class of functions H which are examples of our results is given by

$$H(\xi) = \begin{cases} H_0 & \xi_i \in (a_i,b_i) \\ 0 & \xi_i \notin (a_i-\epsilon, b_i+\epsilon) \end{cases}$$

with H_0, ϵ, a_i, b_i, $i=1,2,3$ positive numbers such that $0 < a_i - \epsilon$, $a_i < b_i$ and $H_0(\sum_{i=1}^3(b_i+\epsilon)^2)^{1/2} < 3/2$. We suppose that $H \in C^1(\mathbb{R}^3)$ and $\|H\|_\infty = H_0$. As in [4], $\|Q\|_\infty < \frac{3}{2}$ and $\frac{\partial Q}{\partial \xi_i} \in L^\infty$ follow.

Remark. No assumptions are made on Γ.

§3. Technical lemmas

We have as in the case $H \equiv$ cte:

Lemma 1. *If the function Q associated to H satisfies $Q \in L^\infty(\mathbb{R}^3,\mathbb{R}^3)$, the Hildebrandt volume $V(X) = \frac{1}{3}\int_B Q(X)\cdot X_u \wedge X_v$ is invariant under orientation preserving reparametrizations of X, i.e. if $g \in C^1(\bar{B},\mathbb{R}^2)$ is a diffeomorphim of B onto a domain $\hat{B}$ with $det(dg) = g_{1u}g_{2v} - g_{2u}g_{1v} > 0$, then $V(X) = V(X\circ g^{-1})$.*

Lemma 2. *Let $H : \mathbb{R}^3 \to \mathbb{R}$ continuous and bounded and $X \in H^2(B, \mathbb{R}^3)$ such that $\Delta X = 2H(X)X_u X_v$ in B. Then (η, σ) are isothermal on ∂B if and only if (u, v) are isothermal in B.*

Proof. Suppose that $|X_\eta|^2 - |Vert X_\sigma|^2 = X_\eta \cdot X_\sigma = 0$ on ∂B, then $|X_u|^2 - |X_v|^2 = X_u \cdot X_v = 0$ on ∂B by calculation. We extend X to $\mathbb{R}^2$ by a reflection, setting

$$Y(u, v) = \begin{cases} X(u, v) & \text{in } \bar{B} \\ X\left(\frac{u}{r^2}, \frac{v}{r^2}\right) & \text{in } \mathbb{R}^2 - \bar{B} \text{ where } u^2 + v^2 = r^2 . \end{cases}$$

A direct computation shows that

$$\begin{cases} \Delta Y = 2H(Y) Y_u \wedge Y_v & \text{in } B \\ \Delta Y = -2H(Y) Y_u \wedge Y_v & \text{in } \mathbb{R}^2 - \bar{B} \end{cases}$$

If we consider the conformal measure function

$$F(u, v) = |Y_u|^2 - |Y_v|^2 - 2i Y_u \cdot Y_v ,$$

we have that F is holomorphic in $\mathbb{C} - \partial B$ and $F \in C^0(\mathbb{C}, \mathbb{C})$, because

$$\begin{aligned} \lim_{\substack{r \to 1 \\ r > 1}} (|Y_u|^2 - |Y_v|^2) &= [|X_u|^2 (v^2 - u^2)^2 \\ &\quad + |X_v|^2 4u^2 v^2 + 2X_u \cdot X_v (v^2 - u^2)(-2uv)] \\ &\quad [|X_u|^2 (-2uv)^2 + |X_v|^2 (u^2 - v^2)^2 + X_u \cdot X_v (u^2 - v^2)(-2uv)] \\ &= (|X_u|^2 - |X_v|^2)[(v^2 - u^2)^2 - 4u^2 v^2] + 2X_u \cdot X_v (u^2 - v^2) 4uv \\ &= 0 = \lim_{\substack{r \to 1 \\ r < 1}} (|Y_u^2 - |Y_v|^2) \end{aligned}$$

and

$$\begin{aligned} \lim_{\substack{r \to 1 \\ r > 1}} Y_u \cdot Y_v &= (|X_u|^2 - |X_v|^2)[(v^2 - u^2)(-2uv)] \\ &\quad + X_u \cdot X_v [-(u^2 - v^2)^2 + 4u^2 v^2] \\ &= 0 = \lim_{\substack{r \to 1 \\ r < 1}} Y_u \cdot Y_v . \end{aligned}$$

Then F is holomorphic in $\mathbb{C}$; but from

$$\begin{aligned} \int_{\mathbb{R}^2} |F(u, v)| &\le 2 \int_{\mathbb{R}^3} (|Y_u|^2 + |Y_v|^2) = 4 \int_B (|Y_u|^2 + |Y_v|^2) \\ &= 4 \int_B (|X_u|^2 + |X_v|^2) < +\infty \end{aligned}$$

we deduce that $F \equiv 0$ and then we have $|X_u|^2 - |X_v|^2 = X_u \cdot X_v = 0$ in B. Conversely, if (u, v) are isothermal in B, so are (r, σ), because $(u, v) \to (r, \sigma)$ is conformal; hence on ∂B, $(r, \sigma) \equiv (\eta, \sigma)$ are isothermal.

§4. A NON EXISTENCE RESULT AND NECESSARY CONDITIONS

Taking into account the case $H = H_0 \in \mathbb{R}$ and the Heinz' non existence result [5], [2], we have the following:

Theorem 3. *Let $\Gamma \subset \mathbb{R}^3$ be a rectifiable Jordan curve of length $L(\Gamma)$, $H \in C(\mathbb{R}^3)$, and suppose that there exists a positive number δ having the properties*

i) $L(\Gamma) < \delta$.

ii) *For any $X \in C(\Gamma)$ there exists a unit vector $\eta_X \in \mathbb{R}^3$ such that*

$$\eta_X \cdot \int_B 2H(X)X_u \wedge X_v \geq \delta \, .$$

Then there is no solution to (P) in $C^1(\bar{B}, \mathbb{R}^3) \cap C^2(B, \mathbb{R}^3)$.

Proof. Suppose that $X \in C^1(\bar{B}, \mathbb{R}^3) \cap C^2(B, \mathbb{R}^3)$ is a solution of (P). Then

$$\begin{aligned} \eta_X \cdot \int_B 2H(X)X_u \cdot X_v &= \eta_X \cdot \int_B \Delta X = \eta_X \cdot \int_{\partial B} \frac{\partial X}{\partial \eta} d\sigma \\ &\leq \int_{\partial B} \left| \frac{\partial X}{\partial \eta} \right| d\sigma = \int_{\partial B} \left| \frac{\partial X}{\partial \sigma} \right| d\sigma = L(\Gamma) < \delta \, . \end{aligned}$$

A contradiction.

Theorem 4. *Let $\Gamma \subset \mathbb{R}^3$ be a rectifiable Jordan curve of length $L(\Gamma)$, $H : \mathbb{R}^3 \to \mathbb{R}$ continuous and bounded, and suppose that $X \in C^1(\bar{B}, \mathbb{R}^3) \cap C^2(B, \mathbb{R}^3)$ is a solution of (P) verifying $\|H(X)X\|_\infty < 1$. Then*

$$D(X) \leq \frac{L(\Gamma)\|X\|_\infty}{2(1 - \|H(X)X\|_\infty)} \, .$$

Proof. We have that $\Delta X = 2H(X)X_u \wedge X_v$ in B. Thus

$$\begin{aligned} 0 &= \int_B [-\Delta X + 2H(X)X_u \wedge X_v] \cdot X \\ &= \int_B [|\nabla X|^2 + 2H(X)X_u \wedge X_v \cdot X] - \int_{\partial B} \frac{\partial X}{\partial \eta} \cdot X d\sigma \\ &\geq 2D(X) - 2\|H(X)X\|_\infty \int_B |X_u \wedge X_v| - \|X\|_\infty \int_{\partial B} \left| \frac{\partial X}{\partial \eta} \right| d\sigma \\ &\geq 2D(X)(1 - \|H(X)X\|_\infty) - \|X\|_\infty \int_{\partial B} \left| \frac{\partial X}{\partial \sigma} \right| d\sigma \\ &= 2D(X)(1 - \|H(X)X\|_\infty) - \|X\|_\infty L(\Gamma) \, . \end{aligned}$$

It follows now that

$$D(X) \leq \frac{L(\Gamma)\|X\|_\infty}{2(1 - \|H(X)X\|_\infty)} .$$

REFERENCES

1. Adams, R., *Sobolev Spaces*, Academic Press.
2. Heinz, E., *On the nonexistence of a surface of constant curvature with finite area and prescribed rectifiable boundary*, Arch. Rat. Mach. Anal. **35** (1969), 249–252.
3. Hildebrandt, S., *Randwertprobleme für Flächen mit vorgeschriebener mittlerer Krümmung und Anwendungen auf die Kapillaritätstheorie. Teil I: Fest vorgegebener Rand*, Math. Z. **112** (1969), 205-213.
4. Lami Dozo, E. & Mariani, M. C., *A Dirichlet problem for an H-system with variable H*, Preprint IAM 181.
5. Struwe, M., *Plateau's Problem and the Calculus of Variations*, Lecture Notes, Princeton University Press.

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