QUASIVARIETIES OF DE MORGAN ALGEBRAS: RCEP

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ABSTRACT. In this note we prove that there is a least strict quasivariety (i.e., a quasivariety which is not a variety) of De Morgan algebras and that such a quasivariety is perhaps the only strict quasivariety enjoying the relative congruence extension property.

§1. INTRODUCTION

For a quasivariety $Q$ and an algebra $A \in Q$, let $\text{Con}_Q(A) = \{ \Theta \in \text{Con}(A) : A/\Theta \in Q \}$, where $\text{Con}(A)$ denotes the set of congruence relations on $A$. The elements of $\text{Con}_Q(A)$ are called $Q$-congruences on $A$. $A$ is said to have relative (to $Q$) congruence extension property (further on RCEP) if for every subalgebra $B$ of $A$, any $Q$-congruence on $B$ is the restriction of a $Q$-congruence on $A$. $Q$ has RCEP if all of its elements have this property. The purpose of this note is to prove that there is a least strict quasivariety of De Morgan algebras and such a quasivariety is perhaps the only strict quasivariety enjoying RCEP. For more results in this direction we refer the reader to [3] and [7]. Recall that a De Morgan algebra is an algebra $(A; \wedge, \vee', 0, 1)$ of type $(2, 2, 1, 0, 0)$ such that the reduct $(A; \wedge, \vee, 0, 1)$ is a bounded distributive lattice and the following identities are satisfied:

$$x'' = x \quad ; \quad (x \lor y)' = x' \land y'.$$

The lattice of subvarieties of De Morgan algebras is a four-element chain $T \subset B \subset K \subset M$ where $T$, $B$, $K$, and $M$ denote respectively the varieties of trivial, Boolean, Kleene and De Morgan algebras. There are three non-trivial subdirectly irreducible De Morgan algebras each of which generates one of the non-trivial varieties above: $B$ is generated by the two-element chain $2 = \{0, 1\}$, $K$ is generated by the three element chain $3 = \{0, a, 1\}$ in which $a' = a$ and $M$ is the variety of De Morgan algebras.

Research supported by the CDCHT (project C-507-91) of the University of the Andes, Mérida, Venezuela.
by the four-element complemented lattice $4 = \{0, a, b, 1\}$ in which $a' = a$ and $b' = b$. $B$ satisfies the identity $x \land x' = 0$ and $K$ satisfies $x \land x' \leq y \lor y'$. For a systematic study of $M$ see [1].

The strategy of the paper is based on the analysis of the two possible cases concerning the variety generated by the strict quasivariety. Section 2 is devoted to the case in which such a variety is $M$. It is proved that under this assumption no strict quasivariety has RCEP. In Section 3 it is proved that, except for the quasivariety generated by the four element chain $C_4 = \{0 < a < a' < 1\}$ (which is the least strict quasivariety of De Morgan algebras), no strict quasivariety such that the least variety containing it is $K$ has RCEP. We still do not know whether or not $Q(C_4)$ has RCEP. Observe that there are not strict quasivarieties contained in $S$.

The basic concepts of universal algebra can be found in [2]. We follow the notation of this book, particularly for the operators on classes of algebras. In addition, by $A \leq_{SD} \prod_{i \in I} A_i$ we mean that $A$ is a subdirect product of the family $\{A_i : i \in I\}$. $A \leq B$ means that $A$ is a subalgebra of $B$. Any class $A$ of algebras such that $Q$ is the least quasivariety containing $A$ is said to generate $Q$ and in this case we write $Q = Q(A)$. An algebra $A \in Q$ is said to be relatively subdirectly irreducible or, $Q$-subdirectly irreducible, if it cannot be subdirectly embedded in a direct product of algebras of $Q$ unless the composite of the embedding with one of the projections is an isomorphism. It can be shown that $A \in Q$ is relatively subdirectly irreducible iff there is a least non-zero $Q$-congruence on $A$. Such a congruence is called the $Q$-monolith of $A$. We denote the class of $Q$-subdirectly irreducible members of $Q$ by $QRSI$.

§2. Quasivarieties That Generate $M$

In this section we prove the following proposition:

**Proposition 2.1.** Let $Q$ be a strict quasivariety such that $H(Q) = M$. Then $Q$ does not have RCEP.

We pave the way for the proof of this proposition with two lemmas.

**Lemma 2.2.** Let $A$ be a homomorphic image of $4$ such that for all $x \in A$, $x' \neq x$. Then, either the algebra depicted in Figure 1, (i) or the one depicted in 1, (ii) is in $IS(A)$

**Proof.** Let $f : A \rightarrow 4$ be a surjective homomorphism. Fix an element $u \in A$ such that $f(u) \notin \{0, 1\}$. Pick $v \in A$ such that $f(u') = f(v)$. Let $a = u \land u'$ and $b = v \land v'$. Clearly, $a$ and $b'$ are not comparable. Let $c = (a \lor b) \land a'$ and $d = (a \lor b) \land b'$. It is routine to verify the following: $c$ and $d$ are not comparable; $c < c'$; $d < d'$; $c' \lor d = c' \lor d' = c \lor d'$; $c \land d = c \land d' = c' \land d$; $c \lor d < c' \lor d'$; $c \land d < c' \land d'$. Thus, the subalgebra of $A$ generated by $c$ and $d$ meets the requirements of the lemma.
Lemma 2.3. Let $A$ be a homomorphic image of $4$ such that there exists $c \in A$ with $c = c'$. Then either $4$ or the algebra depicted in Figure 1, (iii), is isomorphic to a subalgebra of $A$.

Proof. Let $f : A \rightarrow 4$ be a surjective homomorphism. Clearly, $u = f(c) \notin \{0, 1\}$. Let $v$ be the Boolean complement of $u$. Notice that $u' = u$ and $v' = v$. Fix $b \in A$ such that $f(b) = v$. Put $a = b \land b'$ and $d = (c \land a') \lor a$. One checks now that $f(d) = v$. Since $u$ and $v$ are not comparable, so are $c$ and $d$. Using now the hypothesis about $c$ and the fact that $a \leq a'$ one gets $d = d'$. Thus, the subalgebra of $A$ generated by $c$ and $d$ meets the requirements of the lemma.

Proof of Proposition 2.1. By Birkhoff's subdirect representation theorem, there exists $A \in Q - K$ such that one of its homomorphic images is isomorphic to a subalgebra of $4$. In view of the two previous lemmas, one of the algebras depicted in Figure 1 is in $Q$. Let us denote such an algebra by $C$. Let $D$ be the subalgebra of $C$ generated by $t = c \lor d$. The proposition now follows from observing that the $Q$-congruence on $D$ generated by $(0, t')$ coincides with the congruence on $D$ generated by the same element and this $Q$-congruence can not be extended to a $Q$-congruence on $C$.

§3. Quasivarieties that Generate $K$

We start this section proving that there is a least strict quasivariety of De Morgan algebras. Notice that any strict quasivariety contains the variety $B$ of Boolean algebras.

Proposition 3.1. $C_4$ generates the least strict quasivariety of De Morgan algebras.

Proof. Let $Q$ be a strict quasivariety of De Morgan algebras. Clearly $B \subset Q$, so, there exists $A \in Q$ and $a \in A$ such that $0 < d = a \land a'$. Clearly $d \leq d' = a \lor a' < 1$. If $d < d'$, the subalgebra of $A$ generated by $d$ is isomorphic to $C_4$,
so $C_4 \in Q$. If $d = d'$, denote by $D$ the subalgebra of $A$ generated by $d$. As $2 \times D$ has a subalgebra isomorphic to $C_4$, $C_4 \in Q$. Hence the proposition is established. □

We now recall some definitions and results from [4]. Let $L$ be a De Morgan algebra. For a non-empty subset $X$ of $L$, let $X' = \{x' : x \in X\}$. $T(L) \equiv \{t \in L : t \leq t'\} = \{x \land x' : x \in L\}$. Denote by $n(L)$ the ideal of the underlying lattice generated by $T(L)$. $L$ is a Kleene algebra iff $n(L) = T(L)$ iff $T(L)' = \{x \lor x' : x \in L\}$ is a filter [4], Proposition 1.2). $\Theta(n(L))$ (respectively $\Theta(n(L)')$) denote the least $D_{0,1}$-congruence of the underlying lattice which has $n(L)$ (respectively $n(L)'$) as a congruence class (here $D_{0,1}$ denotes the variety of bounded distributive lattices). More precisely, $x \equiv y\Theta(n(L))$ (respectively $\Theta(n(L)')$) iff there exists $j \in n(L)$ (respectively $k \in n(L)'$) such that $x \lor j = y \lor j$ (respectively $x \land k = y \land k$). Let $\beta(L)$ be the least congruence on the De Morgan algebra $L$ such that the quotient algebra $L/\beta(L)$ is a Boolean algebra. Then $\beta(L) = \Theta(n(L)) \lor \Theta(n(L)')$ [4], Theorem 1.3).

Lemma 3.2. Let $Q$ be a strict quasi-variety contained in $K$. Let $L \in Q_{RSI\cdot}$. Let $\alpha$ be the $Q$-monolith of $L$. Then there exist $t, u \in n(L)$ with $t < u$ such that $(t, u)$ generates $\alpha$.

Proof. We first claim that one may choose $a, b$ such that $a \equiv b\Theta(n(L))$ or $a \equiv b\Theta(n(L)')$ with the pair $(a, b)$ generating $\alpha$. To prove the claim, notice that $\beta(L) = \Theta(n(L)) \lor \Theta(n(L)')$ is a $Q$-congruence, so $\alpha \subseteq \Theta(n(L)) \lor \Theta(n(L)')$. Pick $c, d \in L$ with $c < d$ such that $(c, d)$ generates $\alpha$. Thus $c \equiv d\Theta(n(L)) \lor \Theta(n(L)')$. This follows from this that for some $j \in n(L)$ and $k \in n(L)'$, $(c \lor j) \land k = (d \lor j) \land k$. If $c \lor j = d \lor j$ then $c \equiv d\Theta(n(L))$. In this case, take $a = c$ and $b = d$. Otherwise, take $a = c \lor j$ and $b = d \lor j$. In this case $a \equiv b\Theta(n(L)')$. This ends the proof of the claim. Assume now that $a \equiv b\Theta(n(L))$. Observe that since $L$ as a lattice is distributive, either $a \land a' \neq b \land a'$ or $a \land b' \neq b \land b'$ or $a \land a' \neq b \land b'$. If $a \land b' \neq b \land b'$, take $t = a \land b' \land b$ and $u = (a \land b') \lor (b \land b')$. From $a \land b' \equiv b \land b'\Theta(n(L))$ and $b \land b' \in n(L)$ it follows that $a \land b' \in n(L)$. Then, because $n(L)$ is an ideal, $u, t \in n(L)$. The case $a \land a' \neq b \land b'$ is taken care of similarly. If $a \land a' \neq b \land b'$, take $u = (a \land a') \land (b \land b')$ and $t = (a \land a') \lor (b \land b')$. Whatever the case is, $u, t \in n(L)$ and $(u, t)$ generates $\alpha$. Finally, if $a \equiv b\Theta(n(L)')$ then $b' \equiv a'\Theta(n(L))$ and $\alpha$ is also generated by $(b', a')$ and in this situation one can argue as above. Now the proof is complete. □

If $A \in Q$ and $a, b \in A$ then $\Theta^A_Q(a, b)$ denotes the least $Q$-congruence on $A$ which contains the pair $(a, b)$; i.e., the $Q$-congruence generated by $(a, b)$.

Corollary 3.3. In the previous lemma, if $Q$ has RCEP then $t = 0$.

Proof. Assume $t > 0$. As $\alpha$ is the least non-zero $Q$-congruence, $t \equiv u\Theta^L_Q(0, t)$. Now, since $Q$ has RCEP, by Proposition 2.4 of [3], $t \equiv u\Theta^S_Q(0, t)$ where $S = \{0 < t < u < u' < t' < 1\}$ is the subalgebra of $L$ generated by $\{t, u\}$. Observe next that $\Theta^S_Q(0, t) = \Theta^S(0, t)$ because $S/\Theta^S_Q(0, t) \equiv \{0, u, u', 1\} \in Q$. 


(Proposition 3.1). But, as it is easily checked, \( u \not\equiv t \Theta^S(0, t) \), a contradiction. Then \( t \) must be 0. \( \Box \)

**Proposition 3.4.** Let \( Q \neq Q(C_4) \) be a strict quasivariety of Kleene algebras. Then \( Q \) does not have RCEP.

**Proof.** Assume on the contrary that \( Q \) has RCEP. Let \( L \in Q_{RSJ} \) such that \( L \) is not a chain (such an \( L \) must exist; otherwise \( Q \) would be the quasivariety generated by \( C_4 \)) and let \( \alpha \) be the \( Q \)-monolith of \( L \). By Lemma 3.2 and Corollary 3.3 we may pick \( b \in L \) with \( b < c \) such that the pair \((0, b)\) generates \( \alpha \). Pick \( a \in L \) non-comparable to \( b \). Now look at the two possibilities:

\[ a \wedge b = c > 0; \quad a \wedge b = 0. \]

In the first one, \( \alpha \) is also generated by \((0, c)\) and therefore \( b \equiv 0 \Theta^S_Q(0, c) \) where \( S = \{ 0 < c < b < c' < 1 \} \) is the subalgebra of \( L \) generated by \( \{ c, b \} \). It is evident that \( \Theta^S_Q(0, c) = \Theta^S(0, c) \), so \( b \not\equiv 0 \Theta^S(0, c) \) and this is a contradiction. Let us consider now the possibility \( a \wedge b = 0 \). Denote by \( A \) the subalgebra of \( L \) generated by \( a \) and \( b \). Without lost of generality we may assume that either \( a < a' \) or \( a \wedge a' = 0 \). All other possibilities about the comparability of \( a \) and \( a' \) can be reduced to one of these two. Assume first that \( a < a' \).

Observe that \( a \vee b < (a \vee b)' \) because \( n(L) \) is an ideal \((a \vee b) = (a \vee b)' \) is not possible because \( Q \subset \mathcal{K} \). Now, since \( a \) is the least non-zero \( Q \)-congruence and \( Q \) has RCEP, \( b \equiv 0 \Theta^A_Q(0, a) \) (see Figure 2, (i)). Next, notice that \( A/\Theta^A(0, a) \cong C_4 \in Q \); so, \( \Theta^A_Q(0, a) = \Theta^A(0, a) \) and consequently \( b \not\equiv 0 \Theta^A(0, a) \) which is a contradiction. Now we consider the possibility \( a \wedge a' = 0 \). By Corollary 2.5 of [7], \( \Theta(x, 1) = \Theta_{\text{lat}}(x, 1), x \in \{ a, a' \} \), and these two congruences are complement of each other (see [8], Lemma 3.10). Thus \( a' \wedge b > 0 \) \((a' \wedge b = 0 \) implies \( b \equiv 0 \Theta(a', 1) \)). If \( a' \) and \( b \) are not comparable, we proceed as in the earliest case. If \( a' > b \) \((a' \leq b \) is not possible) then \( A \) looks like either the

![Figure 2](image-url)
algebra depicted in Figure 2, (ii) or the one in 2, (iii). If \( a \lor b = b' \) the \( Q \)-congruence generated by \((0, a)\) on the subalgebra of \( A \) generated by \( a \) can not be extended to a \( Q \)-congruence on \( A \), which is a contradiction. If \( a \lor b < b' \), then \( b \equiv 0(\Theta^A_0(0, a) = \Theta^A(0, a)) \), obviously a contradiction. Now the proof of the proposition is complete. \( \Box \)

**Question.** Does \( Q(C_4) \) enjoy RCEP? Proposition 2.9 of [3] can not be used to answer this question in the affirmative because according to Proposition 2.4 of [5], no strict quasivariety of De Morgan algebras is relatively congruence distributive. On the other hand, the method of proof of Fact 2.5 of [3] can not be used to answer it in the negative because \( M \) is not congruence permutable.

**REFERENCES**


(Recibido en julio de 1991)